MATEMATICKÝ ČASOPIS

ROČNÍK 21 1971 ČÍSLO 3

DIMENSIONS OF DISTRIBUTIVE LATTICES

MARTIN GAVALEC, Košice

In this paper the notions of lattice dimension and local dimension for distributive lattices are introduced by a modification of the notion of the dimension of ordered sets. The relations between these three notions are studied. The main result is Theorem 4.1.

0. Notation and terminology

The set-theoretical join and meet are denoted $\cup$, $\cap$, the lattice ones $\vee$, $\wedge$. By $\cup A$ we denote the join of all elements of the set A, analogically for $\cap$, $\vee$, $\wedge$. If X is a set, then $\mathrm{P}(X)$ denotes the set of all subsets of X. An ordered pair of elements x, y is denoted $\langle xy \rangle$, a relation is a set of ordered pairs.

In a lattice, $[xy]$ denotes an interval, i. e. the set $\{z : x \leqslant z \leqslant y\}$, $[x)$ is the set $\{y : x \leqslant y\}$, dually $(x]$. The interval $[xx]$ is called trivial. The denotation $x \prec y$ means that $[xy]$ consists just of two elements, then $[xy]$ is called a prime interval. In a lattice L, the set $\{y : y \in L \;\&\; x \prec y\}$ we denote $\mathrm{N}_L(x)$ or $\mathrm{N}(x)$. If $x = y \wedge u, y \vee u = v$, we write $[xy] \sim [uv]$. The intervals i, j are transposes if either $i \sim j$ or $j \sim i$. The intervals i, j are projective if there are intervals $i_0, i_1, \ldots, i_n$, $n \geqslant 0$ such that i_{k-1}, i_k are transposes for $k = 1, \ldots, n$ and $i = i_0, j = i_n$. The lattice L is locally finite if any interval $[xy]$ in L is finite. For distributive lattices the local finiteness is equivalent to the locally finite length. If L is a lattice, $x \in L$, $b \subseteq L$ and $(\forall\, y,\ z \in b)[y > x \;\&\; (y \neq z \Rightarrow x = y \wedge z)]$, then we say that b is an independent system over x in L. If the dual condition holds, we say that b is an independent system under x in L. The notions used in this paper and not explained here are defined in [1] and [6].

1. Dimensions

1.0. If an ordered set P can be isomorphically embedded into a product of $\mathfrak{k}$ chains ($\mathfrak{k}$ being a cardinal number), but cannot be embedded into a product of less than $\mathfrak{k}$ chains, then the dimension of P is said to be $\mathfrak{k}$, denoted $\dim P =$

$= \mathfrak{k}$. In the case of a lattice we can ask the embedding to be realized by a lattice isomorphism into and we get the notion of the lattice dimension, denoted ldim. It can be easily shown that, if a lattice L is a subdirect product of $\mathfrak{k}$ chains, then

(a) L is distributive,

(b) every independent system b over x in L is of a power less than or equal to $\mathfrak{k}$,

(b′) every independent system b under x in L is of a power less than or equal to $\mathfrak{k}$.

On the other side, by the well-known theorem of Birkhoff, any distributive lattice is a subdirect product of (two-element) chains. Therefore the notion of the lattice dimension is defined just for distributive lattices and the question arises, whether the conditions (a), (b), (b′) imply that the lattice L can be embedded into the product of $\mathfrak{k}$ chains by a lattice isomorphism. For another formulation let us define:

1.1. Definition. *The local dimension of an element x in a lattice L, denoted* $\text{lodim}_L x$, *is the cardinal number* $\sup\{\text{card}\, b : b$ *is an independent system over or under x in L*$\}$. *The local dimension of the lattice L, denoted* lodim L, *is* $\sup\{\text{lodim}_L x : x \in L\}$.

Thus the above problem stands as follows: does the local dimension of a (distributive) lattice coincide with its lattice dimension?

In general the answer is not positive, but we shall prove that for locally finite lattices both dimensions are equal.

First we prove two simple statements on the conditions (b) and (b′).

1.2. Lemma. *If $\mathfrak{k}$ is a finite cardinal, then in a distributive lattice L the conditions* (b) *and* (b′) *are equivalent.*

Proof. Let b be an independent system over x in L, $\text{card}\, b < \aleph_0$. Then let us denote $x' = \bigvee b$ and for $y \in b$ let us set $y' = \bigvee (b - \{y\})$. Then for any $y \in b$ the intervals $[xy]$, $[y'x']$ are transposes and therefore $y' \neq x'$. For $y \neq z$, $y \in b$, $z \in b$ we get $y' \vee z' = x'$ and $y' \neq z'$. Thus $b' = \{y' : y \in b\}$ is an independent system under x' and $\text{card}\, b' = \text{card}\, b$. The converse implication is proved dually.

1.3. Lemma. *In a locally finite lattice L the conditions* (b) *and*

(b″) *for any $x \in L$,* $\text{card}\, N_L(x) \leqslant \mathfrak{k}$

are equivalent.

Proof. For $x \in L$ the set $N_L(x)$ is an independent system over x in L, so that (b) implies (b″). Let (b″) hold, let b be an independent system over x in L. Then for any $y \in b$ there is an element y' less than or equal to y and covering x. For $y \in b$, $z \in b$, $y \neq z$ we have $x = y \wedge z = y' \wedge z'$ and so $y' \neq z'$. Thus $\text{card}\, b \leqslant \text{card}\, N_L(x) \leqslant \mathfrak{k}$ and the lemma is proved.

1.4. Note. The original definition of dimension, given by B. Dushnik and E. W. Miller [2], was a little different from ours. V. Novák [5] showed that this difference is not substantial.

We have mentioned already that for a distributive lattice L lodim $L \leqslant$ $\leqslant$ ldim L holds. Evidently also dim $L \leqslant$ ldim L is true.

Under some conditions on L we can show that lodim $L \leqslant$ dim L. Namely, if b is an independent system over x in a distributive lattice L and card $b = \mathfrak{k}$, then all joins of a finite number of elements of b form a sublattice, which is isomorph to the lattice $\mathrm{P}'(\mathfrak{k})$ of all finite subsets of $\mathfrak{k}$. If L is also complemented (i. e. if L is a Boolean algebra), then analogously L contains a sublattice isomorph to the Boolean algebra $\mathrm{P}''(\mathfrak{k})$ of all finite subsets of $\mathfrak{k}$ and their complements. In these cases, of course, dim $L \geqslant$ dim $\mathrm{P}'(\mathfrak{k})$, dim $\mathrm{P}''(\mathfrak{k})$ respectively. By the method of H. Komm [4] it is easy to show that dim $\mathrm{P}(\mathfrak{k}) =$ dim $\mathrm{P}''(\mathfrak{k}) =$ $= \mathfrak{k}$. Hence if lodim $L \leqslant \aleph_0$ (using $\mathrm{P}(\mathfrak{k}) = \mathrm{P}'(\mathfrak{k})$ for $\mathfrak{k}$ finite) or if L is a Boolean algebra, we have lodim $L \leqslant$ dim L. If L is moreover locally finite, all three dimensions are equal because Theorem 4.1 gives lodim $L =$ ldim L.

