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INVOLUTED RESTRICTIVE BISEMIGROUPS OF BINARY RELATIONS

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Let $\varrho$ and $\sigma$ be binary relations between the elements of two sets $A$ and $B$. Denote by $\text{pr}_1\varrho$ the first projection of $\varrho$ and by $\text{pr}_2\sigma$ the second projection of $\sigma$. Define $\varrho \triangleright \sigma = (\text{pr}_1\varrho \times B) \cap \sigma$, $\varrho \triangleright \sigma = \varrho \cap (A \times \text{pr}_2\sigma)$. Then $\varrho \triangleright \sigma$ and $\varrho \triangleright \sigma$ are binary relations between the elements of the same sets $A$ and $B$. The operations $\triangleright$ and $\triangleright$ are called the restrictive multiplication of the first kind (or the left restrictive multiplication) and the restrictive multiplication of the second kind (or the right restrictive multiplication). A left restrictive semigroup of binary relations is any pair $(\Phi, \triangleright)$ where $\Phi$ is a non empty set of binary relations between elements of some two sets, closed under $\triangleright$. Right restrictive semigroups of binary relations are defined analogously. An algebra $(\Phi, \triangleright, \triangleright)$ where $(\Phi, \triangleright)$ is the left and $(\Phi, \triangleright)$ the right restrictive semigroup of binary relations is called a restrictive bisemigroup of binary relations. Restrictive products of both kinds of two many-to-one (or one-to-one) binary relations are many-to-one (or one-to-one) binary relations, hence one may speak of restrictive semigroups and bisemigroups of (partial) mappings and one-to-one (partial) mappings.

An algebra isomorphic to a (left, right) restrictive semigroup of binary relations is called a (left, right) restrictive semigroup. An algebra isomorphic to a restrictive bisemigroup of binary relations is called a restrictive bisemigroup.

These definitions are due to V. V. Wagner [1]. Restrictive bisemigroups were introduced by the author [2], but the idea goes back to V. V. Wagner [3].

V. V. Wagner [1] proved that left restrictive semigroups are characterized by the following three identities:

1. $(x \triangleright y) \triangleright z = x \triangleright (y \triangleright z)$ (associativity)
2. $x \triangleright x = x$ (idempotency)
3. $x \triangleright y \triangleright z = y \triangleright x \triangleright z$ (left pseudocommutativity)

Operations satisfying (3) are also called right normal.

V. V. Wagner [1] also proved that every left restrictive semigroup is iso-
morphic to a left restrictive semigroup of partial mappings. The author [2] proved that every left restrictive semigroup is isomorphic to a left restrictive semigroup of one-to-one partial mappings, and is isomorphic to a left restrictive semigroup of partial mappings of a set onto another set [4].

Dually, right restrictive semigroups are characterized by the identities:

\( (x \triangleleft y) \triangleleft z = x \triangleleft (y \triangleleft z), \)
\( x \triangleleft x = x, \)
\( x \triangleleft y \triangleleft z = x \triangleleft z \triangleleft y. \)

The author [2] proved that restrictive bisemigroups are characterized by the identities (1)—(6) and the identity:

\( (x \triangleright y) \triangleleft z = x \triangleright (y \triangleleft z) \) (associativity of the pair \((\triangleright, \triangleleft)\))

In what follows the product \((x \triangleright y) \triangleleft z\) will be denoted by \(x \triangleright y \triangleleft z\).

In view of possible applications the most interesting class of restrictive bisemigroups is the class of restrictive bisemigroups of one-to-one partial mappings. If \(V\) is a differentiable manifold and \(K\) is its coordinate atlas (i.e., the set of all admissible coordinate systems) then \(K\) is closed under \(\triangleright\) and \(\triangleleft\) and \((K, \triangleright, \triangleleft)\) is a restrictive bisemigroup. The present author has proved (unpublished) that two differentiable manifolds with isomorphic restrictive bisemigroups of coordinate systems are isomorphic (i.e., diffeomorphic). Hence, a rather complex structure of a differentiable manifold may be completely defined by two rather trivial idempotent associative algebraic operations.

