Matematický časopis

Ján Jakubík

Quasiorder on Systems of Directed Sets

Matematický časopis, Vol. 24 (1974), No. 2, 173--175,176--177

Persistent URL: http://dml.cz/dmlcz/126601

Mat. čas. 24, 1974, No 2, 173—177

QUASIORDER ON SYSTEMS OF DIRECTED SETS

JÁN JAKUBÍK

Dedicated to Professor Štefan SCHWARZ on the occasion of his sixtieth birthday

Let α be an infinite cardinal and let $D(\alpha)$ be the set of all non-isomorphic types of directed (= upper-directed) sets X with card $X \leqq \alpha$. For $X, Y \in D(\alpha)$ we put $X \leqq_1 Y$ if there exists an isomorphism of X into Y. The relation $\leqq_1$ is a quasiorder on the set $D(\alpha)$ and we denote by $D^1(\alpha)$ the corresponding partially ordered set ([2], Chap. II) consisting of equivalence classes $X^1 = \{Y \in D(\alpha) : X \leqq_1 Y \text{ and } Y \leqq_1 X\}$ for given $X \in D(\alpha)$.

The quasiorder $\leqq_1$ (for order types that need not be directed) was investigated by Laver [5]. A different quasiorder relation between directed sets (based on the notion of cofinality) was studied by J. Schmidt [6], Ginsburg and Isbell [3] and Isbell [4].

For $X^1, Y^1 \in D^1(\alpha)$ we denote by $U^1(X^1, Y^1)$ and $L^1(X^1, Y^1)$ the set of all upper bounds or lower bounds, respectively, of the set $\{X^1, Y^1\}$ in $D^1(\alpha)$. If $U^1(X^1, Y^1)$ posesses a least element, then it will be denoted by $X^1 \vee Y^1$; the symbol $X^1 \wedge Y^1$ has an analogous meaning. In this note there is investigated the existence of the elements $X^1 \vee Y^1$ and $X^1 \wedge Y^1$ in $D^1(\alpha)$ (cf. also [2], Problem 78).

The sets $U^1(X^1, Y^1)$ and $L^1(X^1, Y^1)$ are nonempty for any $X, Y \in D(\alpha)$. In fact, let $Z \in D(\alpha)$ with card $Z = 1$; then $Z^1 \in L^1(X^1, Y^1)$. The ordinal sum of two disjoint partially ordered sets X and Y will be denoted by $X \oplus Y$ ([2], p. 108). The partially ordered set $X \oplus Y$ is directed if and only if Y is directed. If $X, Y \in D(\alpha)$, then for $Z = X \oplus Y$ we have obviously $Z^1 \in U^1(X^1, Y^1)$.

For any cardinal γ we denote by ω_γ the least ordinal β with the property that the power of the set of all ordinals smaller than β equals γ.

Theorem 1. *Let γ be an infinite cardinal. Let A be a directed set that is not linearly ordered,* card $A = \gamma_1$, $\gamma_1 < \gamma \leqq \alpha$. *Let $B = \omega_\gamma$. The partially ordered set $U^1(A^1, B^1)$ has a minimal element C^1 such that C^1 is not the least element of $U^1(A^1, B^1)$ (hence $A^1 \vee B^1$ does not exist in $D^1(\alpha)$).*

Proof. Put $C = A \oplus B$; then we have $C^1 \in U^1(A^1, B^1)$. Let $E \in D(\alpha)$ such that $E^1 \in U^1(A^1, B^1)$, $E^1 \leqq {}_1C^1$. There exists an isomorphism φ of E into C. Further there exists an isomorhism φ_1 of A into E and an somorphism φ_2 of B into E. Put

$$A' = \varphi(\varphi_1(A)), \quad B' = \varphi(\varphi_2(B)).$$

Since card $A' = \gamma_1$, there is $b_1 \in B$ such that $a < b_1$ for each $a \in A'$ in C. Because B' is isomophic to ω_γ there is $b_2 \in B'$ with $b_1 \leqq b_2$. Let

$$B'' = \{b \in B' : b \geqq b_2\}, \quad E' = \varphi^{-1}(A' \cup B'').$$

Then B'' is isomorphic to ω_γ and hence E' is isomorhic to C, whence $C \leqq_1 E$. Therefore C^1 is a minimal element of the partially ordered set $U^1(A^1, B^1)$.

