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PRODUCTS OF VECTOR MEASURES

CHARLES SWARTZ

1. Introduction

In [4], [5] and [8] M. Duchon and I. Kluvanek have discussed the notion of the product of two vector measures where the product in each case is taken to be the tensor product. In [9] and [25] the product of vector measures is also considered but in these papers the product is taken to be the inner product in a Hilbert space; a somewhat similar situation is considered in [28]. A notion of the product of operator-valued measures is considered in [15] and [19]. In this paper we consider a general notion of the product of two vector measures and attempt to give conditions which will furnish positive results of the nature of those given by Duchon and Kluvanek in [8] for the ε-tensor product.

Let X_1, X_2 and Z be locally convex Hausdorff spaces and let $b : X_1 \times X_2 \to Z$ be a separately continuous bilinear map. (The bilinearity assumption is made for convenience; b could also be taken to be sesquilinear, [9], [25].) Let S_1, S_2 be non-void sets and let Σ_1 and Σ_2 ($\mathscr{A}_1$ and $\mathscr{A}_2$) be σ-algebras (algebras) of subsets of S_1 and S_2 respectively. For any family $\mathfrak{T}$ of subsets of a non-void set, let a $a(\mathfrak{T})$ denote the algebra generated by $\mathfrak{T}$ and let $\sigma(\mathfrak{T})$ denote the σ-algebra generated by $\mathfrak{T}$.

If $\mu_i : \mathscr{A}_i \to X_i$ (or $\mu_i : \Sigma_i \to X_i$) is a finitely additive set function, the product, $\mu_1 \times \mu_2$, of μ_1 and μ_2 (with respect to b) is defined on $\mathscr{A}_1 \times \mathscr{A}_2$ (or $\Sigma_1 \times \Sigma_2$) by $\mu_1 \times \mu_2(A_1 \times A_2) = b(\mu_1(A_1), \mu_2(A_2))$, $A_i \in \mathscr{A}_i$ (or $A_i \in \Sigma_i$). If $H \in a(\mathscr{A}_1 \times \mathscr{A}_2)$ (or $a(\Sigma_1 \times \Sigma_2)$), then $H = \bigcup_{i=1}^{n} A_i \times B_i$ where $\{A_i \times B_i\}$ are pairwise disjoint with $A_i \in \mathscr{A}_i$ and $B_i \in \mathscr{A}_2$ (or $A_i \in \Sigma_1$ and $B_i \in \Sigma_2$) ([16], 33. E), and we may extend $\mu_1 \times \mu_2$ to $a(\mathscr{A}_1 \times \mathscr{A}_2)$ by setting $\mu_1 \times \mu_2(H) = \sum_{i=1}^{n} b(\mu_1(A_i), \mu_2(B_i))$. Then $\mu_1 \times \mu_2$ is a finitely additive Z-valued set function on $a(\mathscr{A}_1 \times \mathscr{A}_2)$ (or $a(\Sigma_1 \times \Sigma_2)$).

The various types of products of vector measures treated in the literature fit into the abstract setup above. For example, in [8] $Z = X_1 \hat{\otimes}_\varepsilon X_2$ and the bilinear map is that given by the tensor product while in [4] Z is $X_1 \hat{\otimes}_\pi X_2$. In [9] and [25], $X_1 = X_2 = H$, where H is a Hilbert space, Z is the scalar field, an the map b is taken to be the inner product on $H \times H$ (if H is complex, b is of course sesquilinear rather than bilinear, but this annoyance causes no real difficulties). Also in [19], I. Kluvanek and M. Kovarikova consider the product of spectral measures: in [19] X is a B-space and $X_1 = X = Z = B(X)$, where $B(X)$ denotes the B-space of bounded linear operators on X, and $b(T, S) = TS$ is the composition of T and S. A similar product is utilized in [15].