The author [4] proved that a restrictive bisemigroup is isomorphic to a restrictive bisemigroup of partial mappings if and only if it satisfies the identity:

\( (x \triangleleft y) \triangleright x = x \triangleleft y \)

and is isomorphic to a restrictive bisemigroup of one-to-one partial mappings if and only if it satisfies the identities (8) and

\( x \triangleleft (y \triangleright x) = y \triangleright x. \)

Every restrictive bisemigroup \((\mathcal{P}, \triangleright, \triangleleft)\) of binary relations is ordered by the inclusion relation \(\subset\). An ordered restrictive bisemigroup \((A, \triangleright, \triangleleft, \leq)\) is isomorphic to a restrictive bisemigroup of binary relations ordered by the inclusion relation if and only if the order \(\leq\) satisfies the following conditions:

\( v \preceq w, x \preceq y \rightarrow v \triangleright x \preceq w \triangleright y, \)

(stability of \(\leq\))
(11) \[ v \leq w, \ x \leq y \rightarrow v \triangleleft x \leq w \triangleleft y, \]
(12) \[ x \triangleright y \leq y, \]
(13) \[ x \triangleright y \leq x, \]
(14) \[ x \leq y \rightarrow y \triangleright x \triangleright y = x. \]

This was proved in [4].

The present note is devoted to restrictive bisemigroups of binary relations between the elements of two equal sets (the so-called homogeneous binary relations). If \( \sigma \subseteq A \times A \) is a binary relation between the elements of a set \( A \) then \( \sigma^{-1} \) denotes the converse of \( \sigma \): \( (a_1, a_2) \in \sigma^{-1} \leftrightarrow (a_2, a_1) \in \sigma. \)

An involuted set of binary relations is any pair \((\Phi, -1)\) where \( \Phi \) is a non-empty set of homogeneous binary relations closed under the operation \(-1\) of conversion. An involuted bisemigroup of binary relations is an algebra \((\Phi, \triangleright, \triangleleft, -1)\) where \((\Phi, \triangleright, \triangleleft)\) is a restrictive bisemigroup of binary relations and \((\Phi, -1)\) is an involuted set of binary relations. We consider also ordered involuted restrictive bisemigroups of binary relations, i.e., algebraic systems \((\Phi, \triangleright, \triangleleft, -1, <)\) where \((\Phi, \triangleright, \triangleleft, -1)\) is an involuted restrictive bisemigroup of binary relations and \((\Phi, <)\) is a set of binary relations ordered by the inclusion relation.

Let \( x \) and \( y \) denote binary relations, \( x^{-1} \) and \( y^{-1} \) their converses and \( \leq \) denote the inclusion relation. It is easy to verify the following properties:

(15) \[ x \leq y \rightarrow x^{-1} \leq y^{-1}, \text{(stability of } \leq \text{ under conversion)} \]
(16) \[ (x^{-1})^{-1} = x, \text{(involutivity)} \]
(17) \[ (x \triangleright y)^{-1} = y^{-1} \triangleleft x^{-1}. \]

The identities (16) and (17) imply

(18) \[ x \triangleright y = (y^{-1} \triangleleft x^{-1})^{-1}, \]
(19) \[ x \triangleleft y = (y^{-1} \triangleright x^{-1})^{-1} \]

i.e., each one of the restrictive operations is a derived operation from the other one and from conversion \(-1\). Identities (18) and (2), (5) imply both (16) and (17). If (16) and (17) are satisfied, then (1) is equivalent to (4), (2) to (5), (3) to (6), (8) to (9). If (15)—(17) are satisfied, then (10) is equivalent to (11), (12) to (13). If (1), (2), (7), (15)—(17) are satisfied, then (14) is equivalent to each of the formulas:

(20) \[ x \leq y \rightarrow y \triangleright x = x, \]
(21) \[ x \leq y \rightarrow x \triangleleft y = x. \]
Let (1), (2), and (7) be satisfied. If (14) is true and \( x < y \) then \( y > x = y > (y > x < y) = (y > y) > x < y = (x < y) = x \). If (16) and (17) are satisfied and (20) is true then \( x < y \) imply \( y > x = y > (y > x < y) = (y > y) > x < y = (x < y) = x \).

