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ON THE EXTENSION OF A MEASURE ON LATTICES

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Every measure \( \gamma \) defined on a subalgebra \( R \) of a \( \sigma \)-complete Boolean algebra \( H \) can be extended to a measure \( \gamma \) on the smallest \( \sigma \)-algebra \( S \) over \( R \). In the present paper we prove this theorem for a certain type of not necessarily distributive lattices (\( \sigma \)-continuous, orthocomplemented, modular).

We have only two new definitions. All other definitions will be used according to [2]. If \( b \) is an element of a complemented lattice \( H \), then we denote by \( C(b) \) the set of all complements of the element \( b \). A non-empty subset \( R \) of an orthocomplemented lattice \( S \) is called a lattice ring if \( a \cup b, a \cap b, a \cap b^\perp \in R \) for any \( a, b \in R \). A lattice \( \sigma \)-ring is a \( \sigma \)-complete lattice ring.

A real-valued function \( \gamma \) defined on a lattice ring \( R \) is called a measure if it fulfills the following three conditions\( ^{(1)} \):

1. If \( x_n \not\in x, x_n \in R \) \((n = 1, 2, \ldots), x \in R \), then \( \lim_{n \to \infty} \gamma(x_n) = \gamma(x) \).
2. \( \gamma(x) + \gamma(y) = \gamma(x \cup y) + \gamma(x \cap y) \) for every \( x, y \in R \).
3. \( \gamma(0) = 0 \) and \( \gamma(x) \geq 0 \) for every \( x \in R \).

**Theorem 1.** Let \( H \) be a \( \sigma \)-continuous, modular, complemented (orthocomplemented) lattice. Let \( R \) be a sublattice of \( H \) and let for any \( a, b \in R \) and any \( b' \in C(b) \) the following holds: \( a \cap b' \in R \) \((a \cap b \in R \) for any \( a, b \in R \)). Let \( \gamma \) be a finite measure on \( R \).

Then there exists a set \( N \subset R \) and a finite real-valued function \( \tilde{\gamma} \) on \( N \) with the following properties:

4. \( N \) is a conditionally \( \sigma \)-complete sublattice of \( H \).
5. \( \tilde{\gamma} \) is an extension of \( \gamma \), i.e. \( \tilde{\gamma}(a) = \gamma(a) \) for \( a \in R \).
6. \( \tilde{\gamma}(x) + \tilde{\gamma}(y) = \tilde{\gamma}(x \cup y) + \tilde{\gamma}(x \cap y) \) for any \( x, y \in N \).
7. \( \tilde{\gamma} \) is a non-negative and non-descendent function.
8. If \( x_n \in N \), \( x_n \not\in x (x_n \not\in x) \) and \( \{\gamma(x_n)\} \) is bounded, then \( x \in N \) and \( \tilde{\gamma}(x) = \lim_{n \to \infty} \tilde{\gamma}(x_n) \).
9. Let \( \gamma^*(x) = \inf_{b \in B} \gamma_0(b) \) for \( x \in H \), where the infimum is taken over all

\( ^{(1)} \) Cf. Theorem 4.
elements $b \geq x$ such that there exists a sequence $\{a_n\}$ of elements of $R$, $a_n \not\in b$. Here $\gamma_0(b) = \lim_{n \to \infty} \gamma(a_n)$. Then $\tilde{\gamma}(x) = \gamma^*(x)$ for all $x \in N$.

Proof. Denote by $B$ the set of all $b \in H$ for which there exists a sequence $\{a_n\}$ of elements of $R$ such that $a_n \not\in b$. Let $c \leq d$, $c, d \in B$, $c_n \not\in c$, $d_n \not\in d$, $c_n, d_n \in R$. From (1), (2) and $\sigma$-continuity of $H$ it follows that

$$\gamma(c_m) = \lim_{n \to \infty} \gamma(c_m \cap d_n) \leq \lim_{n \to \infty} \gamma(d_n)$$

hence

(10) \[ \lim_{n \to \infty} \gamma(c_n) \leq \lim_{n \to \infty} \gamma(d_n). \]

Hence we can define the function $\gamma_0$ on the set $B$ by the equality

$$\gamma_0(b) = \lim_{n \to \infty} \gamma(a_n),$$

where $a_n \not\in b$, $a_n \in R$ ($n = 1, 2, \ldots$). The function $\gamma_0$ is non-negative, non descendant, subadditive and coincident on $R$ with $\gamma$.

