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GROUPS OF ORDER WHICH IS INDIVISIBLE BY A FIXED PRIME

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Many of the results obtained by Käte Fenchel [2] in the investigation of groups of odd order hold with some restrictions for infinite groups, too. We consider now the possibility of this generalization.

In the structure of groups whose order is odd, that is, indivisible by 2, the prime 2 plays a major role. It can be shown, e. g. that the order of a finite group G is odd if and only if there exists for any $a \in G$ one and only one $x \in G$ such that $a = x^2$. In view of the generalization we consider instead of groups of odd order the more general concept of groups whose order cannot be divided by an arbitrarily fixed prime number.

As far as possible the groups are denoted by capitals, the elements by small Roman, while the real numbers by small Greek letters. The following symbols are used:

e is the unit element of a group;
π is the arbitrary fixed prime;
$[G]$ is the order of group G;
$[G : H]$ is the index of subgroup H in G;
$\{a\}$ is the cyclic subgroup generated by the element a;
$\omega(a)$ is the order of element a;
(a) is the class of elements conjugate with a;
$a^x = x^{-1}ax$;
Z_a is the centralizer of element a;
$Z(H)$ is the center of subgroup H;
$+$ denotes the union of disjoint sets;
$\varkappa_{(a)}$ is the number of elements in the class (a);

Theorem 0.1. *If in the product $(a_1)(a_2)\dots(a_\nu)$ of the classes $(a_1), (a_2), \dots, (a_\nu)$ of group G an element $b \in G$ is contained ϱ-times, then for any $x \in G$ the element b^x is also contained ϱ-times.*

Proof. Let

$$
\begin{aligned}
&a_1^{x_{11}} \dots a_\nu^{x_{1\nu}} = b\,,\\
&a_1^{x_{21}} \dots a_\nu^{x_{2\nu}} = b\,,\\
&\vdots\\
&a_1^{x_{\varrho 1}} \dots a_\nu^{x_{\varrho\nu}} = b
\end{aligned}
$$

be all those products which yield b in the product $(a_1) \dots (a_\nu)$. It follows that the products

$$
\begin{aligned}
&a_1^{x_{11}x} \dots a_\nu^{x_{1\nu}x} = b^x,\\
&\vdots\\
&a_1^{x_{\varrho 1}x} \dots a_\nu^{x_{\varrho\nu}x} = b^x
\end{aligned}
$$

represent every possible production of b^x.

Using the notation $\varkappa_{(a_1)(a_2)\dots(a_\nu)(b)}$ for the number ϱ in Theorem 0.1. we write

$$\text{card}\,[(a_1)\,(a_2)\dots(a_\nu)] = \sum_{(b)} \varkappa_{(a_1)(a_2)\dots(a_\nu)(b)}\,.$$

(For the case $\nu = 2$ see [1, p. 58.]) If $(a_1) = (a_2) = \dots (a_\nu) = (a)$, then

$$\varkappa_{(a_1)\ a\ )\dots(a_\nu)(b)} = \varkappa_{(a)^\nu(b)}\,.$$

1. Fundamental properties

Theorem 1.1. *The order of a finite group G is indivisible by π if and only if the group has no element of order divisible by π.*

Proof. If $[G]$ cannot be divided by π and $a \in G$, then by Lagrange's theorem $[\{a\}] = \omega(a)$ is also indivisible by π.

If no element of G is of order divisible by π, then the divisibility of $[G]$ by π is in contradiction with Cauchy's theorem.

Remark 1.1. The concept of a group whose order is indivisible by π can be generalized by Theorem 1.1. for those torsion groups in which (though they may be infinite themselves) the order of every element is a finite natural number, relatively prime to π.

Theorem 1.2. *Let G be a torsion group. The following properties are equivalent:*

(a) *G has no element of order which is divisible by π.*

(b) *There exists for any $a \in G$ one and only one $x \in G$ such that $x^\pi = a$.*

(c) *For any fixed $x \in G$, any $h \in G$ can be uniquely produced in the form $h = axax \dots xa$ where $a \in G$, and a is contained π-times, while x occurs $(\pi - 1)$-times in the product.*

(d) *If any $g \in G$ is fixed, there exists for any element $x \in G$ one and only one $a \in G$ such that $g = a \,.\, a^x \,.\dots.\, a^{(x^{\pi-1})}$.*

Proof. The proof is given cyclically, as (a) → (b) → (c) → (d) → (a).

