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NOTE ON MODULAR AND DISTRIBUTIVE EQUALITIES IN LATTICES

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In the paper [4] F. Šik studies a sublattice \( \langle a, b, c \rangle \) of a lattice \( S \) that is generated by the triple of the elements \( a, b, c \in S \). He investigates the properties of this sublattice when instead of some modular or distributive identity in the lattice \( S \) there holds a corresponding equality only for the triple \( a, b, c \). F. Šik considers the following equalities:

**Modular:**

\[
(1) \quad (a \lor b) \land c = a \lor (b \land c), \quad \text{where } a \leq c,
\]
\[
(1^*) \quad (a \land b) \lor c = a \land (b \lor c), \quad \text{where } a \geq c,
\]
\[
(2) \quad a \lor [b \land (c \lor a)] = (a \lor b) \land (a \lor c),
\]
\[
(2^*) \quad a \land [b \lor (c \land a)] = (a \land b) \lor (a \land c) .
\]

**Distributive:**

\[
(3) \quad a \lor (b \land c) = (a \lor b) \land (a \lor c),
\]
\[
(3^*) \quad a \land (b \lor c) = (a \land b) \lor (a \land c),
\]
\[
(4) \quad (a \land b) \lor (b \land c) \lor (c \land a) = (a \lor b) \land (b \lor c) \land (c \lor a) .
\]

In this paper we shall study some other equalities:

\[
(5) \quad [a \lor (b \land c)] \land (b \lor c) = [a \land (b \lor c)] \lor (b \land c) ,
\]
\[
(6) \quad (b \lor c) \land [a \lor (b \land c)] = (a \land b) \lor (b \land c) \lor (c \land a) ,
\]
\[
(6^*) \quad (b \land c) \lor [a \land (b \lor c)] = (a \lor b) \land (b \lor c) \land (c \lor a) ,
\]
\[
(7) \quad (b \land c) \lor [a \land (b \lor c)] = (a \land b) \lor (b \land c) \lor (c \land a) ,
\]
\[
(7^*) \quad (b \lor c) \land [a \lor (b \land c)] = (a \lor b) \land (b \lor c) \land (c \lor a) .
\]

It is easy to prove that a lattice \( S \) is modular if and only if there holds the equality (5) for all triples of elements in \( S \) (otherwise if the identity (5) is satisfied in \( S \)).

Similarly a lattice \( S \) is distributive if and only if the identity (6) or (6*) is satisfied in \( S \).

Now, let \( x, y, z \) be elements of a lattice \( S \). The symbol \( x y z \) expresses that the following is satisfied:

\[
(x \land y) \lor (y \land z) = y = (x \lor y) \land (y \lor z) .
\]

Then for \( a, b, c \in S \) we shall denote
There holds

**Theorem 1.** For elements \(a, b, c\) of a lattice \(S\) the equality (5) is satisfied if and only if \(B(a, X, Y) \neq \emptyset\), where \(X\) is a left-hand side, \(Y\) is a right-hand side of the equality (5).

**Proof.** 1. If the equality (5) is satisfied, then clearly \(X \in B(a, X, Y)\).
2. Assume that (5) is not satisfied. Therefore

\[
[a \setminus (b \lor c)] \lor (b \land c) < [a \lor (b \land c)] \land (b \lor c).
\]

Let us denote

\[
p = b \land c, \quad q = a, \quad r = b \lor c.
\]

Then

\[
p \leq r, \quad p \lor (q \land r) = (b \land c) \lor [a \land (b \lor c)] < [(b \land c) \lor a] \land (b \lor c) = (p \lor q) \land r.
\]

Now, we denote

\[
Y = p \lor (q \land r) = (b \land c) \lor [a \land (b \lor c)],
X = (p \lor q) \land r = [(b \land c) \lor a] \land (b \lor c),
D = q \land r = a \land (b \lor c),
E = p \lor q = a \lor (b \land c).
\]

From [1, II, 9, proof of Theorem 9.3] it follows that \(E > X > Y > D\). \(E > q > D, X \parallel q \parallel Y\) form a “pentagonal” sublattice of \(S\).

Now, let us suppose that there exists \(z \in B(a, X, Y)\). Thus

\[
(X \land z) \lor (z \land Y) : = z : = (X \lor z) \land (z \lor Y).
\]

From \(Y < X\) it follows \(X \land z = z, \ z = z \lor Y\), then it is \(Y \leq z \leq X\). Thus

\[
X = E \land X = (a \lor z) \land (z \lor Y) = z = (a \land z) \lor (z \land Y) = D \lor Y = Y,
\]

a contradiction.