For an arbitrary element $d \in H$ we define

$$\gamma^*(d) = \inf \{\gamma_0(b) : d \leq b \in B\}.$$ 

The function $\gamma^*$ is an extension of $\gamma_0$, non descendant (and hence also non-negative) and subadditive.

Now we shall prove the following property of $\gamma$:

(11) If $y_n, z_n \in R$, $y_n \not\in y \in H$, $z_n \not\in z \in H$, $z \leq y$, then $\inf \gamma(z_n) \leq \sup \gamma(y_n)$.

By the definition $y \in B$, hence

(12) \[ \gamma^*(y) = \gamma_0(y) = \lim \gamma(y_n) = \sup \gamma(y_n). \]

Now we shall distinguish two cases: complemented resp. orthocomplemented lattices.(2)

Let $H$ be complemented and let $x \cap y' \in R$ for any $x, y \in R$ and any $y' \in C(y)$. Then we have:

(13) If $z_n \not\in z, z_n \in R, z \in H$, then there exist a sequence $\{z'_n\}$ and an element $z'$ such that $z' \in C(z)$, $z'_n \in C(z_n)$ ($n = 1, 2, \ldots$) and $z'_n \not\in z'$.(3)

It follows from the $\sigma$-continuity of $H$ that $z_1 \cap z'_n \not\in z_1 \cap z'$. Since $z_1 \cap z'_n \in R$, we have by (12)

(14) \[ \gamma^*(z_1 \cap z') = \lim \gamma^*(z_1 \cap z'_n), \quad z_1 \cap z'_n \not\in z_1 \cap z'. \]

(2) If $H$ is orthocomplemented we suppose less about $R$.

(3) [2], I., Lemma 1.9 and 1.13.
If $H$ is orthocomplemented, then clearly (13) holds and hence (14) too. Now let $z_n \wedge z$ and $z_n' \in C(z_n)$, $z' \in C(z)$ be those complements for which (14) holds (we examine simultaneously both cases, complemented and orthocomplemented). From (14) it follows that

$$
\gamma^*(z_1 \cap z') = \lim \gamma^*(z_1 \cap z_n') = \lim (\gamma(z_1) - \gamma(z_n)) = 
\gamma(z_1) - \lim \gamma(z_n).
$$

Evidently

$$
\gamma^*(z_1) = \gamma^*(z \cup (z_1 \cap z')) \leq \gamma^*(z) + \gamma^*(z_1 \cap z').
$$

From (15) and (16) it follows that $\gamma^*(z) \geq \lim \gamma^*(z_n)$. Because the opposite inequality is evident, we have

$$
\gamma^*(z) = \lim \gamma(z_n) = \inf \gamma(z_n).
$$

From (17), (12) and from the fact that $\gamma^*$ is non descendant it follows (11).

In [1] the following theorem is proved (Theorem 5): If $\gamma$ is a non descendant function on a sublattice of a $\sigma$-continuous lattice $S$, fulfilling the conditions (2) and (11), then there exists a conditionally $\sigma$-continuous sublattice $N \subset R$ of the lattice $S$ such that the function $\gamma^*$, defined on $N$ by the equality $\gamma^*(d) = \gamma(d)$ fulfills on $N$ the conditions (4), (6), (8) (and evidently also (9)). We have found out that $\gamma^*$ fulfills also (5) and (7).

**Lemma 1.** Let $H$ be a $\sigma$-continuous, orthocomplemented lattice, $R \subset H$ be a lattice ring, $S$ the smallest lattice $\sigma$-ring over $R$ and $M$ the smallest monotonous set over $R$. Then $S = M$.