1. Assume that no element of the torsion group G has an order divisible by π. Let a be an arbitrary element of G and of order $\omega(a) = \alpha$. Since $(\alpha, \pi) = 1$, there exists a natural number ξ such that

$$\xi\pi \equiv 1 \pmod{\alpha} ,$$

that is

(1) $$\xi\pi = \alpha\nu + 1 ,$$

where ν is a positive integer. Let

(2) $$x = a^{\xi}.$$

Then, by (1) and (2) we have

$$x^{\pi} = a^{\xi\pi} = (a^{\alpha})^{\nu} . a = a .$$

Suppose also that

(3) $$y^{\pi} = a ,$$

that is

(4) $$x^{\pi} = y^{\pi}.$$

Using (2) and (3),

$$x \in Z(Z_a)$$
$$a \in Z_y ; \quad y \in Z_a ; \quad y^{-1} \in Z_a ,$$

that is

(5) $$xy^{-1} = y^{-1}x .$$

Using (4) and (5), we find

(6) $$e = y^{\pi}y^{-\pi} = x^{\pi}y^{-\pi} = (xy^{-1})^{\pi}.$$

Let $\omega(xy^{-1}) = \mu$. Because of (6) $\mu|\pi$, but it was postulated that $\mu \nmid \pi$, thus $\mu = 1$, that is

$$xy^{-1} = e ; \quad x = y ,$$

which completely proves the assertion (a) $\to$ (b).

2. Suppose that for any element $a \in G$ of the torsion group G there exists one and only one $x \in G$ such that $x^{\pi} = a$. Consider the elements $x \in G$ and $h \in G$ with $hx = g$. By the postulate, the equation

$$y^{\pi} = g$$

can be solved for y; while the equation

$$y = ax$$

has a solution for a owing to the group axioms. On resubstitution

$$hx = axax \dots xax\,,$$
$$h = axax \dots xa\,,$$

where a is contained π-times on the right side of either equation, while x occurs π-times and $(\pi - 1)$-times respectively. If, in addition,

$$h = bxbx \dots xb$$

where b occurs π-times and x occurs $(\pi - 1)$-times in the product, then

$$hx = (ax)^\pi = (bx)^\pi.$$

Thus, owing to the postulate and to the group axioms,

$$a = b$$

which proves that (b) $\to$ (c).

3. Suppose that for any $x \in G$ and $h \in G$ in the torsion group G there exists uniquely an element $a \in G$ such that $h = ax \dots xa$ where a is contained π-times and x occurs $(\pi - 1)$-times in the product. Let the element $g \in G$ be fixed and consider an arbitrary $x \in G$. The equations

$$gx^{-(\pi-1)} = a \,.\, x^{-1} \,.\, a \,.\, x^{-1} \,.\,\dots\,.\, x^{-1} \,.\, a$$

(a occurs π-times) and

$$g = a \,.\, a^x \,.\, a^{(x^2)} \,.\,\dots\,.\, a^{(x^{\pi-1})}$$

are obviously equivalent and the former can be solved uniquely for a. Thus (c) $\to$ (d).

4. Suppose that for every fixed $g \in G$ of the torsion group G there exists for any $x \in G$ one and only one $a \in G$ such that $g = a \,.\, a^x \,.\,\dots\,.\, a^{(x^{\pi-1})}$ while the property (a) does not hold for G, that is, there exists an element $h \in G$ such that $\omega(h) = \pi\sigma$, where σ is a natural number. Then by Cauchy's theorem there exists an element of order π of subgroup $\{h\}$, that is, there exists some a $\in G$ such that $a \neq e$, $a^\pi = e$. In this case, however, the simultaneous validity of the equations

$$e = e \,.\, e^e \,.\, e^{(e^2)} \,.\,\dots\,.\, e^{(e^{\pi-1})}$$

and

$$e = a \,.\, a^e \,.\, a^{(e^2)} \,.\,\dots\,.\, a^{(e^{\pi-1})}$$

is contrary to what we supposed, that is (d) $\to$ (a) and by this the theorem is proved.

Remark 1.2. If G is not assumed to be a torsion group, the property (a) does not follow from the properties (b), (c) or (d) (though (b), (c), (d), as it can be readily checked, are equivalent). For example, the additive group of rational numbers has the properties (b), (c) and (d), but does not possess the property (a).

Theorem 1.3. *If the order of any element of G is relatively prime to π, then for any $q \in G$ and $q^\pi = g$ it follows that $Z_q = Z_g$.*

Proof. Let x be an arbitrary element of Z_q. Hence $xq = qx$; which, if used successively π-times, gives

$$xg = xq^\pi = q^\pi x = gx$$

that is $x \in Z_q$, thus

$$Z_q \subseteq Z_g \,. \tag{7}$$

Let now y be an arbitrary element of Z_g. Then $yg = gy$, by which

$$(y^{-1}qy)^\pi = y^{-1}gy = y^{-1}yg = q^\pi;$$

thus, because of the postulate and by Theorem 1.2.

$$y^{-1}qy = q$$

that is, $y \in Z_q$ and

$$Z_g \subseteq Z_q \tag{8}$$

hence, the theorem follows from (7) and (8).