At this point there are several natural questions which arise relative to $\mu_1 \times \mu_2$. For example, if $\mu_i : \Sigma_i \to X_i$ is countably additive, is $\mu_1 \times \mu_2$ countably additive on $a(\Sigma_1 \times \Sigma_2)$ and if this is the case, does $\mu_1 \times \mu_2$ have a countably additive extension to $\sigma(\Sigma_1 \times \Sigma_2)$? The examples presented in [9] and [25] show that even in the case when $X_1 = X_2 = H$, a Hilbert space, and b is the inner product, $\mu_1 \times \mu_2$ will not in general be countably additive on $a(\Sigma_1 \times \Sigma_2)$. A similar phenomena occurs when $Z = X_1 \hat{\otimes}_\pi X_2$ ([18]). On the other hand, when $Z = X_1 \hat{\otimes}_\varepsilon X_2$, $\mu_1 \times \mu_2$ will always have a countably additive extension to $\sigma(\Sigma_1 \times \Sigma_2)$, [8]. It should also be pointed out that if either μ_1 or μ_2 has bounded variation, then $\mu_1 \times \mu_2$ always has a countably additive extension to $\sigma(\Sigma_1 \times \Sigma_2)$ regardless of the nature of b, [5]. We consider the question of the countable additivity of $\mu_1 \times \mu_2$ in section 2 and present some fairly general assumptions which guarantee countable additivity on $a(\Sigma_1 \times \Sigma_2)$ and the existence of a countably additive extension to $\sigma(\Sigma_1 \times \Sigma_2)$. In section 3 we consider the case where each $\mu_i : \mathscr{A}_i \to X_i$ is strongly bounded ([2]) and give conditions that insure that $\mu_1 \times \mu_2$ is strongly bounded. In section 4 we consider the question of regularity of the product of two regular vector measures. Finally in the concluding section 5 we give some indication of the necessity of the assumptions made in the preceding three sections.

Before proceeding to the material concerning products of vector measures, we present a lemma concerning scalar measures which will be needed later. Two parts of this lemma appear in [8], but we give entirely different proofs here which present (hopefully) interesting applications of the Dunford-Pettis property ([14]; [11], 9.4). In the lemma and throughout the remainder of thepaper, we use the notation and terminology of [10].

Lemma 1.

(a) *If $\Gamma_i \subseteq ca(\Sigma_i)$ is conditionally weakly compact* ($i = 1, 2$), *then* $\Gamma_1 \times \Gamma_2 = \{\mu_1 \times \mu_2 : \mu_i \in \Gamma_i\}$ *is conditionally weakly compact in* $ca(\sigma(\Sigma_1 \times \Sigma_2))$.

(b) *If $\Gamma_i \subseteq ba(\mathscr{A}_i)$ is conditionally weakly compact ($i = 1, 2$), then $\Gamma_1 \times \Gamma_2$ is conditionally weakly compact in $ba(a(\mathscr{A}_1 \times \mathscr{A}_2))$.*

(c) *Let λ_i be a positive finite measure on Σ_i ($i = 1, 2$). If $\Gamma_i \subseteq L^1(\lambda_i)$ is conditionally weakly compact, then $\Gamma_1 \otimes \Gamma_2$ is conditionally weakly compact in $L^1(\lambda_1 \times \lambda_2)$, where if $f_i \in L^1(\lambda_i)$, $f_1 \otimes f_2 : S_1 \times S_2 \to R$ is given by $(s, t) \to f_1(s)f_2(t)$.*

(d) *If $\Gamma_i \subseteq ca(\Sigma_i)$ is uniformly absolutely continuous with respect to the positive measure $\lambda_i \in ca(\Sigma_i)$, then $\Gamma_1 \times \Gamma_2 \subseteq ca(\sigma(\Sigma_1 \times \Sigma_2))$ is uniformly absolutely continuous with respect to $\lambda_1 \times \lambda_2$.*

Proof: For (a), note the bilinear map $(\mu_1, \mu_2) \to \mu_1 \times \mu_2$ from $ca(\Sigma_1) \times ca(\Sigma_2) \to ca(\sigma(\Sigma_1 \times \Sigma_2))$ is continuous since $v(\mu_1 \times \mu_2)(S_1 \times S_2) = v(\mu_1)(S_1)\, v(\mu_2)(S_2)$ ([10], III. 11. 11). The result now follows from [14], Proposition 1.2.2 or [11], 9.4.3 (c) and the Smulian—Eberlein Theorem ([10], V. 6.1) since $ca(\Sigma_i)$ has the Dunford—Pettis property ([14], 1.4 or [11], 9.4.6 (d)).