Let $F = B \oplus A$; we have $F \in U^1(A^1, B^1)$. There exist uncomparable elements a, a' in F such that for the set $L(a, a')$ of all lower bounds of the set $\{a, a'\}$ we have

$$\text{card } L(a, a') \geqq \text{card } B = \gamma;$$

no pair of uncomparable elements with such property exists in C and hence there is no isomorphism of F into C. Hence C^1 is not the least element of $U^1(A^1, B^1)$.

Let X be a partially ordered set, $x \in X$. We denote $[x) — \{y \in X : y \geqq x\}$. Let M be a partially ordered set with a greatest element m such that card $(M \setminus \setminus \{m\}) \geqq 2$ and any two distinct elements of the set $M \quad \{m\}$ are uncomparable.

Theorem 2. *Let A be a linearly ordered set, $\aleph_0 \leqq$ card $A \leqq \alpha$. Assume that for each $a \in A$ there exists an isomorphism of A into $[a)$. Let $B = M$,* card $B \leqq \alpha$. *The partially ordered set $U^1(A^1, B^1)$ has two distinct minimal elements.*

Proof. (a) Put $C = A \oplus B$ and let $E \in D(\alpha)$, $E^1 \in U^1(A^1, B^1)$, $E^1 \leqq {}_1C^1$. Let φ, φ_1, φ_2, A', B' have an analogous meaning as in the proof of Thm. 1. Let $a' \in A'$. According to the assumption, the set $\{x \in A' : x > a'\}$ is infinite, hence $a' \in A$ and so $A' \subset A$. Let $b' \in B'$, $b' \neq \varphi(\varphi_2(m))$. There exists $b'' \in B'$ such that b' and b'' are uncomparable and from this it follows that b' cannot belong to A and $b' \neq m$; thus $b' \in B \setminus \{m\}$ and therefore $\varphi(\varphi_2(m)) \quad m$. This implies that the set $\varphi^{-1}(A' \cup B') \subset E$ is isomorphic to C and hence $C^1 \leqq_1 \leqq {}_1E^1$, showing that C^1 is minimal in $U^1(A^1, B^1)$.

(b) Put $C = B \oplus A$ and let us use the same denotations as in (a). Analogously as in (a) we have $b' \in B$ for each $b' \in B'$, $b' \neq \varphi(\varphi_2(m))$. Since A' is isomorphic to A, there are elements $a_1, a_2 \in A' \cap A$ with $\varphi(\varphi_2(m)) \leqq a_1 < a_2$. According to the assumption there exists an isomorphism ψ of A' into $A' \cap \cap [a_2)$ (the symbol $[a_2)$ being considered with respect to A). Let $C' = \psi(A') \cup \cup B'$. Then C' is isomorphic to C and $C' \subset \varphi(E)$. Hence $C^1 \leqq {}_1E^1$ and therefore C^1 is minimal in $U^1(A^1, B^1)$.

It is easy to verify that there is no isomorphism of $A \oplus B$ into $B \oplus A$ and no isomorphism of $B \oplus A$ into $A \oplus B$; hence $(A \oplus B)^1$ and $(B \oplus A)^1$ are uncomparable in $U^1(A^1, B^1)$.