Remark 1.3. The reversal of Theorem 1.3. does not hold, e. g. as a counter-example consider the commutative groups of order divisible by π.

2. The special case of $\pi = 2$

Considering now the case of $\pi = 2$, let us see how the theorem 2. and the relation (3) stated in [2] will be satisfied for torsion groups. We shall investigate also the possibility of generalizing the problems under consideration to any π.

Theorem 2.1. *Let G be a torsion group. The order of every element of G is odd if and only if for any $a \in G$, $a \neq e$ we have $(a) \neq (a^{-1})$.*

Proof. Suppose that an arbitrary element of group G is of odd order, yet there exists some $a \in G$ such that $(a^{-1}) = (a)$, that is, there exists some $q \in G$ such that

$$q^{-1}aq = a^{-1} \tag{9}$$

$$aq = qa^{-1}. \tag{10}$$

Then, by (9)

$$a = (a^{-1})^{-1} = q^{-1}a^{-1}q$$
$$a^{-1} = qaq^{-1}.$$

Hence, again by (9), with the notation $q^2 = g$

$$a \in Z_g .$$

By Theorem 1.3. then

$$aq = qa \tag{11}$$

and from (10) and (11)

$$qa^{-1} = qa$$
$$a^2 = e .$$

Since the order of every element of G is odd, $a = e$.

Suppose further that for any element $a \in G$, $a \neq e$ we have $(a^{-1}) \neq (a)$. We have to prove that G has no element of even order. Suppose therefore that $v \in G$ and $\omega(v) = 2\nu$, where ν is a positive integer. Then

$$g = v^\nu \neq e$$

and

$$g^2 = e ,$$
$$g^{-1} = g$$

that is $(g^{-1}) = (g)$, which is a contradiction.

Remark 2.1. It must be pointed out in connection with Theorem 2.1. that the order of every element is assumed to be finite. In fact, it is not true that a group which has an element of infinite order (when, of course, the order of every element cannot be odd) must always contain also other ambivalent class than that of the unit element. Consider, e. g., the infinite cyclic group $G = \{g\}$. If it contained also another than the trivial ambivalent class of elements, there would exist two positive integers η and ϱ satisfying the equation $g^{-\eta} g^{\varrho} g^{\eta} = g^{-\varrho}$. In this case

$$g^{2\varrho} = g^{\varrho} . g^{\varrho} = g^{\varrho} . g^{-\eta+\varrho+\eta} = g^{\varrho} . (g^{-\eta} . g^{\varrho} . g^{\eta}) = g^{\varrho} . g^{-\varrho} = e ,$$

contrary to the infinite order of g.

Theorem 2.2. *Let the order of every element of G be odd. Then each of the elements $g \in G$ determines a system of cosets of centralizers of certain (g-dependent) elements. This system of cosets covers the group G precisely once.*

Proof. Let $g \in G$ be fixed and consider the equation

$$g = a . a^x \tag{12}$$

which by Theorem 1.2. determines for every $x \in G$ precisely one a.

By the known and easily proved theorem $a^x = a^y$, that is $a . a^x = a . a^y$ if and only if $y \in Z_a x$. This means that a solution to (12) for a can be mapped onto that coset of Z_a whose elements produce this element a.

The thus obtained cosets of centralizers cover the group G since by Theorem 1.2. every $x \in G$ determines an a. These cosets are disjoint since, if $a \neq b$, and we postulate that the cosets $Z_a x$ and $Z_b y$ are involved in the covering, it would mean that for $z \in Z_a x \cap Z_b y$ both a and b are determined by z and this would be contrary to Theorem 1.2. (K. Fenchel's proof in this respect is, in fact, superfluous, since the disjointness follows immediately from the preceding assertions.)

Thus, we have obtained a partition of the group G with respect to the element g, which can be written in the form

$$G = Z_g + Z_a x_a + \ldots + Z_s x_s \tag{13}$$

Theorem 2.3. *Let $G = \sum_{\nu \in \Gamma} A_\nu g_\nu$ be a partition of group G where Γ is an arbitrary set of indices, $g_\nu \in G$ and A_ν is a subgroup of G for which $[G : A_\nu] = \alpha_\nu$ is finite. Then*

$$\sum_{\nu \in \Gamma} \frac{1}{\alpha_\nu} \leqq 1 \, .$$

(By definition $\sum_{\nu \in \Gamma} \beta_\nu = \sup \sum_{\nu \in \Gamma_1} \beta_\nu$, where Γ_1 runs over all finite, non-empty subsets of Γ.)

