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A REMARK ON THE OSCILLATORINESS
OF SOLUTIONS OF A NON-LINEAR THIRD-ORDER EQUATION

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In [2] a theorem is given (Theorem 2, p. 250) which gives sufficient conditions for a non-oscillatory solution of the equation

\[(1) \quad x'' + p(t)x' + q(t)x^\alpha = 0,\]

with \(\alpha > 1, \alpha = m/n\), where \(m\) and \(n\) are nondivisible odd natural numbers, to have the properties:

\[
\lim_{t \to \infty} x''(t) = \lim_{t \to \infty} x'(t) = 0, \quad \lim_{t \to \infty} |x(t)| = L \geq 0.
\]

It is further shown (in a Corollary) that under the hypotheses of Theorem 2 (in [2]) with the added assumption \(0 < \varepsilon < q(t)\) we have for a non-oscillatory solution \(x(t)\)

\[
\lim_{t \to \infty} x(t) = 0.
\]

In the present remark it is shown that the hypotheses of Theorem 2 (in [2]) are sufficient for \(L = 0\) and thus for \(\lim x(t) = 0\) to hold. A further theorem is presented which gives sufficient conditions for a non-oscillatory solution \(x(t)\) of (1) with \(\alpha = m/n > 0\), where \(m\) and \(n\) are relatively prime odd natural numbers, to have the property

\[
\lim_{t \to \infty} x(t) = 0
\]

or

\[
\lim_{t \to \infty} \inf |x(t)| = 0.
\]

**Theorem 1.** Let the hypotheses of Theorem 2 in [2] hold, i.e.: Let \(\alpha > 1, \alpha = m/n\), where \(m\) and \(n\) are relatively prime odd natural numbers. Let the functions \(p(t)\) and \(q(t)\) satisfy the following conditions for sufficiently large \(t\):

1) \(q(t)\) is non-negative and continuous:
2) \(p(t), p'(t)\) are continuous and \(p(t) < 0, p'(t) \geq 0;\)
3) for any constants \( A, B \) there exists a \( t_1 > t_0 \) such that for all \( t \geq t_1 \) we have
\[
A + Bt - \int_{t_0}^{t} Q(s) \, ds < 0, \quad \text{where} \quad Q(t) = \int_{t_0}^{t} q(s) \, ds.
\]

Then any non-oscillatory solution \( x(t) \) of the non-linear differential equation (1) has the following properties for large \( t \):

a) \( \text{sgn} \, x(t) = \text{sgn} \, x''(t) \neq \text{sgn} \, x'(t) \), where
\[
\text{sgn} \, x(t) = \begin{cases} 1 & \text{if} \quad x(t) \geq 0, \\ -1 & \text{if} \quad x(t) < 0; \end{cases}
\]

b) \( \lim_{t \to \infty} x''(t) = \lim_{t \to \infty} x'(t) = \lim_{t \to \infty} x(t) = 0; \)

c) \( x(t), x'(t) \) and \( x''(t) \) are monotonous functions.

Proof. We shall prove that \( \lim_{t \to \infty} x(t) = 0 \). Let \( x(t) \) be any non-oscillatory solution of the differential equation (1). Thus there exists a number \( t_1 \geq t_0 \) such that \( x(t) \neq 0 \) for all \( t \geq t_1 \). Since \( -x(t) \) is also a solution of the differential equation (1), without loss of generality, assume that \( x(t) > 0 \) for all \( t \geq t_1 \). Suppose that \( \lim_{t \to \infty} x(t) = L > 0 \). Then from (1) we have:
\[
x''(t) = -p(t)x'(t) - q(t)x^2(t);
\]
now, since for sufficiently large \( t \) \( x'(t) < 0 \), we have
\[
x''(t) \leq -q(t)x^2(t) < -L^2q(t).
\]
Since, by assumption 3), \( \lim_{t \to \infty} Q(t) = +\infty \), this leads to \( x''(t) \to -\infty \) for \( t \to \infty \), which is a contradiction. Thus necessarily \( L = 0 \).