Next we shall study the distributivity of the lattice \(\langle a, b, c \rangle\).

**Lemma.** Let the equalities (6), (6*), or equalities arising from (6), (6*) by some permutation of the elements \(a, b, c\), be satisfied. Then also the corresponding equality (5) and the equality (4) are satisfied for \(a, b, c\).

**Proof.** In an arbitrary lattice \(S\) there are satisfied \(Y \leq X, t \leq t^*\), where \(t\) is a left-hand side, \(t^*\) is a right-hand side of the equality (4). If now

\[
t = X, \quad t^* = Y,
\]

then

\[
t^* = Y \leq X = t,
\]

thus

\[
X = Y, \quad t = t^*.
\]
Let us state the following conditions:

(I) There are satisfied all six distributive equalities that we can obtain from (3), (3*) by permutations of \(a, b, c\).

(II) There is satisfied one of the following conditions:

(i) \(\langle a, b, c \rangle\) satisfies the upper covering condition.

(ii) \(\langle a, b, c \rangle\) satisfies the lower covering condition.

(iii) \(\langle a, b, c \rangle\) is semimodular.

(III) There is satisfied one of six distributive equalities that we can obtain from (6), (6*) by permutations of \(a, b, c\).

Now, the following theorem holds:

**Theorem 2.** Let \(S\) be a lattice, \(a, b, c \in S\). Then the following conditions are equivalent:

(a) There are satisfied (4) and (II).

(b) There are satisfied (I) and \(B(a, b, c) \neq \emptyset\).

(c) \(\langle a, b, c \rangle\) is distributive.

(d) \(\langle a, b, c \rangle\) is modular and one of seven equalities (3), (3*), (4) is satisfied for \(a, b, c\).

(e) There are satisfied (I) and (III).

(f) \(\langle a, b, c \rangle\) is modular and (III) is satisfied.

(g) There are satisfied (II) and one of three couples of mutually dual equalities that can be obtained from (6), (6*) by permutations of \(a, b, c\).

(h) \(\langle a, b, c \rangle\) is modular and one of six equalities (7), (7*) is satisfied for \(a, b, c\).

**Proof.** Equivalences (a) \(\iff\) (b) \(\iff\) (c) \(\iff\) (d) are proved in [4].

(e) \(\Rightarrow\) (c): By (6)
\[
(a \land b) \lor (b \land c) \lor (c \land a) = (b \lor c) \land [a \lor (b \land c)] .
\]

By (I)
\[
(b \lor c) \land [a \lor (b \land c)] = (b \lor c) \land (a \lor b) \land (a \lor c) .
\]
Therefore (I) and (4) are satisfied and hence (c) holds by [2].

(c) \(\Rightarrow\) (e): Evident.

(f) \(\Rightarrow\) (d): Let us denote
\[
t = (a \land b) \lor (b \land c) \lor (c \land a) ,
\]
\[
v = (a \lor b) \land [c \lor (a \land b)] .
\]

Let \(t = v\) hold. Then
\[
c \land t = c \land \{(a \land b) \lor [(b \land c) \lor (c \land a)]\} .
\]
Since \(\langle a, b, c \rangle\) is modular,
\[
c \land t = [c \land (a \land b)] \lor [(b \land c) \lor (c \land a)] = (c \land a) \lor (c \land b) .
\]
Furthermore
\[ c \land v = c \land \{(a \lor b) \land [c \lor (a \land b)]\} = c \land (a \lor b) . \]

Since \( t = v \), \( c \land t = c \land v \); and hence
\[ c \land (a \lor b) = (c \land a) \lor (c \land b) . \]

(c) \( \Rightarrow \) (f): Evident.

(g) \( \Rightarrow \) (a): By Lemma \( t = t^* \).

(c) \( \Rightarrow \) (g): Evident.

(h) \( \Rightarrow \) (f): (7) is satisfied, thus
\[ t = (b \land c) \lor [a \land (b \lor c)] . \]

Now, by modular identity (5) there holds
\[ t = (b \lor c) \land [a \lor (b \land c)] . \]

(c) \( \Rightarrow \) (h): Evident.

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