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CLASSES OF REGULARITY IN SEMIGROUPS

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R. Croisot introduced in [1] the following condition in semigroups: An element $a$ of a semigroup $S$ satisfies the Condition $(m, n)$ if there exists an element $x \in S$ such that

$$a = a^m x a^n,$$

where $m, n$ are non-negative integers and $a^0$ means the void symbol. The set of all elements satisfying the Condition $(m, n)$ is called a class of regularity and will be denoted by $\mathcal{R}_S(m, n)$.

First we state some known relations concerning the classes of regularity (see [2]):

(a) $\mathcal{R}_S(0, 0) = S$.

(b) If $m_1 \geq m_2$ and $n_1 \geq n_2$, then

$$\mathcal{R}_S(m_1, n_1) \subseteq \mathcal{R}_S(m_2, n_2).$$

(c) If $m_1 \geq m_2 \geq 2$, then for any $n$ we have:

$$\mathcal{R}_S(m_1, n) = \mathcal{R}_S(m_2, n).$$

(d) If $n_1 \geq n_2 \geq 2$, then for any $m$ we have:

$$\mathcal{R}_S(m, n_1) = \mathcal{R}_S(m, n_2).$$

(e) $\mathcal{R}_S(1, 2) = \mathcal{R}_S(1, 1) \cap \mathcal{R}_S(0, 2)$.

(f) $\mathcal{R}_S(2, 1) = \mathcal{R}_S(1, 1) \cap \mathcal{R}_S(2, 0)$.

These relations imply that there exist at most nine distinct classes of regularity $\mathcal{R}_S(m, n)$, $0 \leq m \leq 2$, $0 \leq n \leq 2$, connected by relation (b).

There are at most five distinct classes of regularity in commutative semigroups. In these semigroups classes of regularity for which the sum of numbers $m, n$ is equal, coincide. Moreover, in a commutative semigroup $S$ all non-empty classes of regularity are subsemigroups of $S$. The situation in non-commutative semigroups is different. In these semigroups non-empty classes of regularity are not necessarily subsemigroups.
The purpose of this paper is to investigate some sufficient conditions for classes of regularity to be subsemigroups of a given semigroup.

A left (right) ideal $L(R)$ of a semigroup $S$ is called complete if $SL = L$ ($RS = R$).

A left ideal $L$ of a semigroup $S$ is called semiprime if for every element $a \in S$ and an arbitrary integer $n$ the relation $a^n \in L$ implies $a \in L$.

It may occur in some semigroups that some classes of regularity are empty sets. First we state relatively simple conditions for classes of regularity to be non-empty sets.

**Theorem 1.** $\mathcal{R}_S(1, 0)$ ($\mathcal{R}_S(0, 1)$) is non-empty if and only if at least one of the right (left) principal ideals generated by an element of the semigroup $S$ is complete.

**Proof.** (a) Let $\mathcal{R}_S(1, 0) \neq \emptyset$. Let $a \in \mathcal{R}_S(1, 0)$. The right principal ideal generated by $a$ we denote by $(a)_R = a \cup aS$. Then we have: $(a \cup aS)S = aS \cup aS^2 = aS = a \cup aS$, since $a \in aS$. But it means that $(a)_R$ is a complete ideal.

(b) Let the right principal ideal generated by an element $a$ be complete. Therefore, $(a \cup aS)S = aS \cup aS^2 = aS = a \cup aS$. But the last relation implies that $a \in aS$, and it means that $\mathcal{R}_S(1, 0) \neq \emptyset$.

**Theorem 2.** If at least one principal right (left) ideal generated by a square of an element of a semigroup $S$ is semiprime, then $\mathcal{R}_S(2, 0)$ ($\mathcal{R}_S(0, 2)$) is non-empty.

**Proof.** Let a right principal ideal generated by the element $a^2$ be semiprime. Therefore, $a^2 \in (a^2)_R$ implies that $a \in (a^2)_R$ hence $a \in a^2 \cup a^2S$. But the last relation implies that either $a = a^2$, or $a \in a^2S$ and in both cases we obtain that $a \in \mathcal{R}_S(2, 0)$.