Condition (b) can be established exactly as part (a) above (recalling IV. 9.11 of [10]) once the continuity of the bilinear map $(\mu_1, \mu_2) \to \mu_1 \times \mu_2$ is established. Theorem III. 11.11 of [10] cannot be used directly here since the set functions involved are only finitely additive (the proof of this result in [10] uses the Radon—Nikodym Theorem). Suppose $E = \bigcup_{i=1}^{n} A_i \times B_i$ belongs to $a(\mathscr{A}_1 \times \mathscr{A}_2)$ with the union disjoint and also $\{A_i\}_1^n$ pairwise disjoint. Then $|\mu_1 \times \mu_2(E)| \leqslant v(\mu_2)(S_2) \sum_{i=1}^{n} |\mu_1(A_i)| \leqslant v(\mu_1)(S_1)\, v(\mu_2)(S_2)$. By [10], III. 1.5, $v(\mu_1 \times \mu_2)(S_1 \times S_2) \leqslant 4v(\mu_1)(S_1)\, v(\mu_2)(S_2)$ which implies that the bilinear map above is continuous.

Part (c) is established exactly as part (a) using the fact that $L^1(\lambda_i)$ has the Dunford—Pettis property ([11], 9.4.4).

For (d), let $\Gamma_i' = \left\{ \frac{d\mu}{d\lambda_i} : \mu \in \Gamma_i \right\} \subseteq L^1(\lambda_i)$. By [10], IV. 8.11, Γ_i' is conditionally weakly compact in $L^1(\lambda_i)$. By (c), $\Gamma_1' \otimes \Gamma_2'$ is conditionally weakly compact in $L^1(\lambda_1 \times \lambda_2)$. The result now follows from [10], IV. 8.11 and [17], 21.29.

Remark 2. In [8], part (c) is established first and then (a) follows as a corollary. The result in (c) is also an easy consequence of part (a) and Theorem IV. 9.2 of [10]. Part (b) does not appear in [8], and the proof of part (a) presented in [8] cannot be adapted to derive (b) since the countable additivity of the measures in question is used at several points. It may be possible to derive (b) from (a) by using a „Stonespace technique", [10], IV. 9.10.

2. Countable Additivity

In this section we consider the question of countable additivity for the product $\mu_1 \times \mu_2$ of two vector measures. The basic assumption made on the map b is essentially that it be an integral-type bilinear map. This appears to be the difference between the results for the inductive and projective tensor product as given in [8] and [4]. Recall a scalarvalued bilinear map f on $X_1 \times X_2$ is an integral map iff there exist weak*-closed equicontinuous subsets $A_i \subseteq X_i'$ and a regular probability measure m on the Borel sets of $A_1 \times A_2$ (equipped with the weak* topologies) such that

$$(1) \quad f(x, y) = \int_{A_1 \times A_2} \langle x', x \rangle \langle y', y \rangle \, dm(x', y')$$

(see [29], §49 and [27], §7 and 16 for the properties of integral maps). The space of all scalar-valued integral maps on $X_1 \times X_2$ is denoted by $J(X_1, X_2)$; $J(X_1, X_2)$ is the dual of $X_1 \otimes_\varepsilon X_2$ ([29], §49 and [27], §7). Throughout this paper we consider the following two fundamental assumptions on the bilinear map $b : X_1 \times X_2 \to Z$:

(α) $z'b$ is an integral bilinear form for each $z' \in Z'$

(β) for each equicontinuous subset $D \subseteq Z'$, $\{z'b : z' \in D'\}$ is an equicontinuous subset of $J(X_1, X_2)$ (considered as the dual of $X_1 \otimes_\varepsilon X_2$).