If L is not distributive, then the situation dim $L = 2$, lodim $L = \mathfrak{k}$ can occur. We show here an example of such a lattice, which will be even modular and of finite length. L is the lattice given by the following order S on $\mathfrak{k} + 2$: $\mathfrak{k}$ $S\alpha$, αS $\mathfrak{k} + 1$ for any $\alpha \in \mathfrak{k}$. Let L_1 be $\mathfrak{k} + 1$ with the usual ordering of ordinals and let L_2 be $\mathfrak{k} + 1$ dually ordered. Then we define $f(\mathfrak{k}) = \langle 0\mathfrak{k}\rangle$, $f(\mathfrak{k} + 1) =$ $= \langle \mathfrak{k}0\rangle$ and $f(\alpha) = \langle \alpha + 1, \alpha + 1\rangle$. It is easy to see that f is an order isomorphism of L into $L_1 \times L_2$.

1.5. If the local dimension of a distributive lattice L is finite, then the local dimension of any lattice homomorphic image of L is $\leqslant$ lodim L.

Proof. Let f be a lattice homomorphism of L onto L'. Then there is a mapping g of L' into L such that $f(g(x)) = x$ for any $x \in L'$. Now let $M \subseteq L'$ be a finite independent system over $x \in L'$. For any $y \in M$ we define $\bar{y} = \bigvee \{g(z);\ z \in$ $\in M \ \&\ y \neq z\}$ and $\bar{x} = \bigvee \{g(z); z \in M\}$. For $y, z \in M$, $y \neq z$ we have $y \wedge z = x$, therefore $g(y) \wedge g(z) \in f^{-1}(x)$. Thus $g(y) \wedge \bar{y} = \bigvee \{g(y) \wedge g(z);\ z \in M \ \&\ y \neq$ $\neq z\} \in f^{-1}(x)$. On the other hand $g(y) \wedge \bar{z} = g(y) \notin f^{-1}(x)$ for $y \neq z$, $y, z \in M$. Therefore $\bar{y} = \bar{z}$ implies $y = z$ and $\bar{M} = \{\bar{y}; y \in M\}$ is an independent system under $\bar{x}$ of the same cardinality as M. By Lemma 1.2 the proof is finished.

1.6. Note. If lodim $L \geqslant \aleph_0$, then the local dimension of a lattice homomorphic image of L can be $>$ lodim L (even for L distributive), as the following example shows. Let L be the Boolean algebra of all subsets of a countable set S, let J be the ideal of all finite subsets of S. It is well-known that there is an uncountable system F of infinite subsets of S such that $x \cap y$ is finite for $x, y \in F$, $x \neq y$. The canonical homomorphism f of L onto L/J maps F onto uncountable independent system in L/J, *i.* e. lodim $L/J > \aleph_0$, although lodim $L = \aleph_0$.

2. Independent systems

2.0. In this section we give some lemmas on transposed and projective intervals and independent systems in distributive lattices.

2.1. Lemma. *In a distributive lattice the following statements are equivalent*:

(i) $[x_1y_1] \sim [x_1 \vee x_2, y_1 \vee y_2]$ & $[x_2y_2] \sim [x_1 \vee x_2, y_1 \vee y_2]$,

(ii) $[x_1 \wedge x_2, y_1 \wedge y_2] \sim [x_1y_1]$ & $[x_1 \wedge x_2, y_1 \wedge y_2] \sim [x_2y_2]$,

(iii) $[x_1 \wedge x_2, y_1 \wedge y_2] \sim [x_1y_1]$ & $[x_1y_1] \sim [x_1 \vee x_2, y_1 \vee y_2]$,

(iv) $x_1 \wedge x_2 = x_1 \wedge y_2 = y_1 \wedge x_2$ & $y_1 \vee y_2 = x_1 \vee y_2 = y_1 \vee x_2$.

Proof. Let (i) hold. Then $x_1 \vee y_2 = (y_1 \wedge (x_1 \vee x_2)) \vee y_2 = (y_1 \vee y_2) \wedge \wedge ((x_1 \vee x_2) \vee y_2) = (y_1 \vee y_2)$, analogously $y_1 \vee x_2 = y_1 \vee y_2$. Now we have $x_1 \vee (y_1 \wedge y_2) = (x_1 \vee y_1) \wedge (x_1 \vee y_2) = y_1 \wedge (y_1 \vee y_2) = y_1$ and $x_1 \wedge \wedge (y_1 \wedge y_2) = (x_1 \wedge y_1) \wedge y_2 = x_1 \wedge y_2 = (x_1 \wedge (x_1 \vee x_2)) \wedge y_2 = x_1 \wedge \wedge ((x_1 \vee x_2) \wedge y_2) = x_1 \wedge x_2$, i. e. $[x_1 \wedge x_2, y_1 \wedge y_2] \sim [x_1y_1]$. By dual and analogous considerations we see that (i) $\Rightarrow$ (ii) & (iii) & (iv), (ii) $\Rightarrow$ (i) & (iii) & & (iv).

Let (iii) hold. Then $x_1 \wedge y_2 = (x_1 \wedge y_1) \wedge y_2 = x_1 \wedge (y_1 \wedge y_2) = x_1 \wedge x_2$ and $x_1 \wedge x_2 = (y_1 \wedge (x_1 \vee x_2)) \wedge x_2 = y_1 \wedge ((x_1 \vee x_2) \wedge x_2) = y_1 \wedge x_2$, i. e. using duality we have (iii) $\Rightarrow$ (iv).

At last let us assume (iv). Then $y_1 \wedge (x_1 \vee x_2) = (y_1 \wedge x_1) \vee (y_1 \wedge x_2) = x_1 \vee (x_1 \wedge x_2) = x_1$, $y_1 \vee (x_1 \vee x_2) = (y_1 \vee x_1) \vee x_2 = y_1 \vee x_2 = y_1 \vee y_2$, thus (iv) $\Rightarrow$ (i).

2.2. Lemma. *In a distributive lattice the intervals* $[x_1y_1]$, $[x_2y_2]$ *are projective iff* $[x_1y_1] \sim [x_1 \vee x_2, y_1 \vee y_2]$ and $[x_2y_2] \sim [x_1 \vee x_2, y_1 \vee y_2]$.