**Theorem 3.** The class of regularity $\mathcal{R}_S(1, 1)$ ($\mathcal{R}_S(2, 1)$, $\mathcal{R}_S(1, 2)$, $\mathcal{R}_S(2, 2)$) is a non-empty set if and only if the semigroup $S$ contains at least one idempotent.

**Proof.** (a) If $a = axa$ ($a = a^2xa$, $a = axa^2$, $a = a^2xa^2$) then $ax(a^2x, xa^2, a^2xa)$ is an idempotent of $S$.

(b) The statement is evident.

Remark 1. It is easy to prove that if $S$ is a semigroup then $\mathcal{R}_S(1, 0)$ is a left ideal of $S$, or $\mathcal{R}_S(0, 1) = \emptyset$ and $\mathcal{R}_S(0, 1)$ is a right ideal of $S$ or $\mathcal{R}_S(0, 1) = \emptyset$.

It can occur that some of the sets $\mathcal{R}_S(1, 0)$ and $\mathcal{R}_S(0, 1)$ coincides with the semigroup $S$. We state here one such case.

A semigroup $S$ is called left (right) simple, if $S$ contains no left (right) ideal different from $S$. A semigroup $S$ with zero $0$ is called left (right) $0$-simple if $S^2 \neq 0$ and if $0$ is the unique proper left (right) ideal of $S$.

In [3] it is proved that a semigroup $S(S \neq 0)$ is left simple (left $0$-simple) if and only if for every $a \in S(a \neq 0, a \in S)$ we have $Sa = S$. 

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Remark 2. We can prove easily that if a semigroup $S$ is left simple or left 0-simple, then $S = R_S(0, 1)$.

A simple example can be used in order to show that the preceding condition is only a necessary but not a sufficient one.

Now we show some sufficient conditions in order that other classes of regularity be subsemigroups.

**Theorem 4.** Let $S$ be a semigroup, $R_S(1, 1) \neq \emptyset$ and let one of the following conditions be fulfilled:

(a) The product of any two elements of $R_S(1, 1)$ is an idempotent.
(b) $R_S(1, 1) = R_S(1, 0) \cap R_S(0, 1)$.
(c) The set of all idempotents of $S$ is a subsemigroup of $S$.
(d) Any two idempotents of $S$ commute. Then $R_S(1, 1)$ is a subsemigroup of $S$ and in the case of (d) $R_S(1, 1)$ is an inverse subsemigroup of $S$.

**Proof.**
(a) The statement is evident.
(b) The statement follows from Remark 1.
(c) Let $a, b \in R_S(1, 1)$. Therefore $a = axa$, $b = byb$ for some $x, y \in S$. It is easy to prove that $ax, xa, by, yb$ are idempotents of $S$. Then: $ab = (axa)$ $(byb) = a(xa)(byb)$. According to the assumption the product of two idempotents is an idempotent too therefore $(xa)(by)$ is an idempotent. Hence we have:

$$ab = a(xa)(by) b = a(xaby) b = a(xaby)^2 b =$$

$$= a(xaby)(xaby) b = (axa)(by)(xa)(byb) =$$

$$= ab(yx) ab = ab . z . ab, \text{ where } z = yx \in S.$$

(d) Let $e_1, e_2$ be idempotents of $S$ such that $e_1 . e_2 = e_2 . e_1$. Then $(e_1 . e_2)$ $(e_1 . e_2) = e_1(e_2 . e_1) e_2 = e_1(e_1 . e_2) e_2 = (e_1 . e_1) (e_2 . e_2) = e_1 . e_2$. It follows that the condition (c) is fulfilled and therefore $R_S(1, 1)$ is a subsemigroup of $S$. From [3] it is known that a semigroup $S$ is inverse if all elements of $S$ are regular and if any two idempotents of $S$ commute. But $R_S(1, 1)$ consists only of regular elements of $S$, and according to the assumption any two idempotents of $S$ commute, hence (c) implies that $R_S(1, 1)$ is a subsemigroup of $S$.