Of course, when Z is the scalar field, (α) and (β) are equivalent. Examples are presented following Theorem 3 illustrating circumstances when (α) and (β) are valid

Theorem 3. *Let $\mu_i : \Sigma_i \to X_i$ be countably additive ($i = 1, 2$).*

(a) *If condition (α) is satisfied, then $\mu_1 \times \mu_2$ is weakly countably additive on $a(\Sigma_1 \times \Sigma_2)$.*

(b) *If condition (β) is satisfied, then $\mu_1 \times \mu_2$ is countably additive on $a(\Sigma_1 \times \Sigma_2)$ and has a countably additive extension from $\sigma(\Sigma_1 \times \Sigma_2)$ to $\tilde{Z}$, the completion of Z.*

Proof: Let $z' \in Z$ and $H = \bigcup_{i=1}^{n} E_i \times F_i \in a(\Sigma_1 \times \Sigma_2)$ with the union disjoint and $E_i \in \Sigma_1$, $F_i \in \Sigma_2$. Since $z'b$ is integral, there exist weak*-closed equicontinuous subsets $A_i \subseteq X_i'$ and a regular probability measure $m (= m_{z'})$ on $A_1 \times A_2$ such that $z'b(x, y) = \int_{A_1 \times A_2} \langle x', x \rangle \langle y', y \rangle \, dm(x', y')$ for $x \in X_1$, $y \in X_2$. Hence,

$$(2) \quad \langle z', \mu_1 \times \mu_2(H) \rangle = \sum_{i=1}^{n} \int_{A_1 \times A_2} \langle x', \mu_1(E_i) \rangle \langle y', \mu_2(F_i) \rangle \, dm(x', y') \leq$$

$$\leq \int_{A_1 \times A_2} \sum_{i=1}^{n} v(x' \mu_1)(E_i) \, v(y' \mu_2)(F_i) \, dm(x', y') \leq$$

$$< \int_{A_1 \times A_2} v(x'\mu_1) \times v(y'\mu_2)(H)\, dm(x', y') \leqslant$$

$$\leqslant \sup_{(x', y') \in A_1 \times A_2} v(x'\mu_1) \times v(y'\mu_2)(H) m_{z'}(A_1 \times A_2).$$

Now $\{x'\mu_1 : x' \in A_1\}$ and $\{y'\mu_2 : y' \in A_2\}$ are conditionally weakly compact in $ca(\Sigma_1)$ and $ca(\Sigma_2)$ ([30], Corollary of Theorem 2) so $\{v(x'\mu_1) : x' \in A_1\} = \Gamma_1$ and $\{v(y'\mu_2) : y' \in A_2\} = \Gamma_2$ are also conditionally weakly compact ([10], IV. 8.10). By Lemma 1, $\Gamma_1 \times \Gamma_2$ is conditionally weakly compact in $ca(\sigma(\Sigma_1 \times \Sigma_2))$, and therefore $\Gamma_1 \times \Gamma_2$ is uniformly countably additive ([10], IV 9.1). If $\{H_n\}$ is a sequence in $a(\Sigma_1 \times \Sigma_2)$ which decreases to $\emptyset$, then (2) implies $\langle z', \mu_1 \times \mu_2(H_n)\rangle \to 0$ so that $\langle z', \mu_1 \times \mu_2(.)\rangle$ is countably additive on $a(\Sigma_1 \times \Sigma_2)$. Hence $\mu_1 \times \mu_2$ is weakly countably additive on $a(\Sigma_1 \times \Sigma_2)$ and (a) follows.

