Daniel Palumbín

On a Certain Type of Decompositions of Complete Graphs into Factors with Equal Diameters

*Matematický časopis*, Vol. 22 (1972), No. 3, 235--242

Persistent URL: [http://dml.cz/dmlcz/126516](http://dml.cz/dmlcz/126516)

---

**Terms of use:**

© Mathematical Institute of the Slovak Academy of Sciences, 1972

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use.*

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* [http://project.dml.cz](http://project.dml.cz)
ON A CERTAIN TYPE OF DECOMPOSITIONS OF COMPLETE GRAPHS INTO FACTORS WITH EQUAL DIAMETERS

DANIEL PALUMBÍNY, Zvolen

Paper [2] deals with the existence of a decomposition of the complete graph into factors with given diameters.

In the present paper we shall study the existence of a decomposition of the complete graph into factors with equal diameters if both, the number \( h \) of factors and the difference \( s \) between the number of vertices and the diameter of factors are given. We shall solve this problem for \( s = 1, 2 \) if \( h \geq 2 \) and for \( h = 2 \) if \( s \geq 1 \); we give some results in the case of \( s = 3 \), too.

Lemma 4 was proved by Š. Znám (unpublished). I wish to thank Š. Znám for his kind permission to publish Lemma 4 in the present paper, as well as for his suggestions used in it.

* *

All graphs considered in the present paper are undirected, finite, connected, without loops and multiple edges. The complete graph with \( n \) vertices will be (like in [2]) denoted by \( \langle n \rangle \). By a factor of a graph \( G \) we mean a subgraph of \( G \) containing all vertices of \( G \). By a decomposition of a graph \( G \) into factors we mean such a system of factors of \( G \) that every edge of \( G \) is contained in exactly one factor of the system. The diameter \( d \) of \( G \) is the maximum of the set of all distances \( \varrho_G(x, y) \) between the pairs of vertices \( (x, y) \) of \( G \).

For our further considerations we shall need some results of [2]. In [2] the symbol \( F(d_1, d_2, \ldots, d_h) \) means the smallest natural number \( n \) such that the graph \( \langle n \rangle \) can be decomposed into \( h \) factors with diameters \( d_1, d_2, \ldots, d_h \); if such a natural number does not exist, then \( F(d_1, d_2, \ldots, d_h) = \infty \).(1) In [2] the following statements were proved:

---

(1) In this place in [2], the cardinal number is considered. In [2] the problem of a decomposition of the complete graph into two factors was solved completely. In [3] it was proved that if \( h \geq 3 \) and \( d_i \geq 2 \) for \( i = 1, 2, \ldots, h \), then \( F(d_1, d_2, \ldots, d_h) \neq \infty \). For this reason it is sufficient to consider the case of the natural number only.

235
(a) (Theorem 1.) If the complete graph $\langle n \rangle$ ($n > 1$) is decomposable into $h$ factors with diameters $d_1, d_2, \ldots, d_h$, then for $N > n$ the complete graph $\langle N \rangle$ is also decomposable into $h$ factors with the diameters $d_1, d_2, \ldots, d_h$.

(b) (The second part of Theorem 2.) Let the natural numbers $h, n, d_1, d_2, \ldots, d_h$ be given. If the complete graph $\langle n \rangle$ is decomposable into $h$ factors with the diameters $d_1, d_2, \ldots, d_h$, then

$$2h \leq n.$$ 

(c) (Corollary 2 of Theorem 2.) Let $h, n$ and $d$ be natural numbers. If $\langle n \rangle$ is decomposable into $h$ factors with equal diameters $d$, then

$$n^2 - (2h + 1)n \leq h(s^2 + s - 4),$$

where $s = n - d$.

(d) (A special case of Theorem 5.) $F(2, 2) = 5, F(3, 3) = 4$ and $F(d, d)$ otherwise.