Theorem 3. *Let A be a linearly ordered set, $2 < \operatorname{card} A \leqq \alpha$. Assume that there is $a \in A$ such that there does not exist any isomorphism of A into $[a)$. Let B M. There are partially ordered sets $C, F \in D(\alpha)$ such that $C^1, F^1 \in U^1(A^1, B^1)$ and C^1, F^1 have no common lower bound in $U^1(A^1, B^1)$.*

Proof. Let $C = B \oplus A$. Let $a_1 \in A$ such that there does not exist any isomorphism of A into $[a_1)$. Let B' be isomorphic to B and such that (a) if a_1 is the greatest element of A, then $B' \cap A = \{a_1\}$, $m \neq a_1$; (b) if a_1 is not the greatest element of A, then $B' \cap A = \{a_1, m\}$, $m = a_2 \in A$, $a_2 > a_1$. We consider the following partial order in $F = A \cup B'$. The partial orders in A and B' remain unchanged. In the case (a), m is the greatest element of F and if $b' \in B' \setminus \{a_1, m\}$, $a \in A$, $a \neq a_1$, then the elements a, b' are uncomparable. In the case (b), for any $b' \in B'$, $b' \neq m$ and any $a \in A$, $a \neq a_1$ we put $b' < a$ $(b' > a)$ if and only if $a_1 < a$ $(a_1 > a)$. Obviously we have $C^1, F^1 \in U^1(A^1, B^1)$. Assume that $E^1 \in U^1(A^1, B^1)$, $E^1 \leqq_1 C^1$, $E^1 \leqq_1 F^1$. Then there exist subsets C' and F' of C and F, respectively, that are isomorphic to E. Further there is an isomorphism ψ of B into C'; let $B' = \psi(B)$. For $c' \in C'$ we denote by $[c')'$ the set $[c') \cap C'$, where $[c')$ is taken with respect to C; we use a similar notation $[f')'$ for elements of F'. Each element $b_1 \in B'$, $b_1 \neq \psi(m)$ has the following property

(i) there is an element $b \in C'$ that is uncomparable with b_1 and either $[b_1)'$ contains a subset isomorphic to A or there is an element $b_2 \in C'$ such that b_2 is uncomparable with b_1 and $[b_2)'$ contains a subset isomorphic to A. According to the way in which we have chosen the element a_1 and constructed the partially ordered set F, no element of F' has the property (i), therefore the partially ordered sets C' and F' are not isomorphic, which is a contradiction.

From the Theorems 1, 2 and 3 we obtain as a corollary

Theorem 4. *Let $A \in D(\alpha)$ such that either* (i) *A is a chain and* $\operatorname{card} A > 2$, or (ii) *$A$ is not a chain and* $\operatorname{card} A < \alpha$. *Then there is $B \in D(\alpha)$ such that the oin A^1 B^1 does not exist in $D^1(\alpha)$.*

Theorem 5. *Let $A \in D(\alpha)$ be a chain. If A is finite, then $A^1 \wedge B^1$ exists in $D^1(\alpha)$ for each $B \in D(\alpha)$. If A is infinite, then there is $B \in D(\alpha)$ such that $A^1 \wedge B^1$ does not exist in $D^1(\alpha)$.*

Proof (a). Let A be finite, $B \in D(\alpha)$ and let $\beta = \sup \operatorname{card} X$ where X runs over the system of all linearly ordered subsets of B. If $\beta \geqq \operatorname{card} A$, then $A^1 \leqq_1 B^1$ and $A^1 \wedge B^1 = A^1$. If $\beta < \operatorname{card} A$, let C be a linearly ordered set with $\operatorname{card} C = \beta$; clearly $C^1 = A^1 \wedge B^1$. (b) Further assume that A is infinite and let B_0 be the set of all pairs (m, n) where m, n are positive integers with $n \leqq m$. For (m, n), $(j, k) \in B_0$ we put $(m, n) \leqq (j, k)$ if and only if $m = j$ and $n \leqq k$. Let $B = B_0 \cup \{m_0\}$, $m_0 \notin B_0$ and let m_0 be the greatest element in B. Let $E \in D(\alpha)$, $E \leqq_1 A$, $E \leqq_1 B$. From the first relation we obtain that E is a chain and from the second it follows that E is finite because each linearly ordered subset of B is finite. There exists a linearly ordered subset E' of B with $\text{card}\, E' > \text{card}\, E$; then we have $E' \leqq_1 A$, $E' \leqq_1 B$, $E \leqq_1 E'$ and E' non $\leqq_1 E$. Hence $A^1 \wedge B^1$ does not exist.