Theorem 2. Let \( \alpha = m/n > 0 \), where \( m \) and \( n \) are relatively prime odd natural numbers. Let the functions \( p(t), p'(t) \) and \( q(t) \) be continuous and for sufficiently large \( t_0 \) let for all \( t \geq t_0 \)
\[
p(t) \geq 0, \quad q(t) \geq 0, \quad p'(t) \leq 0.
\]
If for any constants \( A \) and \( B \)
\[
\lim_{t \to \infty} (A + Bt - \int_{t_0}^{t} Q(s) \, ds) = -\infty,
\]
where \( Q(t) = \int_{t_0}^{t} q(s) \, ds \), then a solution \( x(t) \) of (1) for which
is either oscillatory or \( \lim_{t \to \infty} x(t) = 0 \).

**Proof.** Let \( x(t) \) be any non-oscillatory solution of the differential equation (1) satisfying (3). Thus there exists a number \( t_1 \geq t_0 \) such that \( x(t) = 0 \) for all \( t \geq t_1 \). Since \(-x(t)\) is also a solution of the differential equation (1), assume without loss of generality, that \( x(t) > 0 \) for all \( t \geq t_1 \). Then from (1) we have

\[
\frac{x''(t)}{x(t)} + \frac{1}{2} \frac{x'^2(t)}{x^2(t)} + \int_{t_1}^{t} \frac{p(s)x'(s)}{x^2(s)} \, ds + \frac{1}{2} \frac{\alpha(x+1)}{\alpha} \int_{t_1}^{t} \frac{x^3(s)}{x^{x+1}(s)} \, ds = K_1 - \int_{t_1}^{t} q(s) \, ds.
\]

An integration from \( t_1 \) to \( t \geq t_1 \) equality (4) gives

\[
\frac{x'(t)}{x(t)} + \int_{t_1}^{t} \frac{(t-s)p(s)x'(s)}{x^2(s)} \, ds + \frac{\alpha(x+1)}{2} \int_{t_1}^{t} \frac{(t-s)x^3(s)}{x^{x+2}(s)} \, ds \leq K_2 + K_1 t - \int_{t_1}^{t} Q(s) \, ds.
\]

This implies that there is no number \( t_2 \) such that \( x'(t) \geq 0 \) holds for any \( t \geq t_2 \). Thus we have two possibilities:

1) There exists a number \( t_2 \geq t_1 \) such that \( x'(t) \leq 0 \) for any \( t \geq t_2 \).
2) For any \( t_2 \) there exists a number \( t_3 \geq t_2 \) such that \( x'(t_3) > 0 \). Now let \( t_2 \) be such number that for all \( t \geq t_2 \geq t_1 \) we have \( K_2 + K_1 t - \int_{t_1}^{t} Q(s) \, ds < 0 \). We shall prove that then we have \( x'(t) \leq 0 \) for any \( t \geq t_2 \), i.e. the possibility 2) does not hold. Let \( t_3 \geq t_2 \) be such number that \( x'(t_3) > 0 \) and let \( x'(t_4) = 0 \) for any \( t_1 \geq t_1, t_4 < t_3 \).

Then from (1) we have:

\[
x''(t)x(t) - \frac{1}{2} x'^2(t) + \frac{1}{2} \frac{p(t)x^2(t)}{x(t)} + \int_{t_4}^{t} q(s)x^{x+1}(s) \, ds =
\]
\[
= x''(t_0)x(t_0) - \frac{1}{2} x'^2(t_0) + \frac{1}{2} p(t_0)x^2(t_0) + \frac{1}{2} \int_{t_0}^{t} p'(s)x^2(s) \, ds ,
\]
thus for all \( t \geq t_0 \)
\[
x''(t)x(t) - x'^2(t) \leq x''(t)x(t) - \frac{1}{2} x'^2(t) \leq 0
\]
and therefore for all \( t \geq t_1 \)
\[
\frac{d}{dt} \left[ \frac{x'(t)}{x(t)} \right] \leq 0.
\]
An integration from \( t_4 \) to \( t_3 \) gives
\[
\frac{x'(t_3)}{x(t_3)} \leq \frac{x'(t_4)}{x(t_4)} = 0,
\]
which is impossible, because \( x'(t_3) > 0 \). Hence \( x'(t) \leq 0 \) for all \( t \geq t_2 \). Thus \( x(t) \) is a non-increasing function with a finite lower bound so that \( \lim_{t \to \infty} x(t) = L \geq 0 \).