Proof. One of the implications is trivial. The proof of the other will be done by induction. Let us suppose that $[x_1y_1] \sim [x_1 \vee x, y_1 \vee y]$, $[xy] \sim [x_1 \vee x, y_1 \vee y]$ and that $[xy]$, $[x_2y_2]$ are transposes. We shall show then that $[x_1y_1] \sim \sim [x_1 \vee x_2, y_1 \vee y_2]$, $[x_2y_2] \sim [x_1 \vee x_2, y_1 \vee y_2]$. Let $[x_2y_2] \sim [xy]$ hold. We have $x_2 \leqslant x$, $x_1 \leqslant x_1 \vee x_2 \leqslant x_1 \vee x$ and $x_1 = y_1 \wedge x_1 \leqslant y_1 \wedge (x_1 \vee x_2) \leqslant \leqslant y_1 \wedge (x_1 \vee x) = x_1$, thus $y_1 \wedge (x_1 \vee x_2) = x_1$. Further we have $y_1 \vee (x_1 \vee \vee x_2) = (y_1 \vee x_1) \vee x_2 = y_1 \vee x_2 = y_1 \vee (y_2 \wedge x) = (y_1 \vee y_2) \wedge (y_1 \vee x) = = (y_1 \vee y_2) \wedge (y_1 \vee y) = y_1 \vee y_2$; $(x_1 \vee x_2) \wedge y_2 = (x_1 \vee x_2) \wedge (x_1 \vee x) \wedge \wedge y_2 = (x_1 \vee x_2) \wedge (x_1 \vee x) \wedge y \wedge y_2 = (x_1 \vee x_2) \wedge x \wedge y_2 = (x_1 \vee x_2) \wedge \wedge x_2 = x_2$; $(x_1 \vee x_2) \vee y_2 = x_1 \vee y_2 = (y_1 \wedge (x_1 \vee x)) \vee y_2 = (y_1 \vee y_2) \wedge \wedge (x_1 \vee x \vee y_2) = (y_1 \vee y_2) \wedge (x_1 \vee y) = (y_1 \vee y_2) \wedge (y_1 \vee y) = y_1 \vee y_2$. In the case $[xy] \sim [x_2y_2]$ the proof goes dually according to Lemma 2.1.

2.3. Lemma. *In a distributive lattice let the intervals* $[x_1y_1]$, $[x_2y_2]$ *be projective.*

(i) If $x_1 = x_2$, *then the intervals are equal.*

(ii) If $y_1 \leqslant x_2$, *then the intervals are trivial.*

Proof. From $x_1 = x_2$ we get $y_1 = x_1 \vee y_1 = x_2 \vee y_1 = x_1 \vee y_2 = x_2 \vee y_2 = y_2$, using Lemma 2.1, 2.2.

From $y_1 \leqslant x_2$ we get $y_1 = x_1 \vee y_1 \leqslant x_1 \vee x_2$. By the previous lemma we have $x_1 = y_1 \wedge (x_1 \vee x_2) = y_1$, q. e. d.

2.4. Lemma. *If, in a distributive lattice, a, b are independent systems over x, y respectively, than there exist independent systems c, d over $x \wedge y$, $x \vee y$ respectively, such that* $\operatorname{card} a + \operatorname{card} b = \operatorname{card} c + \operatorname{card} d$.

Proof. Let us set $a_1 = \{z : z \in a \;\&\; (x \vee y) \wedge z > x\}$,

$b_1 = \{z : z \in b \;\&\; (x \vee y) \wedge z > y\}$.

As for $z \in a$ $(x \vee y) \wedge z = x \vee (y \wedge z) \geqslant x$ holds, it is
$a_2 = a - a_1 = \{z : z \in a \;\&\; (x \vee y) \wedge z = x\}$ and analogously
$b_2 = b - b_1 = \{z : z \in b \;\&\; (x \vee y) \wedge z = y\}$.
We define $a_1' = \{y \wedge z : z \in a_1\}$, $b_1' = \{x \wedge z : z \in b_1\}$,

$a_2' = \{y \vee z : z \in a_2\}$, $b_2' = \{x \vee z : z \in b_2\}$,

$a_2'' = \{z : z \in a_2' \;\&\; (\forall t)[t \in b_2' \Rightarrow z \wedge t = x \vee y]\}$,
$b_2'' = \{z : z \in b_2' \;\&\; (\forall t)[t \in a_2' \Rightarrow z \wedge t = x \vee y]\}$,
$c_0 = \{z \wedge t : z \in a_2 \;\&\; t \in b_2 \;\&\; (y \vee z) \wedge (x \vee t) > x \vee y\}$,
$d_0 = \{(y \vee z) \wedge (x \vee t) : z \in a_2 \;\&\; t \in b_2 \;\&\; z \wedge t \in c_0\}$,
$c = c_0 \cup a_1' \cup b_1'$, $d = d_0 \cup a_2'' \cup b_2''$.

If $z \in a_1$, then $x < x \vee (y \wedge z)$ and from $[x \wedge y, y \wedge z] \sim [x, x \vee (y \wedge z)]$ as well as $x \wedge y < y \wedge z$ follows.
Analogously we get $x \wedge y < x \wedge z$ for $z \in b_1$.

If $z \in a_2$, then $[xy] \sim [x \vee y, y \vee z]$ and again we have $x \vee y < y \vee z$ and analogously $x \vee y < x \vee z$ for $z \in b_2$.

If $z \in a_2$, $t \in b_2$, $u = (y \vee z) \wedge (x \vee t) > x \vee y$, then let us denote $z' = z \wedge u$, $t' = t \wedge u$. We have then
$(x \vee y) \wedge z' = (x \vee y) \wedge z \wedge u = x \wedge u = x$,
$(x \vee y) \vee z' = y \vee (z \wedge u) = (y \vee z) \wedge (y \vee u) = (y \vee z) \wedge u = u$.
So $[xz] \sim [x \vee y, u]$ and analogously $[xt'] \sim [x \vee y, u]$. As $z' \vee t' = (z \wedge u) \vee (t \wedge u) = (z \vee t) \wedge u = u$ and $z \wedge t = (z \wedge u) \wedge (t \wedge u) = z \wedge t \wedge u = z \wedge t \wedge (y \vee z) \wedge (x \vee t) = z \wedge t$ holds, by Lemma 2.1 we get $[x \wedge y, z \wedge t] \sim [xz']$, therefore $[x \wedge y, z \wedge t]$, $[x \vee y, u]$ are projective and $x \wedge y < z \wedge t$. Using Lemma 2.3 we can see that $\operatorname{card} c_0 = \operatorname{card} d_0$.

Now we have proved $z \in c \Rightarrow x \wedge y < z$ and $z \in d \Rightarrow x \vee y < z$.
If $z \in a_1$, $t \in a_1$, $z \neq t$, then $(y \wedge z) \wedge (y \wedge t) = y \wedge (z \wedge t) = x \wedge y$.
If $z \in a_1$, $t \in b_1$, then $(y \wedge z) \wedge (x \wedge t) = (x \wedge z) \wedge (y \wedge t) = x \wedge y$.
If $s \in a_1$, $z \in a_2$, $t \in b_2$, then $(y \wedge s) \wedge (z \wedge t) = (s \wedge z) \wedge (y \wedge t) = x \wedge y$.
If $u \in a_2$, $z \in a_2$, $u \neq z$, $v \in b_2$, $t \in b_2$, then $(u \wedge v) \wedge (z \wedge t) = (u \wedge z) \wedge \vee (v \wedge t) = x \wedge v \wedge t = x \wedge (x \vee y) \wedge v \wedge t = x \wedge y \wedge t = x \wedge y$.

The last four implications togehter with three analogous ones prove that c

is an independent system over $x \wedge y$ and that $\operatorname{card} a_1 = \operatorname{card} a_1'$, $\operatorname{card} b_1 = b_1'$, $a_1' \cap b_1' = a_1' \cap c_0 = b_1' \cap c_0 = 0$.