Let \((A, >, <)\) be a restrictive bisemigroup. Define a binary relation \( \alpha \) between the elements of \( A \) by the formula:

\[
(a_1, a_2) \in \alpha \iff a_1 > a_2 < a_1 = a_1.
\]

It is known [2, 4] that \( \alpha \) is an order relation. It is called the *canonical order relation* of the restrictive bisemigroup. This order relation satisfies the conditions (10)–(14). If \((A, >, <)\) satisfies (8) or (9), then \( \alpha \) is the only order relation satisfying (10)–(14).

Our main result is the following

**Theorem of representation.** Every algebra \((A, >, <, ^{-1})\) satisfying the identities (1)–(3), (7), (16), (17) is isomorphic to an involuted restrictive bisemigroup of binary relations. If this algebra satisfies also (8) then it is isomorphic to an involuted restrictive bisemigroup of one-to-one partial transformations.

Every algebraic system \((A, >, <, ^{-1}, \leq)\), where \( \leq \) is an order relation, is isomorphic to an ordered involuted restrictive bisemigroup of binary relations, if this system satisfies (1)–(3), (7), (10), (12), (15)–(17), (20).

**Proof.** Let \((A, >, <, ^{-1}, \leq)\) satisfy (1)–(3), (7), (10), (12), (15)–(17), (20). Then it satisfies (1)–(7), (10)–(21). Let \( F(A) \) be a free group over the alphabet \( A \) such that for every \( a \in A \) \( a^{-1} \) is the group inverse of \( a \). Let \( R \) be any faithful simply transitive representation of \( F(A) \) by permutations of some set \( B \) (say, let \( R \) be the representation of \( F(A) \) by translations).

Clearly, \((A, >, <)\) is a restrictive bisemigroup. Let \( e_1 \) and \( e_2 \) be its canonical equivalence relations [2, 4], i.e., \( a_1 \equiv a_2(e_1) \iff a_1 > a_2 = a_2 > a_1 = a_1; a_1 \equiv a_2(e_2) \iff a_1 < a_2 = a_2 < a_1 = a_2 \). The identities (16) and (17) imply that \( a_1 \equiv a_2(e_1) \iff a_1^{-1} = a_2^{-1} \). In what follows we write \( e \) instead of \( e_1 \). \( e(a) \) denotes the \( e \)-class containing \( a \).

Let \( \theta_{(a)} \) be a one-to-one mapping of \( B \) onto some set \( b_{e(a)} \) and sets \( b_{e(a_1)} \) and \( b_{e(a_2)} \) be disjoint if \( e(a_1) \neq e(a_2) \). Denote \( C = \bigcup_{a \in A} b_{e(a)} \). Define \( \varphi_a = \Theta_{e(a^{-1})} \circ R(a) \circ \Theta_{e(a)}^{-1}, P(a) = \bigcup_{b \leq a} \varphi_b \).

Then \( P \) is an isomorphism of \((A, >, <, ^{-1}, \leq)\) onto some ordered involutive restrictive bisemigroup of binary relations between the elements of \( C \), i.e., the following formulas hold:
\( P(a_1) \triangleright P(a_2) = P(a_1 \triangleright a_2) , \)
\( P(a_1) \triangleleft P(a_2) = P(a_1 \triangleleft a_2) , \)
\( \frac{-1}{P(a)} = P(a^{-1}) , \)
\( P(a_1) \subset P(a_2) \iff a_1 \leq a_2 . \)

If (26) holds and \( P(a_1) = P(a_2) , \) then \( P(a_1) \subset P(a_2) , P(a_2) \subset P(a_1) . \) Hence \( a_1 \leq a_2 \) and \( a_2 \leq a_1 , \) i.e. \( a_1 = a_2 \) and \( P \) is one-to-one.

For the proof of (23) we need the formula:
\( b \leq a_1 \triangleright a_2 \iff (\forall a) [b \equiv a(\varepsilon) \land a \leq a_1 \land b \leq a_2] . \)

Here \( \forall \) is the existence quantifier and \( \land \) is the conjunction.