Let the natural numbers $s \geq 1$, $h \geq 2$ be given. Our aim is to determine all natural numbers $d$ such that the complete graph $\langle d + s \rangle$ is decomposable into $h$ factors with equal diameters $d$, for every pair $(s, h)$. Let us denote by the symbol $D_{s,h}$ the set of natural numbers $d$ assigned to a certain pair $(s, h)$ in this way.(1) First (Theorem 1) we prove that for every pair $(s, h)$ there exists such a natural number $d$, i.e. the set $D_{s,h}$ is not empty.

**Theorem 1.** Let natural numbers $s, h, d$ be given such that $h \geq 2$ and $d = 2h - 1$. Then the complete graph $\langle d + s \rangle$ is decomposable into $h$ factors with equal diameters $d$.

**Proof.** According to [1], p. 91, every complete graph $\langle 2h \rangle$ can be decomposed into $h$ factors with equal diameters $2h - 1$. From Theorem 1 of [2] it follows that an arbitrary complete graph with a greater number of vertices can be decomposed in this way.

In the following we prove five lemmas.

**Lemma 1.** Let a natural number $h \geq 3$ be given. Then every complete graph $\langle 2h \rangle$ is decomposable into $h$ factors with equal diameters $2h - 2$.

**Proof.** The decomposition of $\langle 6 \rangle$ into 3 factors with the diameter 4 is shown in Fig. 1.

(a) Let $h \geq 4$ be an even number, i.e. $h = 2r$, where $r \geq 2$, then the factor

---

(1) Thus $\langle n \rangle$ can be decomposed into $h$ factors with equal diameters $d$ obviously if and only if $d \in D_{n-a,h}$. 

236
$F_1 (F_{r+1})$ has the form shown in Fig. 2 (Fig. 3). The factors $F_j (F_{r+j})$, $2 \leq j \leq r$ can be obtained from $F_1 (F_{r+1})$ by replacing each vertex $i$ by the vertex $i + j - 1 \pmod{2h}$.

(b) Let $h > 4$ be an odd number, i.e. $h = 2r - 1$, where $r \geq 3$, then the factor $F_1 (F_{r+1})$ has the form shown in Fig. 4 (Fig. 5). The factors $F_j (F_{r+j})$, $2 \leq j \leq r (2 \leq j \leq r - 1)$ can be obtained from $F_1 (F_{r+1})$ by replacing each vertex $i$ by the vertex $i + j - 1 \pmod{2h}$.

![Fig. 1](image1)

![Fig. 2](image2)

![Fig. 3](image3)

![Fig. 4](image4)

![Fig. 5](image5)
We can easily check that every edge of the graph \(2K\) is contained in exactly one of the factors \(F_i (i = 1, 2, \ldots, h)\) and consequently the system of subgraphs \(F_t\) forms a decomposition of \(2K\) into \(h\) factors. E. g. in case (a) this follows from the fact that if we draw \(2K\) in the form of a regular polygon with all its diagonals, then the union of the factors \(F_1\) and \(F_{r+1}\) consists just of the edges (diagonals) parallel to \((1, 4r), (2, 4r), (1, 2r)\) and \((1, 2r + 1)\), and cyclic permutations of vertices correspond to rotations of \(F_1\) and \(F_{r+1}\). It is also evident that each of the factors \(F_t\) has the diameter \(2h - 2\).

**Lemma 2.** Let a natural number \(h \geq 3\) be given. Then every complete graph \(2K\) is decomposable into \(h\) factors with equal diameters \(2h - 3\).

**Proof.** The factor \(F_1\) has the form shown in Fig. 6. The remaining factors \(F_t (i = 2, 3, \ldots, h)\) can be obtained from \(F_1\) by cyclic permutations of vertices. The rest of the proof is similar to that in Lemma 1.