If $z \in a_2$, $t \in a_2$, $z \neq t$, then $(y \vee z) \wedge (y \vee t) = y \vee (z \wedge t) = x \vee y$.

If $z \in a_2$, $t \in b_2$, $y \vee z \in a_2''$, then $(y \vee z) \wedge (x \vee t) = x \vee y$.

If $s \in a_2$, $y \vee s \in a_2''$, $z \in a_2$, $t \in b_2$, $z \wedge t \in c_0$, then $y \vee z \notin a_2''$, $s \neq z$ and $(y \vee s) \wedge ((y \vee z) \wedge (x \vee t)) = ((y \vee s) \wedge (y \vee z)) \wedge (x \vee t) = (y \vee (s \wedge z)) \wedge (x \vee t) = (x \vee y) \wedge (x \vee t) = x \vee y$.

If $u \in a_2$, $z \in a_2$, $u \neq z$, $v \in b_2$, $t \in b_2$, $u \wedge v \in c_0$, $z \wedge t \in c_0$, then $((y \vee u) \wedge (x \vee v)) \wedge ((y \vee z) \wedge (x \vee t)) = ((y \vee u) \wedge (y \vee z)) \wedge ((x \vee v) \wedge (x \vee t)) = (y \vee (u \wedge z)) \wedge (x \vee (v \wedge t)) = (x \vee y) \wedge (x \vee (v \wedge t)) = x \vee y$.

These four implications together with four analogous ones show that d is an independent system over $x \vee y$ and that $\operatorname{card} a_2 = \operatorname{card} a_2'$, $\operatorname{card} b_2 = \operatorname{card} b_2'$, $a_2'' \cap b_2'' = a_2'' \cap d_0 = b_2'' \cap d_0 = 0$, $\operatorname{card} d_0 = \operatorname{card} (a_2' - a_2'') = \operatorname{card} (b_2' - b_2'')$.

Now we have $\operatorname{card} a = \operatorname{card} a_1 + \operatorname{card} a_2 = \operatorname{card} a_1' + \operatorname{card} a_2' = \operatorname{card} a_1' + \operatorname{card} a_2'' + \operatorname{card} (a_2' - a_2'')$, analogously for $\operatorname{card} b$, and then $\operatorname{card} a + \operatorname{card} b = \operatorname{card} a_1' + \operatorname{card} b_1' + \operatorname{card} (a_2' - a_2'') + \operatorname{card} a_2'' + \operatorname{card} b_2'' + \operatorname{card} (b_2' - b_2'') = \operatorname{card} c + \operatorname{card} d$. The proof is complete.

2.5. Note. Although in this paper Lemma 2.4 will be used only for covering systems $N(x)$, $N(y)$ and the proof in the special case would be simpler, it is formulated more generally because it may be useful for other purposes, e. g. for the eventual generalizing of Theorem 3.18 (see also Note 4.4).

3. Finite case

3.0. In this section $\mathfrak{k}$ is a fixed finite non-zero cardinal number and L is a locally finite distributive lattice with $\operatorname{lodim} L = \mathfrak{k}$, i. e. L is a locally finite lattice satisfying the conditions (a), (b″). Then the following statement holds:

3.1. Theorem. *There exist congruences R, Q in L such that*

(i) $\operatorname{lodim} L/R = 1$,

(ii) $\operatorname{lodim} L/Q < \mathfrak{k}$,

(iii) *R, Q are orthogonal, i. e. $R \cap Q$ is the identity on L.*

The proof on Theorem 3.1 is given in several steps.

3.2. Definition. *Let us define an equivalence R' in L^2 as follows: $\langle x_1 y_1 \rangle R' \langle x_2 y_2 \rangle$ iff $[x_1 y_1]$, $[x_2 y_2]$ are projective prime intervals in L.*

Note. If $\langle x_1 y_1 \rangle R' \langle x_2 y_2 \rangle$, then by Lemma 2.2 all the conditions in 2.1 are fulfilled and by Lemma 2.3 we have $y_1 \not\leq x_2$, $y_2 \not\leq x_1$.

For $x, y \in L$ we shall denote $A(x, y) = \{\langle x_1 y_1 \rangle : \langle x_1 y_1 \rangle R' \langle xy \rangle\}$, $A' = \{A(x, y) : [xy] \text{ is a prime interval in } L\}$.

For $A \in A'$ let us denote $A^0 = \{x : (\exists y)[\langle xy \rangle \in A]\}$, $A^1 = \{y : (\exists x)[\langle xy \rangle \in A]\}$.

The structure of $A(x, y)$ is described by

3.3. Lemma. *If $A \in A'$, then*
(i) $A^0 \cap A^1 = 0$,
(ii) *A^0, A^1 are convex sublattices in L,*
(iii) *A is a lattice isomorphism of A^0 onto A^1,*
(iv) $\operatorname{lodim} A^0 = \operatorname{lodim} A^1 < \mathfrak{k}$.

Proof. From the definitions and the note in 3.2 we immediately get (i). By the Lemmas 2.1, 2.2, A^0, A^1 are sublattices in L. For the proof of the convexity of A^0 let $x_1 \leqslant x_2 \leqslant x_3$, $x_1, x_3 \in A^0$. Hence there exists y_1 such that $\langle x_1 y_1 \rangle \in A$. The assumption $y_1 \leqslant x_2$ gives $y_1 \leqslant x_3$, which is impossible by Note 3.2. From $y_1 \not\leqslant x_2$ we get $y_1 \wedge x_2 = x_1$ and $[x_1 y_1] \sim [x_2, y_1 \vee x_2]$. Therefore $\langle x_2, y_1 \vee \vee x_2 \rangle \in A$ and $x_2 \in A^0$. The convexity of A^1 is proved dually.

From Lemma 2.3 we can see that A is a one-to-one mapping of A^0 onto A^1, preserving the lattice operations by Lemma 2.1. Thus (iii) has been checked. At last (iv) follows from (i), (ii), (iii) and Lemma 1.3.

The extent of $A(x, y)$ in L is shown by the following

3.4. Lemma. *If $A \in A'$, $z \in L$, then*
(i) *$x \in A^0$, $x \prec z$ implies $\langle xz \rangle \in A$ or $z \in A^0$,*
(ii) *there exist $x \in A^0$, $y \in A^1$ such that $z \leqslant x$ or $z \geqslant y$.*

Proof. From $x \in A^0$, $\langle xy \rangle \in A$ for some $y \in L$ follows. If $x \prec z \neq y$, then $[xy] \sim [z, y \vee z]$ and $z \in A^0$.

For the proof of (ii) let $\langle xy \rangle \in A$. We denote $w = (z \vee x) \wedge y = (z \wedge y) \vee x$. There is $x \leqslant w \leqslant y$, thus $x = w$ or $w = y$. If $x = w$ holds, then $[xy] \sim \sim [x \vee z, y \vee z]$ and $x \vee z \in A^0$. In the case $w = y$ we have analogously $y \wedge z \in A^1$.

