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ON TENSOR PRODUCT OF VECTOR MEASURES IN LOCALLY COMPACT SPACES

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Introduction. In this article we are concerned with some properties of vector measures in the product of two locally compact spaces, assuming we are given a Borel vector measure on each of the factor spaces. Our approach is that used in [14] and [2].

We prove, for example, that if μ and ν are regular vector Borel measures on the locally compact spaces X and Y, then there exists one and only one vector regular Borel measure ϱ on $X \times Y$ which extends the inductive tensor product $\mu \check{\otimes} \nu$ of μ and ν. This result is useful in the case whenever the Borel sets fail to „multiply", because in such a case, if μ and ν are Borel measures, the product $\mu \times \nu$, resp. $\mu \check{\otimes} \nu$, resp. $\mu \hat{\otimes} \nu$ as defined in [1], [13], [7], resp. [10], resp. [8], [9], may fail to be a Borel measure for want of sufficient domain.

1. Natations and preliminaries. Let X be a locally compact (Hausdorff) topological space. The class of Baire sets in X is the sigma ring generated by the compact G_δ' s, and is denoted $\mathscr{B}_0(X)$. The class of Borel sets in X is the sigma ring generated by the compact sets, and is denoted $\mathscr{B}(X)$. The class of weakly Borel sets in X is the sigma ring generated by the closed (or equivalently, open) sets, it is a sigma algebra, and is denoted $\mathscr{B}_w(X)$. This terminology is drawn from [1] and [13].

If S and T are any two sigma rings, the $S \times T$ denotes the sigma ring generated by all „rectangles" of the form $E \times F$, with „sides" in S and T, respectively [1, Theorem 35.2] or [13, p. 140].

For any pair of locally compact spaces X and Y, the following relations hold:

(1) $$\mathscr{B}_0(X) \times \mathscr{B}_0(Y) = \mathscr{B}_0(X \times Y) \ [1, \text{p. } 179 \text{ or } 13, \text{p. } 222]\,.$$

(2) $$\mathscr{B}(X) \times \mathscr{B}(Y) \subset \mathscr{B}(X \times Y) \ [1, \text{p. } 118]\,.$$

(3) $$\mathscr{B}_w(X) \times \mathscr{B}_w(Y) \subset \mathscr{B}_w(X \times Y) \ [1, \text{p. } 118]\,.$$

In general, the inclusion in (2) or (3) is proper (cf. [13, p. 261] and [14]).

For the rest of the paper, M and N denote locally convex spaces with a topology defined by a family $\{|\cdot|_p\}$ $p \in P$, $\{|\cdot|_q\}$, $q \in Q$ of seminorms, respectively; $\tilde{M}$ and $\tilde{N}$ denote their completion, respectively.

Let $\mathscr{R}(X)$ be a ring of subsets of X and $\mu : \mathscr{R}(X) \to M$ an additive set function. We say that μ is regular if for any $E \in \mathscr{R}(X)$ and for any $\varepsilon > 0$ there exists for every $p \in P$, a compact set C in $\mathscr{R}(X)$ and an open set U in $\mathscr{R}(X)$ with $C \subset E \subset U$ such that

$$|\mu(H)|_p < \varepsilon$$

for every H in $\mathscr{R}(X)$ with $H \subset U - C$ (cf. [3], [4], [5], [6] and [15]).

A vector Baire measure on X is a vector measure $\mu_0 : \mathscr{B}_0(X) \to M$. A vector Borel, resp. weakly Borel measure on X is a vector measure $\mu : \mathscr{B}(X) \to M$, resp. $\mathscr{B}_w(X) \to M$.

It is known that every vector Baire measure is regular ([15, Lemma 1] or [6]) and there exists a unique regular Borel, resp. regular weakly Borel measure $\mu : \mathscr{B}(X) \to \tilde{M}$, resp. $\mathscr{B}_w(X) \to \tilde{M}$ such that $\mu(E) = \mu_0(E)$ for $E \in \mathscr{B}_0(X)$.

Proposition. *If μ is a regular vector Borel measure extending a vector measure $\mu_0 : \mathscr{B}_0(X) \to M$, then μ takes its values in M.*

Proof. For every continuous linear functional $x' \in M'$ we have, for a suitable real number t,

$$x'\mu_0(B_0) \geqq t \text{ for every } B_0 \in \mathscr{B}_0(X) .$$

Since $x'\mu$ is a regular Borel scalar measure, for every Borel set B, there exists a Baire set $F_{x'}$ such that $x'\mu\,(B \triangle F_{x'}) = 0$ [1, Sect. 68]. Hence $x'\mu\,(B) = x'\mu\,(F_{x'}) = x'\mu_0\,(F_{x'}) \geqq t$ for every $B \in \mathscr{B}(X)$. Therefore every closed half-space in M containing $\{\mu_0(B_0) : B_0 \in \mathscr{B}_0(X)\}$ contains $\{\mu(B) : B \in \mathscr{B}(X)\}$.