**Corollary.** If a semigroup $S$ contains only one idempotent, then $R_S(1, 1)$ is an inverse subsemigroup of $S$.

The following examples of semigroups show that the conditions (b), (d) are not necessary ones

**Example 1** Let $S = \{a, b, c, d\}$ be a semigroup with the multiplication table:
\[ g(s(l, 1), 0) = @ \]
\[ s(0, 1) = \{a, b, c, d\}, \]
\[ s(l, 1) = \{a, b, d\}, \]
but \( s(l, 1) \) is a subsemigroup.

**Example 2.** Let \( S = \{a, b, c, d\} \) be a semigroup with the multiplication table:

\[
\begin{array}{cccc}
  & a & b & c & d \\
 a & a & a & a & a \\
b & a & b & b & b \\
c & a & b & b & c \\
d & a & b & c & d \\
\end{array}
\]

\( R_S(1, 0) = R_S(0, 1) = \{a, b, c, d\}, \)
\( R_S(1, 1) = \{a, b, d\}, \) but \( R_S(1, 1) \) is a subsemigroup.

**Remark 3.** Elements of \( R_S(1, 1) \) have one-sided identities of the form: \( ax, xa \). Elements of \( R_S(2, 0) \) have right identities of the form \( ax \). But we cannot assert that all one-sided identities of elements of \( R_S(1, 1), R_S(2, 0) \) and \( R_S(0, 2) \) have such a form.

**Example 3.** Let \( S = \{a, b, c, d\} \) be a semigroup with the following multiplication table:

\[
\begin{array}{cccc}
  & a & b & c & d \\
 a & a & a & c & d \\
b & a & a & c & d \\
c & c & c & d & a \\
d & d & d & a & c \\
\end{array}
\]

\( R_S(1, 1) = \{a, c, d\}. \) \( c = cxc \) for the unique element \( x = d, \) \( dc = cd = a. \)
The element \( dc \) is a right (and also a left) identity of the element \( c, \) but for the element \( b \) we have moreover: \( cb = c. \)

Left (right) identities of elements of \( R_S(1, 1) \) are called left (right) regular identities. But for one-sided identities of elements of \( R_S(2, 0) \) and \( R_S(0, 2) \) no special name is used. Therefore, for our need we introduce:

**Definition 1.** Left identities of an element \( a \in R_S(0, 2) \) of the form \( xa \) and right identities of an element \( a \in R_S(2, 0) \) of the form \( ax \) are called local left identities, and local right identities respectively, or shortly local one-sided identities.

**Theorem 5.** Let \( S \) be a semigroup, \( R_S(2, 0) \neq \emptyset \) and let any of the following conditions be fulfilled:
(a) The product of any two elements of $\mathcal{R}_S(2, 0)$ is an idempotent.

(b) The product of local right identities of the elements $a, b \in \mathcal{R}_S(2, 0)$ is a right identity of the element $ab$.

(c) Every local right identity of any element of $\mathcal{R}_S(2, 0)$ belongs to the centre $Z$ of the semigroup $S$. Then $\mathcal{R}_S(2, 0)$ is a subsemigroup of $S$.

Proof. (a) The statement is evident.

(b) Let $a, b \in \mathcal{R}_S(2, 0)$, therefore $a = a^2x$, $b = b^2y$, and $x, y \in S$. Then $a = a(ax)$, $b = b(by)$. According to the assumption we have $ab = ab(ax)(by)$, $ba = ba(by)(ax)$. Then $ab = (ab)(ax)(by) = a(ba)(xy)(by) = (ab)(ab)(yax)(xy) = (ab)^2(yax)(xy) = (ab)^2z$, where $z = (yax)(xy) \in S$.

(c) We shall show that (c) implies (b). Let $a, b \in \mathcal{R}_S(2, 0)$. Then $ab = a(ax)b(by) = a(bax)(by) = (ab)(axy)$. Hence the proof follows from (b).