3.5. Definition. *Let us define the relations P', P'' in A' as follows: For A, $B \in A'$ we set $\langle AB \rangle \in P'$ if and only if*
(i) $A^1 \cap B^0 \neq 0$,
(ii) $A^0 \cap B^1 = 0$.

Further we denote as $\overline{AB}$ the convex hull of $A^1 \cup B^0$ and set $\langle AB \rangle \in P''$ if and only if $\langle AB \rangle \in P'$ and
(iii) $\operatorname{lodim} \overline{AB} < \mathfrak{k}$.

We examine some properties of P':

3.6. Lemma. *If $\langle AB \rangle \in P'$, then*
(i) $A \neq B$,
(ii) $A^0 \cap B^0 = A^1 \cap B^1 = 0$,
(iii) *for any $y \in A^1$ there is $x \in B^0$ such that $y \leqslant x$,*
(iv) *$\langle xy \rangle \in A$, $\langle uv \rangle \in B$ implies $x \not\geqslant u$, $y \not\geqslant v$,*
(v) *if $x \leqslant y$, $x \in B^0$, $y \in A^1$, then x, $y \in A^1 \cap B^0$.*

Proof. From 3.3(i), 3.5(i) we easily get (i).

Let be $x \in A^0 \cap B^0$, then $\langle xy \rangle \in A$, $\langle xz \rangle \in B$ for some $y, z \in L$. By (i) we have $y \neq z$, therefore by 3.4(i) we have $z \in A^0$, which contradicts 3.5(ii). Analogously we prove $A^1 \cap B^1 = 0$.

Let $y \in A^1$, by 3.5(i) there is $v \in A^1 \cap B^0$, $\langle vw \rangle \in B$. Let us denote $x = y \vee v$. By 3.3(ii) we have $x \in A^1$ and so the assumption $w \leqslant x$ would give by 3.3(ii) $w \in A^1$, which is a contradiction to (ii). Thus $w \nleqslant x$ and $[vw] \sim [x, w \vee x]$, $x \in B^0$.

If $y \in A^1$, $v \in B^1$, then (iii) gives us $z \in B^0$ such that $z \geqslant y$. Accordingly, by the note in 3.2 we have $y \ngeqslant v$.

At last let $x \leqslant y$, $x \in B^0$, $y \in A^1$, then $\langle xz \rangle \in B$ for some $z \in L$. By (iv) we have $z \nleqslant y$, hence $[xz] \sim [y, z \vee y]$ and $y \in B^0$. Analogously $x \in A^1$.

3.7. Lemma. *If $x \prec y$, $x \in A^1$, $y \notin A^1$, then for $B = A(x, y)$, $\langle AB \rangle \in P'$ holds.*

Proof. We have $x \in A^1 \cap B^0$ and therefore 3.5(i) is fulfilled. As $x \in A^1$, there is $u \in L$ such that $\langle ux \rangle \in A$. If we assume $A^0 \cap B^1 \neq 0$, then there are $\langle u'x' \rangle \in B$, $\langle x'y' \rangle \in A$. By 3.3(ii) we have $x \vee y' \in A^1$, therefore $y \vee x' \neq x \vee y'$ holds, otherwise we had $y \in A^1$ by 3.3(ii). Further we have $x \vee u'$, $u \vee x' \leqslant y \vee x'$ and $x \vee u'$, $u \vee x' \leqslant x \vee y'$, so that denoting $w = (x \vee u') \vee (u \vee x')$, $z = (y \vee x') \wedge (x \vee y')$ we get $x \vee u' \leqslant w \leqslant z \leqslant y \vee x'$. We have shown that there cannot be $y \vee x' = z = x \vee y'$, therefore, e. g., $z < y \vee x'$ (the case $z < x \vee y'$ is analogous). By Lemma 2.2 $x \vee u' \prec y \vee x'$ holds, therefore $x \vee u' = w = z$. But then $u \vee x' \leqslant x \vee u' \leqslant x \vee y'$, $u \vee x' \prec x \vee y'$ holds and we have $u \vee x' = x \vee u'$ or $x \vee u' = x \vee y'$. In the first case we get $u \leqslant x \leqslant u \vee x'$, $u \in A^0$, $u \vee x' \in A^0$ and by the convexity of A^0 also $x \in A^0$, which is a contradiction with 3.3(i) because $x \in A^1$. In the second case we have $u' \leqslant x' \leqslant y' \leqslant x \vee y' = x \vee u'$, $u' \in B^0$, $x \vee u' \in B^0$, hence $x' \in B^0$, a contradiction. We have therefore proved 3.5(ii) and the proof is finished.

3.8. Lemma. *If $\langle AB \rangle \in P'$, then*

(i) *$z \in \overline{AB}$ exactly when there are $y \in A^1$, $x \in B^0$ such that $y \leqslant z \leqslant x$,*

(ii) *$x \in \overline{AB}$, $x \prec y$ imply $\langle xy \rangle \in B$ or $y \in \overline{AB}$.*

Proof. If $y \in A^1$, $x \in B^0$, $y \leqslant z \leqslant x$ hold, then evidently $z \in \overline{AB}$. To prove the converse implication it suffices to show that the set S of all such z is a convex sublattice in L, containing $A^1 \cup B^0$. But that follows from 3.3(ii) and 3.6(iii).

Now let $x \in \overline{AB}$, $x \prec y$, then by (i) there are $u \in A^1$, $v \in B^0$ such that $u \leqslant x \leqslant v$. If $y \leqslant v$, then $y \in \overline{AB}$. If $y \nleqslant v$, then $[xy] \sim [v, y \vee v]$ and by 3.4(i) we have $\langle v, y \vee v) \rangle \in B$ and $\langle xy \rangle \in B$, or $y \vee v \in B^0$ and $y \in \overline{AB}$.

3.9. Lemma. *If $x, y \in L$ and card $N_L(x) =$ card $N_L(y) = \mathfrak{k}$, then* card $N_L(x \wedge y) =$ card $N_L(x \vee y) = \mathfrak{k}$.

Proof. Follows from Lemma 2.4.