Let \( b \leq a_1 \triangleright a_2 . \) Denote \( a = b \triangleright a_1 . \) By (12), \( a \leq a_1 \) and \( b \leq a_1 \triangleright a_2 \leq a_2 . \) By (1) and (2), \( b \triangleright a = b \triangleright b \triangleright a_1 = b \triangleright a_1 = a . \) By (1) and (3) and (20),

\( a \triangleright b = b \triangleright a_1 \triangleright b = a_1 \triangleright b = a_1 \triangleright (a_1 \triangleright a_2) \triangleright b = \left( a_1 \triangleright a_1 \right) \triangleright a_2 \triangleright b = b , \) hence, \( b = a(\varepsilon) . \)

Let now \( (\forall a) [b \equiv a(\varepsilon) \land a \leq a_1 \land b \leq a_2] . \) By (10), \( b = a \triangleright b \leq a_1 \triangleright b \leq a_1 \triangleright a_2 , \) i.e., \( b \leq a_1 \triangleright a_2 . \) We have proved (27).

\( P(a_1) \triangleright P(a_2) = (\bigcup_{a \leq a_1} \varphi_a) \triangleright (\bigcup_{b \leq a_2} \varphi_b) = \bigcup_{a \leq a_1, b \leq a_2} \varphi_a \triangleright \varphi_b = \bigcup_{a = b(\varepsilon), b \leq a_2} \varphi_a \triangleright \varphi_b = \bigcup_{b \leq a_1 \triangleright a_2} \varphi_b = P(a_1 \triangleright a_2) . \) In the proof we used the following property: \( \varphi_a \triangleright \varphi_b = \emptyset \) if \( \varepsilon(a) \not= \varepsilon(b) \), and \( \varphi_a \triangleright \varphi_b = \varphi_b \) otherwise.

The proof of (24) is quite analogous. It may also be obtained from (23) and (25). Using (19) as an axiom and as a property of binary relations we obtain

\[ P(a_1) \triangleright P(a_2) = \frac{-1}{P(a_2)} = \frac{-1}{P(a_1)} = \frac{-1}{P(a_2)} \triangleright P(a_1) \]

\[ = P((a_2^{-1} \triangleright a_1^{-1})^{-1}) = P(a_1 \triangleleft a_2) . \]

\[ \varphi_a^{-1} = \Theta_{\varepsilon(a)}^{\circ} \frac{-1}{R(a)} \circ \Theta_{\varepsilon(a)}^{-1} = \Theta_{\varepsilon(a)} \circ R(a^{-1}) \circ \Theta_{\varepsilon(a)}^{-1} = \varphi_a . \]

Hence, \( \frac{-1}{P(a)} = \frac{-1}{\bigcup_{b \leq a} \varphi_b} = \bigcup_{b \leq a} \varphi_b^{-1} = \bigcup_{b^{-1} \leq a^{-1}} \varphi_b = P(a^{-1}) . \)

Let \( P(a_1) \subset P(a_2) . \) Then \( \varphi_a \subset P(a_1) \subset P(a_2) , \) therefore \( \varphi_a \cap \varphi_b \not= \emptyset \) for some \( b \leq a_2 . \) It means that \( R(a_1) \cap R(b) \not= \emptyset . \)

(1) The proof of this fact may be found in [4], p. 118.
$R(a_1) = R(b)$ and $a_1 = b$. Hence $a_1 \leq a_2$. If $a_1 \leq a_2$ then evidently $P(a_1) \subseteq P(a_2)$.