**Proof.** Since $S$ is a monotonous set, we have $M \subset S$. For the proof of the reverse inclusion it suffices to prove that $M$ is a ring. Let $\circ$ be an arbitrary operation of $\cup$, $\cap$. Let $x \in R$ be an arbitrary but fixed element, $G = \{y \in M : x \circ y \in M\}$. Evidently $G \supset R$, $G$ is a monotonous set, hence $G \supset M$. Hence for each $x \in R$ and each $y \in M$ we have $x \circ y \in M$. Now let us take a fixed $y \in M$ and put $K = \{x \in M : x \circ y \in M\}$. Since according to the previous $K$ is a monotonous set and $K \supset R$, we have $K \supset M$, hence $M$ is closed under the lattice operations. Similarly we prove that $a \cap b^k \in M$ for all $a, b \in M$.

**Lemma 2.** Let $H$ be a $\sigma$-continuous, orthocomplemented, modular lattice. Let $R \subset H$ be a lattice ring, $\gamma$ a finite measure on $R$, $S$ the smallest lattice $\sigma$-ring over $R$, $\gamma^*$ the function defined in Theorem 1 by (9). Let $F$ be the smallest set over $R$ fulfilling the following condition:

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(4) A set $K \subset H$ is monotonous if it contains the supremum and the infimum of every monotonous (i.e. non descendant, or non ascendent) sequence of $K$. 

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(a) If either $x_n \not\sim x$ or $x_n \not\prec x$ and \{\(y^*(x_n)\)\} is bounded, $x_n \in F$ ($n = 1,2, \ldots$), then $x \in F$.\(^{(5)}\)

Then

\[(18) \quad F = \{d \in S : y^*(d) < \infty\}.\]

Proof. According to Lemma 1 we have $S = M$, where $M$ is the smallest monotonous set over $R$. Clearly $R < F < N \cap M$. Because $y^*$ is finite on $N$, we have $F \subset \{d \in M : y^*(d) < \infty\}$. Let $d \in M$, $y^*(d) < \infty$. According to the definition of $y^*$, there exists an element $e \in B$ for which $d \leq e$, $y^*(e) < \infty$, $e = \bigcup_{n=1}^{\infty} a_n$, where $a_n \in R$. Put $P = \{f \in M : f \cap e \in F\}$. Evidently $P \supset R$. $P$ is a monotonous set, because $0 \leq f \cap e = e$ and $y^*(e) < \infty$ for all $f \in M$. Hence $P \supset M$ and $d = d \cap e \in F$. Therefore we have proved (18).

**Theorem 2.** Let $H$ be a $\sigma$-continuous, orthocomplemented, modular lattice. Let $R < H$ be a lattice ring, $\gamma$ a finite measure on $R$, $S$ the smallest lattice $\sigma$-ring over $R$. Then there exists a measure $\tilde{\gamma}$ on $S$ that is an extension of $\gamma$.

Proof. Put $\tilde{\gamma}(d) = y^*(d)$ for all $d \in S$. $\tilde{\gamma}$ is a extension of $\gamma$, it is non-negative and $\tilde{\gamma}(0) = 0$. It remains to be proved that $\tilde{\gamma}$ satisfies the conditions (1) and (2).

Let $\{x_n\}$ be a sequence of elements of $S$, $x_n \not\sim x$. Clearly $\lim \tilde{\gamma}(x_n) \leq \tilde{\gamma}(x)$, hence the equality holds, if $\lim \tilde{\gamma}(x_n) = \infty$. Let $\lim \tilde{\gamma}(x_n) < \infty$. Then by (18) it is $x_n \in F \subset N$ for all $n$ and $\{\tilde{\gamma}(x_n)\}$ is bounded. Hence according to Theorem 1 we have $\tilde{\gamma}(x) = \lim \tilde{\gamma}(x_n)$ and the property (1) is proved.