3.10. Lemma. *If for $A \in A'$ there is $B \in A'$ such that $\langle AB \rangle \in P'$, then there is $C \in A'$ such that $\langle AC \rangle \in P''$.*

Proof. Let us denote $B' = \{B : \langle AB \rangle \in P'\}$. For any $B \in B'$ there is $x_B \in \in A^1 \cap B^0$. Let B'' be a finite subset of B', we denote $x' = \bigvee \{x_B : B \in B''\}$ and by 3.3(ii) we have $x' \in A^1$. Then by 3.6(v) we get $x' \in A^1 \cap B^0$ for any $B \in B''$. Therefore card $B'' \leqslant \mathfrak{k}$ must hold. As B'' was an arbitrary finite subset of B', also B' must be finite. Let us suppose that for any $B \in B'$, lodim $\overline{AB} = \mathfrak{k}$ holds. Then by 3.8(i) for any $B \in B'$ there are $u_B \in A_1$, $v_B \in B^0$, $z_B \in [u_B, v_B]$ such that $\mathrm{N}_L(z_B) \subseteq [u_B, v_B]$ and card $\mathrm{N}_L(z_B) = \mathfrak{k}$. We denote $u = \bigwedge \{u_B : : B \in B'\}$, $z = \bigwedge \{z_B : B \in B'\}$ and have $u \in A^1$, $u \leqslant z$. By 3.9 we get $\mathrm{N}_L(z) = = \mathfrak{k}$. *Hence according to 3.3(iv),* $\mathrm{N}_L(z) \subseteq A^1$ *cannot be true, therefore there exists* $z' \notin A^1$, $z \prec z'$. *It means that there exist also* $x \in A^1$, $y \notin A^1$ *such that* $x \prec y$, $u \leqslant x \leqslant z$. *Denoting* $C = A(x, y)$ *we get by 3.7 that* $\langle AC \rangle \in P'$ *and therefore* $C \in B'$. *Thus we have* $x \leqslant z \leqslant z_C \prec w \leqslant v_C$ *for any* $w \in \mathrm{N}_L(z_C)$. *But* $x \in C^0$, $v_C \in C^0$ *give by 3.3(ii) a contradiction with 3.3(iv), card* $\mathrm{N}_L(z_C) = \mathfrak{k}$.

3.11. Definition. *Let Q' be the relation in A', which is the reflexive and transitive hull of the relation P'', i. e. Q' is the meet of all relations in A', fulfilling*
(i) $\langle AA \rangle \in Q'$,
(ii) $\langle AB \rangle \in P''$ *implies* $\langle AB \rangle \in Q'$,
(iii) $\langle AB \rangle \in Q'$ *and* $\langle BC \rangle \in Q'$ *imply* $\langle AC \rangle \in Q'$
for any $A, B, C \in A'$.

Note. One can easily see that $\langle AB \rangle \in Q'$ exactly when there exist $A_0, A_1, \ldots, A_n \in A'$ with $n \geqslant 0$ such that $\langle A_{i-1}A_i \rangle \in P''$ for i = 1, …, n and $A = A_0$, $B = A_n$.

3.12. Lemma. *If* $A, B \in A'$, $\langle AB \rangle \in Q'$, *then*
(i) *if* $A \neq B$, *then for any* $y \in A^1$ *there exists* $x \in B^0$ *such that* $y \leqslant x$,
(ii) *if* $\langle BA \rangle \in Q'$, *then* $A = B$,
(iii) *if* $\langle AB \rangle \in P''$, *then there is no* $C \in A'$ *such that* $A \neq C \neq B$, $\langle AC \rangle \in Q'$, $\langle CB \rangle \in Q'$.

Proof. According to the previous note, (i) is proved by repeated using of 3.6(iii).

If $\langle AB \rangle \in Q'$, $\langle BA \rangle \in Q'$, $A \neq B$, then by (i) there exist $y \in A^1$, $u \in B^0$, $v \in B^1$, $x \in A^0$ such that $y \leqslant u \prec v \leqslant x$, which is a contradiction by the note in 3.2.

Let be $\langle AB \rangle \in P''$, $A \neq C \neq B$, $\langle AC \rangle$, $\langle CB \rangle \in Q'$. Then there exists $D \in A'$ such that $\langle CD \rangle \in Q'$, $\langle DB \rangle \in P''$ and by (ii) $A \neq D$ holds. By 3.5(i) there exists $y \in A^1 \cap B^0$ and by (i) we have $u \in D^0$, $v \in D^1$, $w \in B^0$ such that $y \leqslant \leqslant u \prec v \leqslant w$, therefore $u \in B^0$ by the convexity of B^0. But this is a contradiction with 3.6(ii).

3.13. Definition. *By the previous lemma, Q' is a partial order in A'. Using* 3.4(i), 3.8(ii) *and the local finiteness of L it is easy to show that an interval $[AB]$ in (A', Q') is not longer than any interval $[uv]$ with $u \in A^0$, $v \in B^0$* (by 3.12(i) *such u, v always exist if $\langle AB \rangle \in Q'$*).

Throughout this section, M will denote an arbitrary but fixed maximal chain in (A', Q'). By the above remark, M is locally finite and evidently non-empty (we have assumed $\mathfrak{k} \neq 0$).

Now the set $\bar{R}$ is defined as follows:
if A is the least element in M, then $A^0 \in \bar{R}$,
if A is the greatest element in M, then $A^1 \in \bar{R}$,
if $A, B \in M$ and $\langle AB \rangle \in P''$, then $\overline{AB} \in \bar{R}$,
$\bar{R}$ does not contain any other elements.

3.14. Lemma. *$\bar{R}$ is a partition of L, i. e.*
(i) *$a, b \in \bar{R}$ and $a \neq b$ imply $a \cap b = 0$,*
(ii) $\cup \bar{R} = L$.
Denoting by R the equivalence determined by $\bar{R}$ we have
(iii) *R is a congruence in L,*
(iv) *L/R is a chain,*
(v) *if $\langle xy \rangle \in \cup M$, then $\langle xy \rangle \notin R$.*

Proof. Let $a, b \in \bar{R}$, $a \neq b$, $z \in a \cap b$. Let either $a = C^0$, C being the least element in M, or $a = \overline{AC}$, $A, C \in M$, $\langle AC \rangle \in P''$, and either $b = \overline{BD}$, $B, D \in M$, $\langle BD \rangle \in P''$ or $b = B^1$, B being the greatest element in M. First we suppose $\langle CB \rangle \in Q'$. By 3.8(i) we have elements $u \in C^0$, $y \in B^1$ such that $y \leqslant z \leqslant u$. This leads to a contradiction with the note in 3.2 if $C = B$. Therefore $C \neq B$ and by 3.12(i) there are $v \in C^1$, $t \in B^0$ such that $u \prec v \leqslant t$ and again $y \leqslant t$ is a contradiction with the same note. If we suppose $A = B$, we get by 3.12(iii) $C = D$ and $a = b$. Other possibilities for a, b are verified analogously, thus (i) has been proved.

Let $z \in L - \cup \bar{R}$, let $X \in M$. Then by 3.4(ii) there exists $y \in X^1$ such that $y \leqslant z$ (the other case is dual). If X is not the greatest element in M, from the local finiteness of M we get $Y \in M$, $\langle XY \rangle \in P''$ and $\overline{XY} \in \bar{R}$. Hence $y \in \cup \bar{R}$ and by the local finiteness of L we can find $u \in \cup \bar{R}$, $v \notin \cup \bar{R}$, $u \prec v$. There is not $u \in A^0$, A being the least element in M, nor $u \in \overline{AB}$, $\langle AB \rangle \in P''$, $A, B \in M$, because then by 3.4(i), 3.8(ii), respectively, $v \in \cup \bar{R}$ would hold. Thus $u \in A^1$, A being the greatest element in M. Then by 3.7 for $B = A(u, v)$, $\langle AB \rangle \in P'$ is true and therefore by 3.10 there exists $C \in A'$ such that $\langle AC \rangle \in P''$. That is a contradiction with the maximality of A in M and M in (A', Q').