To establish (b), let p be a continuous semi-norm on Z. Set $U = \{z \in Z : p(z) < 1\}$ and let U° be the polar of U in Z'. Since U° is equicontinuous, $\{z'b : z' \in U^\circ\}$ is equicontinuous in $J(X_1, X_2)$ so there exist weak*-closed equicontinuous sets $A_i \subseteq X_i'$ and a bounded family of positive regular measures $\{m_{z'} : z' \in U^\circ\}$ on $A_1 \times A_2$ such that $z'b(x, y) = \int_{A_1 \times A_2} \langle x', x\rangle \langle y', y\rangle\, dm_{z'}(x', y')$ for $x \in X_1$, $y \in X_2$ ([29], p. 502 and [27], remark following 7.11). The estimate in equation (2) becomes $|\langle z', \mu_1 \times \mu_2(H)\rangle| \leqslant M \sup_{(x', y') \in A_1 \times A_2} v(x'\mu_1) \times v(y'\mu_2)(H)$, where M is the bound for $\{m_{z'} : z' \in U^\circ\}$ and $z' \in U^\circ$. Hence

$$(3) \quad p(\mu_1 \times \mu_2(H)) \leqslant M \sup_{(x', y') \in A_1 \times A_2} v(x'\mu_1) \times v(y'\mu_2)(H)$$

for $H \in a(\Sigma_1 \times \Sigma_2)$. Now as in the first part of the proof $\Gamma_i = \{v(x'\mu_i) : x' \in A_i\}$ is conditionally weakly compact so there exists a positive measure $\lambda_i \in ca(\Sigma_i)$ such that Γ_i is uniformly absolutely continuous with respect to λ_i ([10], IV. 9.2). By Lemma 1, $\Gamma_1 \times \Gamma_2$ is uniformly absolutely continuous with respect to $\lambda_1 \times \lambda_2 \in ca(\sigma(\Sigma_1 \times \Sigma_2))$. Thus, equation (3) and Corollary 1 of [3] imply that $\mu_1 \times \mu_2$ has a countably additive extension from $\sigma(\Sigma_1 \times \Sigma_2)$ to $\tilde{Z}$, and (b) is established.

Remark 4. Note equation (2) yields Axiom A of Duchon ([7]), and equation (3) is similar to condition (B) of [19]. In part (a) it can also be asserted that $\mu_1 \times \mu_2$ has a unique extension from $\sigma(\Sigma_1 \times \Sigma_2)$ to Z'' which is countably additive with respect to the topology $\sigma(Z'', Z')$ (see [21] and also [7], Th. 1). In part (b) if Z is sequentially complete, the extension of $\mu_1 \times \mu_2$ will actually have values in Z (see [3]).

Note also the equality in (1) is not actually required but only the inequality $|f(x, y)| \leqslant \int_{A_1 \times A_2} |\langle x', x\rangle \langle y', y\rangle|\, dm(x', y')$; such bilinear maps could be considered to be the bilinear analogue of the quasi-integral operators of [23].

We now present some examples illustrating conditions under which the assumptions (α) and (β) hold.

Example 5. Take $Z = X_1 \hat{\otimes}_\varepsilon X_2$. Then condition ($\beta$) is clearly satisfied so part (b) of Theorem 3 holds. This gives the result of M. Duchon and I. Kluvanek ([8], Theorem) concerning the existence of a countably additive extension of $\mu_1 \times \mu_2$ from $\sigma(\Sigma_1 \times \Sigma_2)$ into Z.

Example 6. Let Z be sequentially complete. Suppose $A_i \subseteq X_i'$ is weak*-closed and equicontinuous and let m be a Z-valued regular countably additive vector measure defined on the Borel sets of $A_1 \times A_2$. Define $b : X_1 \times X_2 \to Z$ by $b(x, y) = \int_{A_1 \times A_2} \langle x', x\rangle \langle y', y\rangle \, \mathrm{d}m(x', y')$, where the integral is taken in the sense of [20]. (Note the integral exists by Theorem 2.2 of [20].) That condition (β) is satisfied follows from Tweddle [30] and the characterization of the equicontinuous subsets of $J(X_1, X_2)$ noted in the proof of Theorem 3 (b). Bilinear maps of this type furnish a vector generalization of the integral bilinear forms of Grothendieck.