![Fig. 6](image)

**Lemma 3.** Let natural numbers \(s \geq 1, h \geq 2\) be given. Then for an arbitrary \(d \in D_{s,h}\) we have:

\[
(3) \quad d \leq \frac{1}{2} (2h + 1 - 2s + \sqrt{4h^2 + 4h(s^2 + s - 3) + 1}) .
\]

**Proof.** The substitution \(n = d + s\) in the inequality (2) gives:

\[
(4) \quad d^2 + d(2s - 2h - 1) + h(4 - 3s - s^2) + s^2 - s \leq 0 .
\]

The left side of (4) is a quadratic function of the variable \(d\). As this function is convex, the solution of the inequality (4) with respect to \(d\) is

\[
d_2 \leq d \leq d_1,
\]

where

\[
d_{1,2} = \frac{1}{2} (2h + 1 - 2s \pm \sqrt{4h^2 + 4h(s^2 + s - 3) + 1})
\]

are the roots of the left-hand side of (4). For natural numbers \(s \geq 1, h \geq 2\), the expression \(4h^2 + 4h(s^2 + s - 3) + 1\) is positive and so the roots \(d_{1,2}\) are real. Thus we have (3).

**Lemma 4.** Let natural numbers \(s \geq 1, h \geq 2\) be given. Then for an arbitrary \(d \in D_{s,h}\) we have:

238
\[
\begin{align*}
    d &\leq 2h - 1 & \text{if } s = 1, \\
    d &\leq 2h - 2 + \binom{s}{2} & \text{if } s \geq 2.
\end{align*}
\]

Proof. If \( s = 1 \), the first relation follows from (3). Now we assume that \( s \geq 2 \). If under the square root on the right-hand side of the inequality (3) we write 4 instead of 1, we have

\[
d < \frac{1}{2}(2h + 1 - 2s + 2\sqrt{h^2 + h(s^2 + s - 3) + 1}).
\]

For \( s \geq 2 \) we have \((s^2 + s - 3)/2 > 1\) and therefore we can replace the unit under the square root on the right-hand side of (5) by the expression \([(s^2 + s - 3)/2]^2\). Then we have

\[
d < 2h - 1 + \frac{1}{2}(s^2 - s).
\]

The right-hand side of the inequality (6) being an integer, we can write

\[
d \leq 2h - 2 + \binom{s}{2}.
\]

This completes the proof of the lemma.

**Lemma 5.** Let natural numbers \( s \geq 1 \), \( h \geq 2 \) be given. Then for an arbitrary \( d \in D_{s,h} \) we have:

\[
d \geq \max [2, (2h - s)].
\]

Proof. The substitution \( n = d + s \) in the inequality (1) gives \( d \geq 2h - s \). To complete the proof of the lemma it is sufficient to take into account that \( d \geq 2 \).

Let us denote

\[
\begin{align*}
    \max D_{s,h} &= G(s, h), \\
    \min D_{s,h} &= g(s, h).
\end{align*}
\]

From Theorem 1 and Lemmas 4 and 5 it follows that \( G(s, h) \) and \( g(s, h) \) are defined for arbitrary integers \( s \geq 1 \) and \( h \geq 2 \). The following two theorems are dealing with the functions \( G \) and \( g \).

**Theorem 2.** Let \( s \geq 1 \) and \( h \geq 2 \) be integers. Then we have:

\[
\begin{align*}
    \text{I. } & G(1, h) = 2h - 1, \\
    \text{II. } & G(2, h) = 2h - 1, \\
    \text{III. } & G(3, h) = \begin{cases} 2h & \text{if } 3 \leq h \leq 5, \\
                          2h + 1 & \text{if } h \geq 6, 
\end{cases} \\
    \text{IV. } & G(s, 2) = 3.
\end{align*}
\]
Proof. I. From Lemma 4 it follows that $G(1, h) \leq 2h - 1$. Therefore it is sufficient to prove that the graph $\langle 2h - 1 + 1 \rangle = \langle 2h \rangle$ is decomposable into $h$ factors with equal diameters $2h - 1$. According to Theorem 1 such a decomposition evidently exists.