For the proof of (iii), (iv) let us take $x, y \in L$.

If x is arbitrary and $y \in A^1$, where A is the greatest element in M, then $x \vee y \geqslant y$ and the assumption $x \vee y \notin A^1$ gives elements $u \in A^1$, $v \notin A^1$

such that $u \prec v$. Again by 3.7, 3.10 we get $C \in A'$, $\langle AC \rangle \in P''$ and that is a contradiction with the maximality of A and M. Therefore $x \vee y \in A^1$.

If either $x \in \overline{AC}$, $A, C \in M$, $\langle AC \rangle \in P''$, or $x \in C^0$, C being the least element in M and if $y \in \overline{BD}$, $B, D \in M$, $\langle BD \rangle \in P''$, then let $\langle CB \rangle \in Q'$. We have $C \neq D$ and by 3.8(i), 3.12(i) we get elements $u \in C^0$, $v \in C^1$, $z \in D^0$, $s \in B^1$, $t \in D^0$ such that $x \leqslant u \prec v \leqslant z$, $s \leqslant y \leqslant t$. Then $s \leqslant y \leqslant x \vee y \leqslant z \vee t$ gives by 3.8(i) $x \vee y \in \overline{BD}$. If $A = B$, then $C = D$ and $x \vee y \in \overline{AC} = \overline{BD}$.

By this and by the dual reasoning we have proved that if $a, b \in \bar{R}$, then either for any $x \in a$, $y \in b$, $x \vee y \in a$ and $x \wedge y \in b$ hold or for any $x \in a$, $y \in b$ we have $x \vee y \in b$, $x \wedge y \in a$. Hence (iii), (iv) have been proved.

Let $\langle xy \rangle \in A \in M$. If A is the only element of M, then evidently $\langle xy \rangle \notin R$. In the other case there is $B \in M$, $\langle AB \rangle \in P''$ (or dually). Then $y \in \overline{AB}$ but $x \notin \overline{AB}$ by 3.8(i) and by the note in 3.2.

3.15. Definition. *We set $\langle xy \rangle \in Q$ iff there exist elements $x_0, x_1, \ldots, x_n \in L$, $n \geqslant 0$ such that $\langle x_{i-1}x_i \rangle \in \bigcup M$ for $i = 1, \ldots, n$ and either $x = x_0$, $y = x_n$ or $x = x_n$, $y = x_0$.*

3.16. Lemma.

(i) *Q is a congruence in L,*

(ii) *if $\langle xy \rangle \in R$ and $\langle xy \rangle \in Q$, then $x = y$,*

(iii) *for $x, y \in L$, $x \prec y$ implies $\langle xy \rangle \in Q \cup R$.*

Proof. The reflexivity and the symmetry of Q follow immediately from 3.15. To prove the transitivity let us suppose $\langle xy \rangle$, $\langle yz \rangle \in Q$. Then there are $x_0, \ldots, x_n, y_0, \ldots, y_m \in L$, $n, m \geqslant 0$ such that $\langle x_{i-1}x_i \rangle$, $\langle y_{j-1}y_j \rangle \in \bigcup M$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$. If $x = x_0$, $y = x_n = y_0$, $z = y_m$ or $z = y_0$, $y = y_m = x_0$, $x = x_n$, then $\langle xz \rangle \in Q$ by 3.15. Hence let $x = x_n$, $y = x_0 = y_0$, $z = y_m$ (the case $x = x_0$, $y = x_n = y_m$, $z = y_0$ is dual). For the proof by induction we assume $x_{i-1} = y_{i-1}$, $1 \leqslant i \leqslant n, m$. Then $\langle x_{i-1}x_i \rangle \in A \in M$, $\langle y_{i-1}y_i \rangle \in B \in M$ gives $A^0 \cap B^0 \neq 0$. As M is a chain, we have $\langle AB \rangle \in Q'$ (or dually). If we suppose $A \neq B$, by the local finiteness of M we have $\langle AC \rangle \in$ $\in Q'$, $\langle CB \rangle \in P''$ for some $C \in A'$. From 3.12(i) we get the elements $u \in C^0$, $v \in C^1$, $w \in B^0$ such that $x_{i-1} \leqslant u \prec v \leqslant w$. By the convexity of B^0 we have $u \in B^0$, which is a contradiction with 3.6(ii). Hence $A = B$ and by 3.3(iii) $x_i = y_i$ holds. We have proved $x = x_n = y_n$ if $n \leqslant m$ or $z = y_m = x_m$ if $m \leqslant n$. By 3.15 then $\langle xz \rangle \in Q$.

Let $\langle xy \rangle \in \bigcup M$, $z \in L$. If $x \vee z \geqslant y$ holds, then $x \vee z = y \vee z$. If $x \vee z \ngeqslant y$, then $[xy] \sim [x \vee z, y \vee z]$ and $\langle x \vee z, y \vee z \rangle \in A(x, y) \in M$. Dually $\langle x \wedge z, y \wedge z \rangle \in Q$ is shown. According to the transitivity of Q, we have proved that Q preserves the lattice operations in L.

Let $x, y \in a \in \bar{R}$, let there exist $x_0, \ldots, x_n$, $n \geqslant 0$ such that $x = x_0$, $y = x_n$, $\langle x_{i-1}x_i \rangle \in \bigcup M$ for $i = 1, \ldots, n$. The convexity of a implies $x_i \in a$ for $i =$

$= 0, \ldots, n$. By 3.14(v) $\langle x_0 x_1 \rangle \in \cup M$ gives $x_1 \notin a$. Therefore $n = 0$, i. e. $x = y$.

For the proof of (iii) let $x, y \in L$, $x \prec y$. If $x \in A^0$, $A \in M$, then by 3.4(i) we have either $\langle xy \rangle \in A$ and $\langle xy \rangle \in Q$ or $y \in A^0$ and $\langle xy \rangle \in R$. If $x \in \overline{AB}$, $\langle AB \rangle \in \in P''$, A, $B \in M$, then by 3.8(ii) we have either $\langle xy \rangle \in B$ and $\langle xy \rangle \in Q$ or $y \in \overline{AB}$ and $\langle xy \rangle \in R$. If $x \in A^1$, A being the greatest element in M, then using 3.7, 3.10 and the maximality of A, M we get $y \in A^1$ and $\langle xy \rangle \in R$.

3.17. Lemma.

(i) *If X, $Y \in L/Q$ and $X \prec Y$, then there exists $x_0 \in X$ such that for any $x \in X$, $x_0 \leqslant x$ there is $y \in Y$ such that $x \prec y$,*

(ii) lodim $L/Q < \mathfrak{k}$.

Proof. Let us take arbitrary $x' \in X$, $y' \in Y$. As $X \prec Y$, then $x' \vee y' \in Y$ and for any $z \in [x', x' \vee y']$ we have $z \in X$ or $z \in Y$; therefore there exist $x_0 \in X$, $y_0 \in Y$ in $[x', x' \vee y']$ such that $x_0 \prec y_0$. Now let $x \in X$, $x_0 \leqslant x$. Then $y_0 \leqslant x$ would give $y_0 \in X$, therefore $y_0 \nleq x$ and denoting $y = x \vee y_0$ we have $[x_0 y_0] \sim [xy]$ and $x \prec y$. Moreover, $[x_0 x] \sim [y_0 y[$ and so $\langle x_0 x \rangle \in Q$ gives $\langle y_0 y \rangle \in Q$ and $y \in Y$.