Example 7. Let Z be sequentially complete. A special subclass of the maps in Example 6 is given as follows. Let $\{t_k\} \in l^1$, $\{e_k'\}$ and $\{f_k'\}$ be equicontinuous sequences in X_1' and X_2' respectively, and $\{z_k\}$ be a bounded set in Z. Let $b : X_1 \times X_2 \to Z$ be given by $b(x, y) = \sum_k t_k \langle e_k', x\rangle \langle f_k', y\rangle z_k$. Such bilinear maps furnish a vector generalization of nuclear bilinear forms ([24] 7.4).

3. Strong Boundedness

In this section we discuss the strong boundedness of the product of two strongly bounded vector-valued set functions. Recall that if $\mathscr{A}$ is an algebra of sets and if $\mu : \mathscr{A} \to Z$ is finitely additive, then μ is strongly bounded if for each continuous semi-norm p on Z there is a positive finitely additive set function λ on $\mathscr{A}$ such that $\lim_{\lambda(A) \to 0} p(\mu(A)) = 0$ (this is the locally convex generalization of the notion of strong boundedness as discussed for B-spaces by Brooks, [2]; see also [3], Theorem 1). The methods and results of this section are quite similar to those of section 2 so we only outline the proofs.

Theorem 8. *Let $\mu_i : \mathscr{A}_i \to X_i$ be strongly bounded ($i = 1, 2$). If condition (β) is satisfied, then $\mu_1 \times \mu_2$ is strongly bounded on $a(\mathscr{A}_1 \times \mathscr{A}_2)$.*

Proof: Let p be a continuous semi-norm on Z and set $U = \{z : p(z) \leq 1\}$. Then U° is equicontinuous and with the notion as in the proof of part (b) of Theorem 3 we obtain the inequality in (3). Since μ_i is strongly bounded there exists a positive $\lambda_i \in ba(\mathscr{A}_i)$ (depending on A_i) such that $\lim_{\lambda_i(B) \to 0} x'\mu_i(B)$

$\to 0$ uniformly for $x' \in A_i$ ([29], Prop. 36.1). As in Theorem 3, $\Gamma_i = \{v(x'\mu_i) : x' \in A_i\}$ is conditionally weakly compact in $ba(\mathscr{A}_i)$ ([10], IV. 9.12) and by Lemma 1, $\Gamma_1 \times \Gamma_2$ is conditionally weakly compact in $ba(a(\mathscr{A}_1 \times \mathscr{A}_2))$. By IV. 9.12 of [10], there exists a positive $\nu \in ba(a(\mathscr{A}_1 \times \mathscr{A}_2))$ such that $\lim_{\nu(H) \to 0} v(x'\mu_1) \times v(y'\mu_2)(H) = 0$ uniformly for $x' \in A_1$, $y' \in A_2$. Thus equation (3), which still holds in this situation, implies that $\mu_1 \times \mu_2$ is strongly bounded on $a(\mathscr{A}_1 \times \mathscr{A}_2)$.

The examples presented following Theorem 3 are likewise applicable to the situation in Theorem 8. In particular, Example 5 shows that Theorem 8 is applicable to the ε-product of two strongly bounded set functions.

4. Regularity

In this section we consider the regularity of the product of two regular vector measures. Because of the many and varied notions of regularity (see, for example, [3]), we will not attempt to discuss all of the possible types of regularity in detail or even consider the difficulties which arise between using Baire and Borel sets ([1], Lemma 57.2 and Exercise 57.16). We consider two different situations in Theorems 9 and 11: after seeing the basic ideas employed the reader can supply the details pertaining to the various types of regularity, etc.

Let S be a locally compact Hausdorff space and let $\mathscr{A}$ be a ring of subsets of S. A finitely additive set function $\mu : \mathscr{A} \to Z$ is regular if for each $A \in \mathscr{A}$ and each neighborhood of zero in Z, U, there exist a compact $K \in \mathscr{A}$, $K \subseteq A$, and an open $G \in \mathscr{A}$, $G \supseteq A$, such that whenever $D \in \mathscr{A}$ and $D \subseteq G - K$, $\mu(D) \in U$. (This is regularity of type R_1 in [3]; see also [22].)