If $(A, \triangleright, \triangleleft, -1)$ is an algebra satisfying (1)—(3) (7), (16), (17), then $(A, \triangleright, \triangleleft, -1, x)$ satisfies (1)—(3), (7), (10), (12), (15)—(17), (20) and is isomorphic to an ordered involuted restrictive bisemigroup of binary relations. If (8) is also satisfied, then the isomorphism $P$ constructed as above is an isomorphism of $(A, \triangleright, \triangleleft, -1, x)$ onto a bisemigroup of one-to-one partial transformations. If (8) is satisfied, then (9) is also satisfied. Let $(c, c_1) \in P(a)$ and $(c, c_2) \in P(a)$. It means that $(c, c_1) \in p_{b_1}$ and $(c, c_2) \in p_{b_2}$ for some $b_i \leq a$ ($i = 1, 2$). We see that $pr_1 p_{b_1} \cap pr_1 p_{b_2} = \emptyset$. It is possible only if $\varepsilon(b_1) = \varepsilon(b_2)$. Therefore, $b_i \triangleright b_j = b_j$ for $i, j = 1, 2$. The order $\triangleleft$ coincides with $\triangleleft$, hence $b_i \triangleright a \triangleleft b_i = b_i$. Using (8) we have: $b_1 = b_1 \triangleright (a \triangleleft b_1) = b_1 \triangleright ((a \triangleleft b_1) \triangleright a) = (b_1 \triangleright a \triangleleft b_1) \triangleright a = b_1 \triangleright a = (b_2 \triangleright b_1) \triangleright a = b_1 \triangleright (b_2 \triangleright a) = b_1 \triangleright b_2 = b_2$. We used the equality $b_2 \triangleright a = b_2$ which may be proved exactly as the equality $b_1 \triangleright a = b_1$. Hence, $p_{b_1} = p_{b_2}$. But $p_{b_1}$ is one-to-one, hence $c_1 = c_2$. In the same way we prove that $(c_1, c) \in P(a)$ and $(c_2, c) \in P(a)$ imply $c_1 = c_2$. It means that $P(a)$ is one-to-one for all $a \in A$. Therefore, if $(A, \triangleright, \triangleleft, -1)$ satisfies (1)—(3), (7), (16), (17) (and (8)) then this algebra is isomorphic to an involuted restrictive bisemigroup of (one-to-one) binary relations, which completes the proof of our theorem.

As an important example of an involuted restrictive bisemigroup satisfying (8) we give the following one: let $G$ be any inverse semigroup (= generalized group, or pseudogroup, or groupoid, or paragroup) $g^{-1}$ be the inverse for $g \in G$. Define $g_1 \triangleright g_2 = g_1 g_1^{-1} g_2$ and $g_1 \triangleleft g_2 = g_1 g_2^{-1} g_2$. Then $(G, \triangleright, \triangleleft, -1)$ is an involuted restrictive bisemigroup satisfying (8).

Here and henceforward an algebra satisfying (1)—(3), (7), (16), (17) is called an involuted restrictive bisemigroup, and an algebraic system satisfying (1)—(3), (7), (10), (12), (15)—(17) is called an ordered involutive restrictive bisemigroup.

We say that a restrictive bisemigroup $(A, \triangleright, \triangleleft)$ admits involution if $(A, \triangleright, \triangleleft, -1)$ is an involuted restrictive bisemigroup for some unary operation $-1$. Clearly, $(A, \triangleright, \triangleleft)$ admits involution if and only if there exists an involutive antiisomorphism of the left restrictive semigroup $(A, \triangleright)$ onto the right restrictive semigroup $(A, \triangleleft)$ (a transformation $\varphi$ is called involutive if $\varphi \circ \varphi$ is an identical transformation).

An algebra $(A, \triangleright, -1)$ is called an involuted left restrictive semigroup if it is isomorphic to an algebra $(\Phi, \triangleright, -1)$ where $(\Phi, \triangleright)$ is a left restrictive semigroup of binary relations and $(\Phi, -1)$ is an involuted set of binary relations. Our Theorem of representation yields the following
Corollary. An algebra \((A, \rhd, ^{-1})\) is an involuted left restrictive semigroup if and only if it satisfies \((1) - (3), (16)\) and the following identity:

\[
(x \rhd (y \rhd z)^{-1})^{-1} = y \rhd (x \rhd z^{-1})^{-1}.
\]

Proof. Clearly \((A, \rhd, ^{-1})\) is an involuted left restrictive semigroup if and only if there exists a binary operation \(\langle\rangle\) on \(A\) such that \((A, \rhd, \langle\rangle, ^{-1})\) is an involuted restrictive bisemigroup, i.e., it satisfies \((1) - (3), (7), (16)\) and \((17)\). From \((17)\) we deduce that \(x \langle y = (y^{-1} \rhd x^{-1})^{-1}\). The identity \((7)\) may be written in the form \((y \rhd z) \langle x^{-1} = y \rhd (z \langle x^{-1})\). This form is equivalent to the identity \((28)\).