II. The proof is analogous.

III. For $s = 3$ from Lemma 4 we have

(7) \[ G(3, h) \leq 2h + 1. \]

The substitution $s = 3$ and $h = 3, 4, 5$ in (3) successively gives: $G(3, 3) \leq 6$, $G(3, 4) \leq 8$, $G(3, 5) \leq 10$, i.e.

(8) \[ G(3, h) \leq 2h \quad \text{if} \quad 3 \leq h \leq 5. \]

(a) As (8) holds, to prove that $G(3, h) = 2h$ if $h = 3, 4, 5$ it is sufficient to decompose the graph $\langle 2h + 3 \rangle$ into $h$ factors with equal diameters $2h$, when $h = 3, 4, 5$. These decompositions are shown in Fig. 7 (see also Fig. 6 of [2]), Figs. 8 and 9.

(b) If $h \geq 6$, the inequality (7) holds.

IV. According to Theorem 5 of [2] we have $F(3, 3) = 4$ and $F(d, d) = \infty$ if $d > 3$. From this and from Theorem 1 of [2] it follows that 3 is the greatest natural number $d$ such that the graph $\langle d + s \rangle$ is decomposable into two factors with equal diameters $d$. Thus $G(s, 2) = 3$.
Theorem 3. Let \( s \geq 1 \) and \( h \geq 2 \) be integers. Then we have:

I. \( g(1, h) = 2h - 1 \),

II. \( g(2, h) = \begin{cases} 
3 & \text{if } h = 2, \\
2h - 2 & \text{if } h \geq 3,
\end{cases} \)

III. \( g(3, h) = 2h - 3 \) if \( h \geq 3 \),

IV. \( g(s, 2) = 2 \) if \( s \geq 3 \).

Proof. The proof of I follows from Lemma 5 and Theorem 1.

II. (a) With respect to Lemma 5 \( g(2, 2) \geq 2 \) holds, but \( g(2, 2) \neq 2 \), which follows from [2] (Theorem 5). In order to prove that \( g(2, 2) = 3 \), it is sufficient to decompose the graph \( \langle 3 + 2 \rangle = \langle 5 \rangle \) into two factors with diameter 3. According to Theorems 5 and 1 of [2] such a decomposition evidently exists.

(b) If \( h \geq 3 \), from Lemma 5 it follows that \( g(2, h) \geq 2h - 2 \). Lemma 1 shows that the lower bound gives the exact value.

III. If \( h \geq 3 \), from Lemma 5 it follows that \( g(3, h) \geq 2h - 3 \). Lemma 2 shows that the lower bound gives the exact value.

IV. With respect to Theorem 5 of [2] \( F(2, 2) = 5 \) holds. From Theorem 1 of [2] it follows that the graph \( \langle 2 + s \rangle = \langle 3 \rangle \) is decomposable into two factors with diameter 2. Thus \( g(s, 2) = 2 \) if \( s \geq 3 \).

Remark. Theorems 2 and 3 allow us to determine the set \( D_{s,h} \) for these cases: \( D_{1,h} = \{2h - 1\} \), \( D_{2,2} = \{3\} \), \( D_{s,2} = \{2, 3\} \) if \( s \geq 3 \) and \( D_{2,h} = \{2h - 2, 2h - 1\} \) if \( h \geq 3 \). In cases \( D_{3,3}, D_{3,4} \) and \( D_{3,5} \) Theorems 2 and 3
determine only the maximum and minimum of the given sets, but by systematic decompositions of the corresponding graphs we can see that \( D_{3,3} = \{3, 4, 5, 6\} \), \( D_{3,4} = \{5, 6, 7, 8\} \) and \( D_{3,5} = \{7, 8, 9, 10\} \).

REFERENCES


Received December 12, 1970

Katedra matematiky a deskriptívnej geometrie
Vysoké školy lesnické a drevárskej
Zvolen

242