Theorem 6. *Let α be a regular cardinal. Let $A \in D(\alpha)$ and let $C(A) = \{\beta: \text{card}\, X = \beta$ for some linearly ordered subset X of $A\}$. Assume that $C(A)$ has no greatest element. Then there is a chain $B \in D(\alpha)$ such that $A^1 \wedge B^1$ does not exist in $D^1(\alpha)$.*

Proof. Let $\{X_i\}(i \in I)$ be the system of all linearly ordered subsets of A. By using the Axiom of choice we can suppose that the set I is linearly ordered. Let B be the set of all pairs (i, x_i) with $i \in I$, $x_i \in X_i$. For (i, x_i), $(j, x_j) \in B$ we put $(i, x_i) \leqq (j, x_j)$, if either $i < j$, or $i = j$ and $x_i \leqq x_j$. Then B is a chain and because α is regular, $B \in D(\alpha)$. In a similar way as in the proof of Thm. 5 (Part (b)) we can now verify that $A^1 \wedge B^1$ does not exist in $D^1(A^1, B^1)$.

We conclude with the following two remarks concerning the properties of the partially ordered set $L^1(A^1, B^1)$ for $A, B \in D(\alpha)$.

The statement analogous to the Thm. 1 fails to be valid, in general, for the partially ordered set $L^1(A^1, B^1)$. Let A be a directed set that is not linearly ordered, $A \in D(\alpha)$ and let $C(A)$ have the same meaning as in Thm. 6. Assume that $\sup C(A) = n < \aleph_0$. Let $B \in D(\alpha)$ be a chain, card $B = \gamma$. Let E be a finite chain with $\text{card}\, E = \min(n, \gamma)$. Then $A^1 \wedge B^1$ does exist in $D^1(\alpha)$ and $A^1 \wedge B^1 = E^1$.

There exist $A, B \in D(\alpha)$ $(\alpha = \aleph_1)$ such that A is a chain and the partially ordered set $L^1(A^1, B^1)$ is not upper-directed. Example:

Let $X = \omega_\gamma$ for $\gamma = \aleph_1$ and let Y be the open interval $(0, 1)$ of real numbers. Let $A = X \oplus Y$, $B = X \cup Y \cup \{m\}$ such that m is the greatest element of B, x and y are uncomparable for any $x \in X$ and any $y \in Y$, and the linear orders in X and Y, respectively, have the original meaning. Then $X^1, Y^1 \in L^1(A^1, B^1)$ and there does not exist $E \in D(\alpha)$ with $E^1 \in L^1(A^1, B^1)$, $X^1 \leqq E^1$, $Y^1 \leqq_1 E^1$. It is easy to verify that X^1 is not maximal in $L^1(A^1, B^1)$. The elements $(X \cup \{m\})^1$ and Y^1 are distinct and maximal in $L^1(A^1, B^1)$.

This example (and also Thms. 2 and 3) shows that $D^1(\alpha)$ fails to be a multilattice (Benado [1]).

REFERENCES

[1] BENADO, M.: Les ensembles partiellement ordonnées et le théorème de raffinement de Schreier. II. Czechosl. Math. J. *5*, 1955, 308—344.
[2] BIRKHOFF, G.: Lattice theory, third edition. Amer. Math. Soc. Colloquium Publications Vol. XXV, 1967, Providence.
[3] GINSBURG, S. and ISBELL, J. R.: The category of cofinal types. I. Trans. Amer. Math. Soc. *116*, 1965, 386—393.
[4] ISBELL, J. R.: The category of cofinal types. II. Trans. Amer. Math. Soc. *116*, 1965, 393—416.
[5] LAVER, R.: On Fraissé's order type conjecture. Ann. Math. *2* (93), 1971, 89—111.
[6] SCHMIDT, J.: Konfinalität. Z. math. Logik und Grundl. Math. *1*, 1955, 271—303.

Received January 3, 1973

Katedra matematiky
Vysokej školy technickej
Švermova 5
040 01 Košice