The property (2) is fulfilled if at least one of the expressions $\tilde{\gamma}(x)$, $\tilde{\gamma}(y)$ is equal to $\infty$. In the reverse case we have $x, y \in F \subset N$ and (2) follows from Theorem 1.

**Theorem 3.** Let $H$ be a $\sigma$-continuous, modular, orthocomplemented lattice. Let $R < H$ be a lattice ring, $\gamma$ a $\sigma$-finite measure on $R$ (i.e. each element of $R$ is majorized by the supremum of a sequence of elements of $R$ of a finite measure). Then there exists a $\sigma$-finite measure $\tilde{\gamma}$ on the smallest lattice $\sigma$-ring over $R$ that is a extension of $\gamma$. The measure $\tilde{\gamma}$ is determined uniquely.

Proof. Put $A = \{e \in R : \gamma(e) < \infty\}$. $A$ is a lattice ring. According to Theorem 2 there exists a measure $\tilde{\gamma}$ on the smallest lattice $\sigma$-ring $S(A)$ over $A$ that is a extension of $\gamma$. We shall prove that $S(A) = S$. We have $S(A) \subset S$ because $A \subset R$. On the other side, each element of $R$ is a supremum of a countable number of elements of $A$. Actually, let $a \in R$, $a \leq \bigcup_{n=1}^{\infty} a_n$, $a_n \in A$. Put

\(^{(5)}\) Since the set $N$ from Theorem 1 has the property (a), there exists such a set $F$. 

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Clearly \( b_n \in A, b_n \not\in \bigcup_{n=1}^{\infty} a_n \). Further \( b_n \cap a \not\subset \bigcup_{n=1}^{\infty} a_n \cap a = a \) because \( H \) is \( \sigma \)-continuous. Hence \( \{b_n \cap a\} \) is a sequence of elements of \( A \) such that \( a \) is its supremum. Therefore \( R \subset S(A), S \subset S(A) \), hence \( \tilde{\gamma} \) is a measure on \( S \).

The measure \( \tilde{\gamma} \) is \( \sigma \)-finite, because the set \( P = \{d \in S : d \leq \bigcup_{n=1}^{\infty} a_n, a_n \in R \} \) is monotonous and it contains \( R \).

Let \( \gamma_1 \) be any measure on \( S \) being a extension of \( \gamma \). Let \( F \) be the smallest set over \( A \) fulfilling the property (x) (see Lemma 2). According to Theorem 1 the set \( Q = \{x \in S : \gamma_1(x) = \tilde{\gamma}(x)\} \) fulfills the property (x), it contains \( A \), hence \( Q \supset F \) and \( \gamma_1 \) coincides with \( \tilde{\gamma} \) on \( F \). Let \( e \in S \). From (18) and the \( \sigma \)-finiteness of \( \tilde{\gamma} \) it follows that there exists a sequence \( \{e_n\}, \ e_n \in F, e_n \not\subset e \).

Therefore \( \gamma_1(e) = \lim_{n \to \infty} \gamma_1(e_n) = \lim_{n \to \infty} \tilde{\gamma}(e_n) = \tilde{\gamma}(e) \).

**Theorem 4.** Let \( H \) be a \( \sigma \)-complete, modular, complemented (resp. ortho-complemented) lattice. Let \( E \) be a sublattice of the lattice \( H \) and let \( a \cap b' \in R \) for all \( a, b \in R \) and \( b' \in C(b) \) (resp. \( a \cap b' \in R \) for all \( a, b \in R \)). Let \( \gamma \) be a non-negative real-valued function on \( R \), \( \gamma(0) = 0 \). Then \( \gamma \) is a measure if and only if \( \gamma \) is \( \sigma \)-additive, i.e. if \( \gamma(\bigcup_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty} \gamma(a_n) \) for any disjoint sequence of elements of \( R \) such that \( \bigcup_{n=1}^{\infty} a_n \in R \).

A sequence \( \{x_n\} \) is called disjoint if for any two disjoint set of indices \( \alpha, \beta \) we have \( \bigcup_{i \in \alpha} x_i \cap \bigcup_{j \in \beta} x_j = 0 \).