2. Inductive tensor product of two regular vector Borel measures. The inductive tensor product of two vector Baire measures $\mu_0 : \mathscr{B}_0(X) \to M$ and $\nu_0 : \mathscr{B}_0(Y) \to N$ (according to [10]) is the unique vector measure $\mu_0 \check{\otimes} \nu_0$ with values in $M \check{\otimes} N$ (the inductive tensor product of M and N [18], cf. [12] or [17] where this is called the topology of bi-equicontinuous convergence) on the sigma ring $\mathscr{B}_0(X) \times \mathscr{B}_0(Y)$ such that

$$(\mu_0 \check{\otimes} \nu_0)\,(E \times F) = \mu_0(E) \check{\otimes} \nu_0(F)$$

for all Baire sets E in X and F in Y.

From the relation (1) we have at once

Theorem 1. *If μ_0 is a vector Baire measure on the locally compact space X with values in M and ν_0 is a vector Baire measure on the locally compact space Y with values in N, then the inductive tensor product vector measure $\mu_0 \check{\otimes} \nu_0$ is a Baire vector measure on the product topological space $X \times Y$ with values in $M \check{\otimes} N$.*

As to the inductive tensor product of the regular Borel vector measures μ and ν, it is according to [10] the unique vector measure $\mu \check{\otimes} \nu$ on the sigma ring $\mathscr{B}(X) \times \mathscr{B}(Y)$ with values in $M \check{\otimes} N$ such that

$$(\mu \check{\otimes} \nu)(E \times F) = \mu(E) \check{\otimes} \nu(F)$$

for all Borel sets E in X and F in Y.

Now if μ_0 and ν_0 are the Baire restrictions of μ and ν, respectively, then the inductive tensor product of μ_0 and ν_0, namely $\mu_0 \check{\otimes} \nu_0$, is according to Theorem 1 a Baire vector measure on $X \times Y$.

If Borel sets do not multiply, $\mu \check{\otimes} \nu$ fails to be a vector Borel measure, but there is a regular vector Borel measure ϱ on $X \times Y$, namely the unique extension of $\mu_0 \check{\otimes} \nu_0$ to a regular vector Borel measure; this always exists ([15, Lemma 1] or [6, Theorem 5]). We must prove that ϱ is an extension of $\mu \check{\otimes} \nu$. Proving this, we may use the procedure used in [14] and [2].

Theorem 2. *If $\mu : \mathscr{B}(X) \to M$ and $\nu : \mathscr{B}(Y) \to N$ are regular vector Borel measures on the locally compact spaces X and Y, respectively, then there exists one and only one regular vector Borel measure on $X \times Y$ with values in $M \check{\otimes} N$ which extends $\mu \check{\otimes} \nu$. This measure is simply the measure ϱ described above.*

It is useful to prove a slightly more general result which we shall need in the sequel (cf. [2] and [14]).

Theorem 3. *Let X and Y be locally compact spaces and suppose that τ is a vector measure on the sigma ring $\mathscr{B}(X) \times \mathscr{B}(Y)$ with values in M such that*

(i) *for each compact set C_1 in X, the correspondence*

$$E_2 \to \tau(C_1 \times E_2) \quad (E_2 \in \mathscr{B}(Y))$$

is a regular vector Borel measure on Y with values in M, and

(ii) *for each compact set C_2 in Y, the correspondence*

$$E_1 \to \tau(E_1 \times C_2) \quad (E_1 \in \mathscr{B}(X))$$

is a regular Borel vector measure on X with values in M. Then τ may be extended to one and only one regular Borel measure ϱ on $X \times Y$ with values in M.

Proof. The uniqueness of ϱ follows from the fact that the domain of definition of τ includes the Baire sets of $X \times Y$ (cf. formula (1) and [15, Lemma 1] and [6, Theorem 5]).

The restriction of τ to the class of Baire sets of $X \times Y$ is a Baire vector measure τ_0. Let ϱ be the unique regular Borel extension of τ_0 which exists according to [15] or [6].

Our problem is to show that

$$(*) \qquad \varrho(E) = \tau(E)$$

for all sets E in $\mathscr{B}(X) \times \mathscr{B}(Y)$.

It is sufficient to show that for every continuous functional x' in M'

$$(**) \qquad x'\varrho(E) = x'\tau(E)$$

for all sets E in $\mathscr{B}(X) \times \mathscr{B}(Y)$. Clearly we may suppose that $x'\varrho$ and $x'\tau$ are real-valued measures; $x'\varrho$ is a finite regular Borel measure, therefore the upper variation $(x'\varrho)^+$ and the lower variation $(x'\varrho)^-$ of $x'\varrho$ are finite non-negative regular Borel measures (cf. [5, § 15, Prop. 23] or [11, III. 5. 12]).