In order to avoid rephrasing the previous material for rings and σ-rings, we assume that each S_i is σ-compact ([10], XI. 3). Let $\mathscr{B}_i$ denote either the σ algebra of Baire sets or Borel sets of S_i. Again the methods employed in this section are similar to those used in Theorem 3 so we do not write out complete details.

Theorem 9. *Let $\mu_i : \mathscr{B}_i \to X_i$ be regular. If condition (β) is satisfied, then $\mu_1 \times \mu_2$ is regular on $a(\mathscr{B}_1 \times \mathscr{B}_2)$ and has a regular extension to $\sigma(\mathscr{B}_1 \times \mathscr{B}_2)$.*

Proof: Let U be a closed absolutely convex neighborhood of 0 in Z and let p be the Minkowski functional of U. With notation as in the proof of Theorem 3(b), we again obtain equation (3). Now each μ_i is regular so there exists a positive regular measure $\lambda_i \in rca(\mathscr{B}_i)$ such that $\Gamma_i = \{v(x'\mu_i) : x' \in A_i\}$ is uniformly absolutely continuous with respect to λ_i ([22]). By Lemma 1, $\Gamma_1 \times \Gamma_2$ is uniformly absolutely continuous with respect to $\lambda_1 \times \lambda_2$, and

$\lambda_1 \times \lambda_2$ is regular on $\sigma(\mathscr{B}_1 \times \mathscr{B}_2)$ ([1], Theorems 56.3, 60.1 and Exercise 62.10). Thus, equation (3) implies $\mu_1 \times \mu_2$ is regular on $a(\mathscr{B}_1 \times \mathscr{B}_2)$ and also that $\mu_1 \times \mu_2$ has a regular extension to $\sigma(\mathscr{B}_1 \times \mathscr{B}_2)$ ([22], Theorem 3).

Remark 10. In particular, Theorem 9 contains Theorem 1 of [6], and the technique used in the proof of Theorem 3 of [6] can be used to show that if each μ_i is a regular Borel measure, then $\mu_1 \times \mu_2$ can be extended to a regular Borel measure on $S_1 \times S_2$.

If the weak topology on a locally convex space Z is used and a vector measure μ with values in Z is regular with respect to the weak topology, we say that μ is weakly regular. Using the methods of part (a) of Theorem 3 we can obtain

Theorem. 11. *Let $\mu_i : \mathscr{B}_i \to X_i$ be weakly regular. If condition (α) is satisfied then $\mu_1 \times \mu_2$ is weakly regular on $a(\mathscr{B}_1 \times \mathscr{B}_2)$.*

Proof: As in the proof of part (a) of Theorem 3, we obtain equation 2 and as above this equation yields the desired conclusion.

5. Necessity

In this concluding section we make some remarks pertaining to the necessity of the assumptions ((α) and (β)) made in the previous theorems. Taking into consideration the counter-examples presented in [9] and [25], it is certainly desirable that some necessary conditions for the existence of product measures be given. We consider the Hilbert space situation as in [9] and [25]. Let H be a real Hilbert space and $A : H \to H$ be a bounded linear operator. Define a bilinear map b on $H \times H$ by $b(x, y) = (x, Ay)$, where (,) is the inner product on H. According to Theorem 3 if b is an integral form, the product of any two H-valued vector measures has a countably additive extension to the σ-algebra generated by the measurable rectangles. Recall that b is an integral form iff the operator A is a nuclear operator ([29], 49.6). These remarks lead to the following conjecture:

Conjecture 12. Suppose b has the property that the product of any two H-valued vector measures has a countably additive extension to the σ-algebra generated by the measurable rectangles (as in the conclusion of Theorem 3 b)). Then A is a nuclear operator.

We have not been successful in establishing this conjecture. We do however present an example which illustrates that the operator A must indeed satisfy some restrictive conditions if b satisfies the condition set forth in Conjecture 12.