The identity \((9)\) is equivalent to

\[
(x \rhd y)^{-1} \rhd y^{-1} = (x \rhd y)^{-1}.
\]

This identity characterizes involuted left restrictive semigroups isomorphic to involuted left restrictive semigroups of one-to-one partial transformations.

In a quite analogous way one can define and characterize involuted right restrictive semigroups.

We will consider now restrictive bisemigroups and semigroups of certain special binary relations.

Let \(\varrho\) be a binary relation between the elements of a set \(A\). If \(B \subset A\), then \(\Delta_B\) denotes the identical binary relation over \(B\), i.e., \((a_1, a_2) \in \Delta_B \iff a_1 = a_2 \in B\). Let \(\text{pr}_{\varrho} = \text{pr}_1 \varrho \cup \text{pr}_2 \varrho\). The binary relation \(\varrho\) is called:

- **equiprojective**, if \(\text{pr}_1 \varrho = \text{pr}_2 \varrho\);
- **partially reflexive**, if \(\Delta_{\text{pr}_{\varrho}} \subset \varrho\);
- **irreflexive**, if \(\varrho \cap \Delta_A = \emptyset\);
- **symmetric**, if \(\varrho^{-1} = \varrho\);
- **transitive**, if \(\varrho \circ \varrho \subset \varrho\);

it is called **partial quasiequivalence** if \(\varrho\) is partially reflexive and symmetric; **partial quasorder** if \(\varrho\) is partially reflexive and transitive; **partial equivalence** if \(\varrho\) is a symmetric partial quasiorder.

Proposition. The following 12 properties of a restrictive bisemigroup are equivalent:

1) it satisfies the identity

\[
x \rhd y = y \lhd x.
\]

It is isomorphic with a restrictive bisemigroup of

2) symmetric irreflexive one-to-one partial transformations;
3) symmetric irreflexive binary relations;
4) symmetric binary relations;
5) symmetric one-to-one partial transformations;
6) partial equivalence relations;
7) partial quasiorder relations;
8) partial quasiequivalence relations;
9) partially reflexive binary relations;
10) equiprojective one-to-one partial transformations;
11) equiprojective binary relations;
12) antisymmetric partial quasiorder relations.

Remark. A binary relation \( Q \) is antisymmetric if \( Q \cap Q^{-1} \subset A \). Antisymmetric partial quasiorder relations may be called partial order relations, but the term "partial order relation" in the current mathematical language is used in another sense (synonymous with "order relation").

Proof. Let \((A, \succ, \triangleleft)\) be a restrictive bisemigroup satisfying (30) and let \( a^{-1} = a \) for every \( a \in A \). Then \((A, \succ, \triangleleft, \lhd)\) turns out to be an involuted restrictive bisemigroup satisfying (8). Let \( P \) be the isomorphism of \((A, \succ, \triangleleft, \lhd)\) constructed in the proof of the Theorem of representation. For every \( a \in A \), \( P(a) \) is a one-to-one partial transformation. Clearly, \( P(a) \) is a symmetric binary relation. Let for some \( c \in C (c, c) \in P(a) \). Then \( (c, c) \in Q(b) \) for some \( b \in A \), \( (b, a) \in \). This means that \( R(b) \cap R(e) \neq \emptyset \), where \( e \) is the identity of the group \( F(A) \). By simple transitivity of \( R \), \( b = e \), whence \( e \in A \). But this is impossible, hence \( P(a) \) is irreflexive.