Now suppose that \( \lim_{t \to \infty} x(t) = L > 0 \). Then (1) yields
\[
x''(t) = x''(t_2) + p(t_2)x(t_2) - p(t)x(t) + \int_{t_2}^{t} p'(s)x(s) \, ds - \int_{t_2}^{t} q(s)x^3(s) \, ds,
\]
where \( t \geq t_2 \). Therefore
\[
x''(t) \leq K_3 - Lx \int_{t_2}^{t} q(s) \, ds
\]
and from this it follows that \( x''(t) \to -\infty \) for \( t \to \infty \), which contradicts the assumption that \( x(t) > 0 \) for \( t \geq t_2 \).

**Theorem 3.** Let \( \alpha = m/n > 0 \), where \( m \) and \( n \) are relatively prime odd natural numbers. Let the functions \( p(t), p'(t), q(t) \) and \( f(t) \) be continuous and for sufficiently large \( t_0 \) let for all \( t \geq t_0 \)
\[
p(t) \geq 0, \quad q(t) \geq 0, \quad p'(t) + |f(t)| \leq 0.
\]
Suppose that (2) holds and that \( x(t) \) is a solution of the equation
\[
x''(t) + p(t)x'(t) + q(t)x^3 = f(t),
\]
for which

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\( x''(t_0)x(t_0) - \frac{1}{2} x'^2(t_0) + \frac{1}{2} p(t_0)x^2(t_0) + \frac{1}{2} \int_{t_0}^{\infty} |f(t)| \, dt \leq 0. \)

Then \( x(t) \) is either oscillatory or \( \lim_{t \to \infty} |x(t)| = 0. \)

**Proof.** Let \( x(t) > 0 \) for all \( t \geq t_1 \geq t_0 \), let \( x(t) \) satisfy (6) and let \( \lim_{t \to \infty} x(t) = L > 0 \). Thus there exists a number \( t_1^* \geq t_1 \) such that \( x(t) \geq L_1 = L/2 \) for all \( t \geq t_1^* \). From (5) we have for \( t \geq t_1^* \geq t_1 \)

\[
\frac{x''(t)}{x^3(t)} + \int_{t_1^*}^{t} \frac{p(s)x'(s)}{x^3(s)} \, ds + \frac{\alpha(x + 1)}{2} \int_{t_1^*}^{t} \frac{x'^3(s)}{x^{x+2}(s)} \, ds \leq K_1 - \int_{t_1^*}^{t} q(s) \, ds + \frac{1}{L_1^2} \int_{t_1^*}^{t} |f(s)| \, ds
\]

which, analogously as in the proof of Theorem 2, implies the existence of \( t_2 \geq t_1^* \) such that for all \( t \geq t_2 \) \( x'(t) \leq 0 \); thus \( \lim_{t \to \infty} x(t) = L. \)

Using (5), we have for \( t \geq t_2 \)

\[
x''(t) \leq K_3 - \frac{L^2}{t_1} \int_{t_1}^{t} q(s) \, ds + \int_{t_1}^{t} |f(s)| \, ds
\]

and using (2), we see that \( x''(t) \to -\infty \) for \( t \to \infty \), which contradicts the assumption that \( x(t) > 0 \) for all \( t \geq t_2 \). Therefore \( \lim_{t \to \infty} x(t) = 0. \)

Now let \( x(t) < 0 \) for all \( t \geq t_1 \geq t_0 \), let \( x(t) \) satisfy (6) and let \( \lim_{t \to \infty} |x(t)| = L > 0 \). Integrating (7) from \( t_1^* \) to \( t \geq t_1^* \), we get

\[
\frac{x'(t)}{x^3(t)} + \int_{t_1^*}^{t} \frac{(t - s)p(s)x'(s)}{x^3(s)} \, ds + \frac{\alpha(x + 1)}{2} \int_{t_1^*}^{t} \frac{(t - s)x'^3(s)}{x^{x+2}(s)} \, ds \leq K_2 + K_1 t - \int_{t_1^*}^{t} Q(s) \, ds.
\]