Let $A : l^2 \to l^2$ be a compact operator with spectral representation $Ax = \sum_k \lambda_k(x, \delta_k)\delta_k$, where $\{\lambda_k\} \in c_0$, $\lambda_k \geq \lambda_{k+1}$, and $\delta_k = \{\delta_{kj}\}_{j=1}^{\infty} \in l^2$ ([12]

19.3; [29], 48: the sequence $\{\delta_k\}$ is used only for convencience, any complete orthonormal sequence will do). To show that A is nuclear amounts to showing that $\{\lambda_k\} \in l^1$ ([12], 21.2; [29], 48). We have not been successful in establishing this fact (which would essentially prove the conjecture for compact operators), but we do show that $\{\lambda_k\} \in l^p$ for every $p > 2$ whenever b satisfies the conditions of the conjecture. This at least shows that the operator A must satisfy some fairly stringent conditions if the condition of the conjecture is satisfied ([24], 8.3).

Let A_1 be the matrix $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and for each $n > 1$, let A_n be the $2^n \times 2^n$ matrix $A_n = [A_{ij}]$, where $A_{11} = A_{12} = A_{22} = -A_{21} = A_{n-1}$. Let B the unitary operator on l^2 defined to be the direct sum of $\{2^{-n/2}A_n\}_1^\infty$ as in [13], and let $[b_{ij}]$ be the matrix of B with respect to $\{\delta_j\}$.

For $n \geqslant 1$, define $x_n = \{a_{nj}\}_{j=1}^\infty \in l^2$ by $a_{nj} = 0$ for $0 \leqslant j \leqslant 2(2^{n-1} - 1)$, $a_{nj} = 1$ for $2^n - 1 \leqslant j \leqslant 2(2^n - 1)$, and $a_{nj} = 0$ for $j > 2(2^n - 1)$. For $1 < r < \infty$ define a sequence $\{t_j\}$ (depending on r) belonging to l^2 by $\{t_j\} = \sum_n 2^{-nr/2} x_n$. (Note $\Sigma |t_j|^2 = \Sigma (2/2^r)^j < \infty$.) The series $\Sigma t_j\delta_j$ and $\Sigma t_j B\delta_j$ are unconditionally convergent in l^2 so we may define two l^2-valued measures μ and ν on the σ-algebra $\mathcal{M}$ of all subsets of the positive integers by $\mu(\{n\}) = t_n B\delta_n$ and $\nu(\{n\}) = t_n\delta_n$. If the product measure $\mu \times \nu$ (with respect to b) has a (finite) countably additive extension to the σ-algebra generated by $\mathcal{M} \times \mathcal{M}$, then $\sum_{n,m} (\mu\{n\}, A\nu(\{m\})) = \sum_{n,m} \lambda_m t_m t_n (B\delta_n, \delta_m) = \sum_{n,m} \lambda_m t_m t_n b_{nm} < \infty$. However $\sum_{n,m} |\lambda_m t_m t_n b_{nm}| = \sum_{n=1}^{\infty} 2^{n(1/2-r)} \sum_{j=2^n-1}^{2^{n+1}-2} \lambda_j \geqslant \sum_{n=1}^{\infty} |\lambda_{2^{n+1}}| 2^{n(3/2-r)}$. For $1 < r < 3/2$, we have $1/(3/2 - r) > 1$ (recall $r > 1$ so $\{t_j\} \in l^2$) which implies $\sum_{n=1}^{\infty} 2^n \lambda_{2^{n+1}}^{1/(3/2-r)} < \infty$. Hence $\sum_{n=1}^{\infty} |\lambda_n|^{1/(3/2-r)} < \infty$ ([26], 3.27). That is, $\{\lambda_j\}$ belongs to $l^{1/(3/2-r)}$ for $1 < r < 3/2$. But $1/(3/2 - r) \to 2$ as $r \to 1$ so that $\{\lambda_j\} \in l^p$ for each $p > 2$.

This example falls far short of establishing the conjecture (even for compact operators), but even this example does show that the operator A must satisfy some restrictions in order to fulfill the hypothesis of the conjecture ([24], 8.3). If the conjecture can be established as stated, this would show that the theorems of sections 2, 3 and 4 are essentially the best general results that can be expected.

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