We proved that 1) \( \rightarrow \) 2). The following implications are trivial:

\[ 2) \rightarrow 3) \rightarrow 4) \rightarrow 11), \quad 2) \rightarrow 5) \rightarrow 10) \rightarrow 11). \]

Define \( Q(a) = P(a) \cup \Delta_{pr} P(a) \). It is a matter of straightforward computation to check up that \( Q \) is an isomorphism of \((A, \succ, \triangleleft, \lhd)\) onto an involuted restrictive bisemigroup of binary relations (if \( Q(a) = Q(b) \), then \( P(a) = Q(a) \setminus \Delta_{pr} Q(a) = Q(b) \setminus \Delta_{pr} Q(b) = P(b) \) and \( a = b \)). Binary relations \( Q(a) \) are symmetric and partially reflexive. Without difficulty one can obtain \( Q(a) \circ Q(a) = Q(a) \), i.e. \( Q(a) \) for every \( a \in A \) is a partial equivalence relation. Hence, 1) \( \rightarrow \) 6).

The following implications are trivial: 6) \( \rightarrow \) 7) \( \rightarrow \) 9) \( \rightarrow \) 11), 6) \( \rightarrow \) 8) \( \rightarrow \) 11) and 12) \( \rightarrow \) 11).

Let \((\Phi, \succ, \triangleleft)\) be a restrictive bisemigroup of equiprojective binary relations between the elements of a set \( B \) and \( \varphi, \psi \in \Phi \). Then \( pr_2(\varphi \triangledown \psi) = pr_1(\varphi \triangledown \psi) = pr_1 \varphi \cap pr_1 \psi = pr_2 \varphi \cap pr_2 \psi \), whence \( \varphi \triangledown \psi \subset (pr_1 \varphi \times B) \cap (B \times pr_2 \varphi) = pr_1 \varphi \times pr_2 \varphi \). This inclusion implies \( \varphi \triangledown \psi = \psi \cap (pr_1 \varphi \times pr_2 \varphi) \). In a quite analogous way we prove that \( \psi \triangleleft \varphi = \varphi \cap (pr_1 \varphi \times pr_2 \varphi) = \varphi \triangledown \psi \), hence, \((\Phi, \succ, \triangleleft)\) satisfies (30) and 11) \( \rightarrow \) 2).

Let us prove now that 2) \( \rightarrow \) 12). Let \( P \) be any isomorphism of a restrictive
bisetemigroup \((A, \triangleright, \triangleleft)\) onto a restrictive bisetemigroup of binary relations between the elements of sets \(B\) and \(C\). Without any loss of generality we may suppose that \(C \cap B = \emptyset\). Let \(D = B \cup C\) and \(Q(a) = P(a) \cup \Delta_{pr}P(a)\). One can easily verify that \(Q(a) = Q(b) \Rightarrow a = b\). Let \((A, \triangleright, \triangleleft)\) satisfy (30). Then \(Q(a) \triangleright Q(b) = (P(a) \triangleright P(b)) \cup (\Delta_{pr}P(a) \triangleright P(b)) \cup (P(a) \triangleright \Delta_{pr}P(b)) \cup \Delta_{pr}P(a) \triangleright \Delta_{pr}P(b)\). Hence \(prP(a) \cap prP(b) = (pr_1P(a) \cap pr_1P(b)) \cup (pr_2P(a) \cap pr_2P(b)) = pr_1(P(a) \triangleright P(b)) \cup pr_2(P(b) \triangleleft P(a)) = pr_1(P(a) \triangleright b) \cup pr_2(P(a) \triangleright b) = prP(a \triangleright b)\), whence \(Q(a) \triangleright Q(b) = Q(a \triangleright b)\). Quite analogously, \(Q(a) \triangleleft Q(b) = Q(a \triangleleft b)\). Clearly, for every \(a \in A\), \(Q(a)\) is an antisymmetric partial quasiorder relation.

This completes the proof of our Proposition.

The representation \(P\) considered at the end of the proof yields

**Corollary 1.** Every restrictive bisetemigroup is isomorphic with a restrictive bisetemigroup of irreflexive transitive binary relations.

**Corollary 2.** Every left restrictive semigroup is isomorphic to a left restrictive semigroup of binary relations of any type mentioned in the Proposition.

**Proof.** Let \((A, \triangleright)\) be a left restrictive semigroup. Define \(a \triangleleft b = b \triangleright a\) for every \(a, b \in A\). Then \((A, \triangleright, \triangleleft)\) is a restrictive bisetemigroup satisfying the first condition of the Proposition, whence the corollary follows.

**REFERENCES**


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