**Proof.** 1. Let \( \gamma \) be a measure. From the property (2) it follows that \( \gamma(x \cup y) = \gamma(x) + \gamma(y) \) for any two disjoint elements \( x, y \). This leads by induction \( \gamma(\bigcup_{i=1}^{n} x_i) = \sum_{i=1}^{n} \gamma(x_i) \) for any finite disjoint set \( \{x_1, \ldots, x_n\} \) of elements. Finally, let \( \{x_n\} \) be any disjoint sequence of elements of \( R \). Put \( y_n = \bigcup_{i=1}^{n} x_i \). Then \( y_n \not\subset \bigcup_{n=1}^{\infty} x_n \), hence

\[
\gamma(\bigcup_{n=1}^{\infty} x_n) = \lim_{n \to \infty} \gamma(y_n) = \lim_{n \to \infty} \gamma(\bigcup_{i=1}^{n} x_i) = \lim_{n \to \infty} \sum_{i=1}^{n} \gamma(x_i) = \sum_{n=1}^{\infty} \gamma(x_n).
\]

2. Let \( \gamma \) be \( \sigma \)-additive. First we shall prove (1). Let \( x_n \not\subset x, x_n \in R, x \in R \). Put \( y_1 = x_1, y_n = x_n \cap x_{n-1}^c (n = 2, 3, \ldots) \) in the case of the orthocomplementarity of \( H \). In the case of the complementarity let us construct \( x'_n \in C(x_n) \) arbitrarily and put \( y_n = x_n \cap x_{n-1}^c (n = 2, 3, \ldots) \). In both cases \( x = \bigcup_{n=1}^{\infty} x = \bigcup_{n=1}^{\infty} y_n \). Therefore

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\[ y(x) = y(\bigcup_{n=1}^{\infty} y_n) = \sum_{n=1}^{\infty} y(y_n) = \lim_{n} \sum_{i=1}^{n} y(y_i) = \lim \left( y(x_1) + \sum_{i=1}^{n} (y(x_i) - y(x_{i-1})) \right) = \lim y(x_n), \]

if \( y(x_n) < \infty \) for all \( n \). If \( y(x_n) = \infty \) for at least one \( n \), then the equality \( y(x) = \lim y(x_n) \) follows from the fact that \( y \) is not decreasing.

The proof will be completed if we prove (2). Let \( x, y \in R, (x \cap y)' \in C(x \cap y) \). Since \( x \cap y \leq x, x \cap y \leq y \) we have

\[
x = (x \cap y) \cup [x \cap (x \cap y)'], \quad y = (x \cap y) \cup [y \cap (x \cap y)'].
\]

From it and from the additivity of \( y \) we get (under the assumptions for complemented lattices)

\[
\begin{align*}
\gamma(x) + \gamma(y) &= \gamma(x \cap y) + \gamma(x \cap (x \cap y)') + \gamma(x \cap y) + \gamma(y \cap (x \cap y)') = 2\gamma(x \cap y) + \gamma((x \cap (x \cap y)') \cup [y \cap (x \cap y)']) = \gamma(x \cap y) + \gamma(z),
\end{align*}
\]

where \( z = (x \cap y) \cup [x \cap (x \cap y)'] \cup [y \cap (x \cap y)'] \). If \( H \) is orthocomplemented then we take \( (x \cap y)' = (x \cap y)^1 \) and (19) holds for this complement.

As \( x \cap (x \cap y)' \leq x, y \cap (x \cap y)' \leq y \), we have \( z \leq x \cup y \). Further, according to the modular law \( (x \cap y \leq x) \) we have

\[
z = \{x \cap [(x \cap y) \cup (x \cap y)'] \} \cup [y \cap (x \cap y)'] = x \cup [y \cap (x \cap y)'] \geq x.
\]

Symmetrically, we have \( z \geq y \), hence \( z \geq x \cup y \). Hence we proved that \( z = x \cup y \). From it and (19) 2 follows.

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