By (i) for each compact set C_1 in X, the correspondences

$$E_2 \to (x'\tau)^+(C_1 \times E_2)\,, \quad E_2 \to (x'\tau)^-(C_1 \times E_2)\,,$$

$(E_2 \in \mathscr{B}(Y))$ are regular Borel measures on Y; similarly, by (ii) for each compact set C_2 in Y, the correspondences

$$E_1 \to (x'\tau)^+(E_1 \times C_2), \quad E_1 \to (x'\tau)^-(E_1 \times C_2)\,,$$

$(E_1 \in \mathscr{B}(X))$, are regular Borel measures on X. Now ϱ and τ agree on the Baire sets of $X \times Y$ both being extensions of τ_0, hence we may pose $(x'\tau)^+ = (x'\varrho)^+$, $(x'\tau)^- = (x'\varrho)^-$ for Baire sets in $X \times Y$. Since Theorem 3 holds for non-negative measures (cf. [2, Theorem 3]), we have $(x'\tau)^+ = (x'\varrho)^+$, $(x'\tau)^- = (x'\varrho)^-$ for all Borel sets in $X \times Y$, therefore

$$x'\varrho(E) = x'\tau(E)$$

for all sets E in $\mathscr{B}(X) \times \mathscr{B}(Y)$.

Theorem 3 is proved.

Proof of Theorem 2. We apply Theorem 3 to the product vector measure $\tau = \mu \check{\otimes} \nu$; conditions (i) and (ii) are verified using the fact that

$$\tau(E_1 \times E_2) = \mu(E_1) \check{\otimes} \nu(E_2), \quad |\tau(E_1 \times E_2)|^{\vee}_{p,q} = |\mu(E_1)|_p\, |\nu(E_2)|_q$$

for all rectangles with Borel sides.

The next theorem shows that if $\mu \check{\otimes} \nu$ is nonzero, then no regular Borel extension of $\mu \check{\otimes} \nu$ is possible, unless μ and ν are both regular (cf. [14]).

Theorem 4. *If there exists a nonzero regular vector Borel measure $\varrho : \mathscr{B}(X \times Y) \to M \check{\otimes} N$ which extends $\mu \check{\otimes} \nu$, then both μ and ν are regular.*

Proof. (Cf. [14]). If ϱ is a regular Borel extension of $\mu \check{\otimes} \nu$, it is a regular extension of the Baire measure $\mu_0 \check{\otimes} \nu_0$. It follows from Theorem 2 that ϱ extends $\mu' \check{\otimes} \nu'$, where μ' and ν' are the regular Borel extensions of μ_0 and ν_0, respectively. Hence $\mu' \check{\otimes} \nu' = \mu \check{\otimes} \nu$, and thus

$$\mu'(E_1) \check{\otimes} \nu'(E_2) = \mu(E_1) \check{\otimes} \nu(E_2)$$

for all Borel sets E_1 in X and E_2 in Y. Since ϱ, and hence $\mu \check{\otimes} \nu$, are nonzero, it follows that $\mu = \mu'$ and $\nu = \nu'$.

From Theorem 2 we have

Theorem 5. *Let M be nuclear* ([12, II. 2.1] or [17, III. 7.1]). *If $\mu : \mathscr{B}(X) \to M$ and $\nu : \mathscr{B}(Y) \to N$ are regular vector Borel measures on the locally compact spaces X and Y, respectively, then there exists one and only one regular vector Borel measure on $X \times Y$ with values in $M \hat{\otimes} N$ ($M \hat{\otimes} N$ denotes the projective tensor product, cf.* [12] or [17]), *which extends $\mu \hat{\otimes} \nu$. This measure is the measure ϱ defined as in Theorem* 2.

From Theorem 4 we have

Theorem 6. *Let M be nuclear. If $\varrho : \mathscr{B}(X \times Y) \to M \hat{\otimes} N$ is a nonzero regular vector Borel measure on $X \times Y$ which extends $\mu \hat{\otimes} \nu$, then both μ and ν are regular.*

Proof of Theorems 5 and 6. If M is nuclear then $M \hat{\otimes} N$ and $M \check{\otimes} N$ are topological isomorphic [17, IV. 9.4, Corollary 2].

A bilinear mapping $U : M \times N \to L$, where L is a locally convex space, is said to be hypercontinuous, if the linear mapping induced by it on $M \otimes N$ is continuous under the inductive tensor topology (cf. [18]).

Theorem 2 gives the following

Theorem 7. *Let $U : M \times N \to L$ be a hypercontinuous bilinear mapping and L be a (sequentially) complete space. If $\mu : \mathscr{B}(X) \to M$ and $\nu : \mathscr{B}(Y) \to N$ are regular vector Borel measures on the locally compact spaces X and Y, respectively, then there exists one and only one regular Borel vector measure ϱ on $X \times Y$ with values in L for which*

$$\varrho(E_1 \times E_2) = U(\mu(E_1), \nu(E_2)), \quad E_1 \in \mathscr{B}(X),\ E_2 \in \mathscr{B}(Y) .$$

Proof. If $\bar{\varrho}$ is the unique regular Borel vector measure on $X \times Y$ with values in $M \check{\otimes} N$ which extends $\mu \check{\otimes} \nu$ from Theorem 2 and $\overline{U}$ is a mapping induced by U, we define a set function $\varrho : \mathscr{B}(X) \times \mathscr{B}(Y) \to L$ as follows:

$$\varrho(G) = \overline{U}(\bar{\varrho}(G)) , \quad G \in \mathscr{B}(X) \times \mathscr{B}(Y) ;$$

ϱ is a regular vector Borel measure on $X \times Y$ with values in L [11, IV. 10.8].

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