Since for all \( t \geq t_1^* \) \( x^3 < 0 \) holds, we have from the last inequality that there exists a number \( t_2 \geq t_1^* \) such that \( x'(t) \geq 0 \) for all \( t \geq t_2 \). In fact, let \( x'(t_3) < 0 \) and \( x'(t_4) = 0 \), where \( t_1 \leq t_4 < t_3 \). Then from equation (5) we have:
\[ x''(t)x(t) - \frac{1}{2} x'^2(t) + \frac{1}{2} p(t)x^2(t) \leq x''(t_0)x(t_0) - \frac{1}{2} x'^2(t_0) + \]
\[ + \frac{1}{2} p(t_0)x^2(t_0) + \int_{t_0}^{t} |f(s)| \, ds + \int_{t_0}^{t} [p'(s) + |f(s)|] x^2(s) \, ds , \]

and therefore

\[ x''(t)x(t) - x'^2(t) \leq x''(t)x(t) - \frac{1}{2} x'^2(t) \leq 0 \]

for all \( t \geq t_0 \). If \( t \geq t_4 \), then \( x^2(t) \neq 0 \) and

\[ \frac{x'(t)}{x(t)} \leq \frac{x'(t_4)}{x(t_4)} \]

for all \( t \geq t_4 \). For \( t = t_3 \) we have a contradiction.

This proves the existence of \( t_2 \geq t_1^* \) such that for \( t \geq t_2 \) \( x'(t) \geq 0 \). Then from (5) we have

\[ x''(t) \geq K_3 + L^2 \int_{t_2}^{t} q(s) \, ds - \int_{t_2}^{t} |f(s)| \, ds \]

which, owing to (2) and (6), implies \( x''(t) \to +\infty \) for \( t \to \infty \) which again contradicts the assumption that \( x(t) < 0 \) for \( t \geq t_2 \). This completes the proof.

**Theorem 4.** Let the hypotheses be the same as in Theorem 2 with condition (2) replaced by

(2') \[ \int_{t_0}^{\infty} p(t) \, dt = +\infty . \]

If \( x(t) \) is a solution of the equation (1) which satisfies the condition (3), then it is either oscillatory or \( \lim_{t\to\infty} x(t) = 0 \).

Proof. Suppose that the hypotheses hold and that \( x(t) \) is not oscillatory. Thus there exists a number \( t_1 \geq t_0 \), such that \( x(t) \neq 0 \) for all \( t \geq t_1 \). Then from (1) we have

\[ x''(t)x(t) - \frac{1}{2} x'^2(t) + \frac{1}{2} p(t)x^2(t) \leq x''(t_0)x(t_0) - \frac{1}{2} x'^2(t_0) + \]
\[ + \frac{1}{2} p(t_0)x^2(t_0) + \int_{t_0}^{t} p'(s)x^2(s) \, ds , \]
thus for $t \geq t_1$

$$x''(t)x(t) - x'^2(t) \leq x''(t)x(t) - \frac{1}{2} x'^2(t) \leq -\frac{1}{2} \rho(t)x^2(t)$$

and

$$\frac{d}{dt}\left[\frac{x'(t)}{x(t)}\right] \leq -\frac{1}{2} \rho(t),$$

and also there exists a number $t_2 \geq t_1$ such that $x'(t)x(t) < 0$ for every $t \geq t_2$.

Now let $x(t) > 0$ and $x'(t) < 0$. Then

$$\lim_{t \to \infty} x(t) = L \geq 0$$

and hence $x(t) \geq L$ for all $t \geq t_2$. For all $t \geq t_2$ we have

$$\frac{x'(t)}{x(t)} \geq \frac{x'(t)}{L}$$

from which using (8) and (2') we get $lim_{t \to \infty} x''(t) = -\infty$, which is again contradictory to the assumption that $x(t) > 0$ for all $t \geq t_2$.

Now let $x(t) < 0$ and $x'(t) > 0$. Then

$$\lim_{t \to \infty} x(t) = L \leq 0.$$ 

Analogously as in the first case we prove the impossibility of $lim_{t \to \infty} x(t) = L < 0$.

This completes the proof.

Evidently the following theorem also holds:

**Theorem 5.** Let the hypotheses be the same as in Theorem 3 with condition (2) replaced by (2'). If $x(t)$ is a solution of the equation (5) which satisfies the condition (6), then it is either oscillatory or $lim_{t \to \infty} x(t) = 0$.

REFERENCES


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