Analogously we can prove

Theorem 5'. Let $S$ be a semigroup, $\mathcal{R}_S(0, 2) \neq \emptyset$ and any of the following conditions be fulfilled:

(a) The product of any two elements of $\mathcal{R}_S(0, 2)$ is an idempotent.

(b) The product of local left identities of the elements $a, b \in \mathcal{R}_S(0, 2)$ is a left identity of the element $ab$.

(c) Every local right identity of any element of $\mathcal{R}_S(0, 2)$ belongs to the centre $Z$ of the semigroup $S$. Then $\mathcal{R}_S(0, 2)$ is a subsemigroup of $S$.

Lemma 1. $\mathcal{R}_S(2, 2) = \mathcal{R}_S(2, 1) \cap \mathcal{R}_S(1, 2)$.

Proof. (a) From p. 299, (b) we have $\mathcal{R}_S(2, 2) \subseteq \mathcal{R}_S(2, 1)$, $\mathcal{R}_S(2, 2) \subseteq \mathcal{R}_S(1, 2)$, therefore $\mathcal{R}_S(2, 2) \subseteq \mathcal{R}_S(2, 1) \cap \mathcal{R}_S(1, 2)$.

(b) Let $a \in \mathcal{R}_S(2, 1) \cap \mathcal{R}_S(1, 2)$, hence $a = a^2xa$, $a = aya^2$. Then $a = a^2xa = -a^2xya^2 = a^2(axa^2) = a^2za^2$, where $z = xay \in S$ and it follows that $a \in \mathcal{R}_S(2, 2)$.

Theorem 6. Let $E \subseteq Z$, where $E$ is the set of all idempotents and $Z$ is the centre of a semigroup $S$. Then each of classes of regularity $\mathcal{R}_S(1, 1)$, $\mathcal{R}_S(2, 1)$, $\mathcal{R}_S(1, 2)$, and $\mathcal{R}_S(2, 2)$ is a subsemigroup of $S$, or an empty set.

Proof. The statement that $\mathcal{R}_S(1, 1)$ is a subsemigroup of $S$ under our assumption follows from Theorem 4, (d).

Let now $a, b \in \mathcal{R}_S(2, 1)$, therefore $a = a^2xa$, $b = b^2yb$, for some $x, y \in S$. It is easy to prove that the elements $a^2x$, $b^2y$ are idempotents of $S$. Then

$$ab = (a^2xa)(b^2yb) = (a^2x)a(b^2y)b = (a^2x)(b^2y)(ab) = a(ax)b(by)(ab) =$$

$$= (a^2xa)(ax)(b^2yb)(by)(ab) = a(a^2x)(axb)(b^2y)(by)(ab) =$$

$$= a(a^2x)(ax)(b^2y)(b^2y)(ab) = a(b^2y)(a^2x)(ax)(b^2y)(ab) =$$

$$= (ab)(by)(a^2x)(ax)(b^2y)(ab) = (ab)(a^2x)(by)(ax)(b^2y)(ab) =$$

$$= (ab)(a^2x)(b^2y)(by)(ax)(b^2y)(ab) = (ab)a(ax)(b^2y)(by)(ax)(ab) =$$
Analogously we can prove the statement that $R_S(1, 2)$ is a subsemigroup and the statement, concerning $R_S(2, 2)$ follows from Lemma 1.

Remark 4. From [2] (pp. 139, 424) it is known that an element $a \in S$ is totally regular if and only if $a$ belongs to some subgroup of the semigroup $S$. Moreover, $S$ is totally regular if and only if $S = R_S(2, 2)$. From the above we have:

**Corollary.** Let $\emptyset \neq E \subseteq Z$. Then the union of all subgroups of the semigroup of $S$ is a subsemigroup of $S$.

Remark 5. Other conditions for the classes of regularity $R_S(2, 1)$, $R_S(1, 2)$ and $R_S(2, 2)$ to be subsemigroups of $S$ can be obtained by means of statements (e) (f) quoted in the introduction and Lemma 1, by combining the conditions of Theorem 4 with the conditions of Theorem 5 and Theorem 5', respectively.

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