Let $X \in L/Q$ such that $Y \in B$ implies $X \prec Y$. Then for any $Y \in B$ there is x_Y with the property of x_0 from (i). Let us suppose card $B = \mathfrak{k}$ and denote $x = \bigvee \{x_Y : Y \in B\}$. Then $x \in X$ and for any $Y \in B$ there is $y \in Y$ such that $x \prec y$. By 3.16(iii) then $\langle xy \rangle \in R$ for any $Y \in B$. But this is a contradiction, because by 3.3(iv), 3.5(iii) lodim $a < \mathfrak{k}$ for $a \in \bar{R}$.

3.18. Theorem. *If L is a locally finite distributive lattice, then* lodim $L = = \mathfrak{k} < \aleph_0$ *if and only if L is a subdirect product of $\mathfrak{k}$ chains.*

Proof. Let lodim $L = \mathfrak{k} < \aleph_0$. We go by induction through $\mathfrak{k}$. If $\mathfrak{k} = 1$, then L is a chain and the assertion of the theorem is true. If $k > 1$, we use Theorem 3.1, proved by 3.14(iii), (iv), 3.16(i), (ii), 3.17(ii), and get L as a subdirect product of a chain L/R and of the lattice L/Q, which has its local dimension less than $\mathfrak{k}$ and therefore, by induction assumption, is a subdirect product of less than $\mathfrak{k}$ chains. Thus L itself is a subdirect product of $\leqslant \mathfrak{k}$ chains. The converse implication is trivial.

4. Main theorem

4.0. Here we extend Theorem 3.18 for arbitrary cardinal $\mathfrak{k}$ and then show by an example thas local and lattice dimensions need not be equal.

4.1. Theorem. *If L is a locally finite distributive lattice, then* lodim $L = \mathfrak{k}$ *if and only if L is a subdirect product of $\mathfrak{k}$ achains.*

Proof. One of the implications is trivial. The other is proved by Theorem 3.18 for $\mathfrak{k} < \aleph_0$. Thus let $\mathfrak{k}$ be infinite. A dimension function f on L, i. e.

a function with integer values fulfilling $f(x) + 1 = f(y)$ for any $x, y \in L$, $x \prec y$, can be defined (see e. g. [6]). For $x \in L$ and for an integer i we denote $\mathrm{M}_i(x) = \{y : f(y) = i \ \&\ y$ comparable with $x\}$. If $i = f(x)$, then apparently $\mathrm{M}_i(x) = \{x\}$. As L fulfils (b″) and the dual condition, by induction through i we easily prove card $\mathrm{M}_i(x) \leqslant \mathfrak{k}$ for any integer i (using $\mathfrak{k}^2 = \mathfrak{k}$ for $\mathfrak{k}$ infinite). As $[x) = \cup \{\mathrm{M}_i(x) : i \geqslant f(x)\}$ and dually for $(x]$, we get card $[x)$, card $(x] \leqslant \mathfrak{k}$. But $L = \cup \{(y] : y \in [x)\}$ for any $x \in L$ and therefore card $L \leqslant \mathfrak{k}$. By the Birkhoffs theorem, the lattice L, being distributive, can be embedded into a product $\prod\{L_a : a \in A\}$ of (two-element) chains by a lattice isomorphism g. For any $x, y \in L$, $x \neq y$ we take an index $a = a(x, y) \in A$ such that $g_a(x) \neq g_a(y)$ (here $g_a(x)$ denotes the a-th component of $g(x)$) and set $A' = \{a(x, y) : x, y \in L, x \neq y\}$. Then the mapping h of L into $\prod\{L_a : a \in A'\}$ defined by setting $h_a(z) = g_a(z)$ for any $z \in L$, $a \in A'$ is a lattice isomorphism of L into a product of $\leqslant \mathfrak{k}$ chains.

4.2. Definition. *Let $\mathfrak{k}$ be an infinite cardinal, let $D_\mathfrak{k}$ be the Cartesian product of $\mathfrak{k}$ topological spaces, each of them being the two-point space with the discrete topology. By $C_\mathfrak{k}$ we denote the Boolean algebra of all open regular subsets of $D_\mathfrak{k}$, ordered by inclusion.*

4.3. Theorem. Lodim $C_\mathfrak{k} = \aleph_0$; ldim $C_\mathfrak{k} > \aleph_0$ *if* $\mathfrak{k} > 2^{\aleph_0}$.

Proof. It is easy to construct an infinite system of pairwise disjoint elements in $C_\mathfrak{k}$, therefore lodim $C_\mathfrak{k} \geqslant \aleph_0$ holds. On the other side, it is known (see e. g. [3]) that in the Cartesian product of topological space with countable bases any system of non-empty and pairwise disjoint open sets is countable, which gives lodim $C_\mathfrak{k} \leqslant \aleph_0$. Thus lodim $C_\mathfrak{k} = \aleph_0$.

Let us suppose now that $C_\mathfrak{k}$ is embedded into a product of countably many chains by a lattice isomorphism. As a lattice homorphic image of a Boolean algebra is again a Boolean algebra and the only linearly ordered Boolean algebra is the two-element one, we may assume that any chain in the product contains exactly two elements. Then the cardinality of the product is $2^{\aleph_0}$, but the cardinality of $C_\mathfrak{k}$ is at least $\mathfrak{k}$, which is a contradiction for $\mathfrak{k} > 2^{\aleph_0}$.

4.4. Note. The Boolean algebra $C_\mathfrak{k}$ and the assertion lodim $C_\mathfrak{k} = \aleph_0$ are not new. By this counterexample we did not intend to say that lodim $L =$ ldim L could not hold at all without the local finiteness of L. On the contrary, is seems probable that with some additional conditions, e. g. for $\mathfrak{k}$ finite, Theorem 4.1 could be proved also for (distributive) lattices which are not necessarily locally finite.

REFERENCES

[1] Birkhoff G., *Lattice Theory*, New York 1948.
[2] Dushnik B., Miller E. W., *Partially Ordered Sets*, Amer. J. Math. *63* (1941), 600—610.
[3] Engelking R., Karlowicz M., *Some Theorems of Set Theory and Their Topological Consequences*, Fundam. Math. *57* (1965), 275—285.
[4] Komm H., *On the Dimension of Partially Ordered Sets*, Amer. J. Math. *70* (1948), 507—520.
[5] Novák V., *On the Pseudodimenstion of Ordered Sets*, Czech. Math. Journ. *13* (88), 1963, 587—598.
[6] Szász G., *Einführung in die Verbandstheorie*, Budapest 1962.

Received November 21, 1968

Katedra matematiky
Prírodovedeckej fakulty
Univerzity P. J. Šafárika
Košice