Proof. Let

$$A_1 \cap A_2 \cap \ldots \cap A_\mu = D_\mu$$

By Poincaré's theorem also $[G : D_\mu]$ is finite. Decompose G into cosets of D_μ such that the set $\sum_{\substack{\nu \in \Gamma \\ \nu \neq 1, 2, \ldots, \mu}} A_\nu g_\nu$ decomposes to ϱ_μ cosets and thus

$$[G : D_\mu] = \varrho_\mu + \sum_{\iota=1}^{\mu} [A_\iota : D_\mu] \, .$$

If both sides are divided by the finite $[G : D_\mu]$, then

$$1 = \frac{\varrho_\mu}{[G : D_\mu]} + \sum_{\iota=1}^{\mu} \frac{1}{\alpha_\iota} \geqq \sum_{\iota=1}^{\mu} \frac{1}{\alpha_\iota},$$

hence

$$\sum_{\nu \in \Gamma} \frac{1}{\alpha_\nu} \leqq 1 \, .$$

Remark 2.2. Theorem 2.3. cannot be stated more explicitly since, in general, the relation $\sum_{\nu \in \Gamma} \frac{1}{\alpha_\nu} = 1$ is not true. As a counter-example consider the additive group of integers. Let $A_\nu = \{3^\nu\}$ ($\nu = 1, 2, \ldots$). The cosets $A_\nu + g_\nu$ covering the group G are formed by choosing for g_ν the smallest of the non--negative numbers not covered by the set $\sum_{\iota=1}^{\nu-1} (A_\iota + g_\iota)$. ($g_1 = 0$). Since such a number always exists, the sequence of cosets as well as the sequence of indices $\alpha_\nu = 3^\nu$ are infinite. On the other hand, $\sum_{\nu=1}^{\infty} (A_\nu + g_\nu)$ is a partition of the group for which

$$\sum_{\nu=1}^{\infty} \frac{1}{\alpha_\nu} = \sum_{\nu=1}^{\infty} \frac{1}{3^\nu} = \frac{1}{2}.$$

Theorem 2.4. *Every element of the locally normal group G will be of odd order if and only if for any class (g)*

$$\sum_{(a)} \frac{\varkappa_{(a)(a)(g)}}{\varkappa_{(a)}} \leqq 1 . \tag{14}$$

Proof. If the order of every element of the locally normal group G is odd then by Theorems 2.2. and 2.3., since $[G : Z_a] = \varkappa_{(a)}$,

$$\sum_{a}{}' \frac{1}{\varkappa_{(a)}} \leqq 1 , \tag{15}$$

where the summation is over the elements whose centralizer is involved in the partition (13), that is, those elements which for fixed g are solutions to equation (12).

Keepeng g fixed, let the class (a) be also fixed. By definition, there exist $\varkappa_{(a)(a)(g)}$ products of the form $a_1 a_2$ such that $a_1 \in (a)$ (i. e. of the form $a_1 = a^x$), $a_2 \in (a)$ and $a_1 a_2 = g$. It follows that if the class (a) is fixed, then there exist exactly $\varkappa_{(a)(a)(g)}$ elements a which yield a solution to (12) for x. Thus

$$\sum_{a}{}' \frac{1}{\varkappa_{(a)}} = \sum_{(a)} \frac{\varkappa_{(a)(a)(g)}}{\varkappa_{(a)}} \tag{16}$$

and (14) follows from (15) and (16).

Suppose now that (14) is satisfied for any class (g) of the locally normal group G, yet G has an element of even order. Then, by Theorem 2.1., G must have a class (b) satisfying the condition that $(b) \neq (e)$, $(b^{-1}) = (b)$. Consider the case of $(g) = (e)$, then

$$\sum_{(a)} \frac{\varkappa_{(a)(a)(e)}}{\varkappa_{(a)}} = \frac{\varkappa_{(e)(e)(e)}}{\varkappa_{(e)}} + \frac{\varkappa_{(b)(b)(e)}}{\varkappa_{(b)}} + \sum_{\substack{(a) \\ (a)\neq(e) \\ (a)\neq(b)}} \frac{\varkappa_{(a)(a)(e)}}{\varkappa_{(a)}},$$

that is

$$\sum_{(a)} \frac{\varkappa_{(a)(a)(e)}}{\varkappa_{(a)}} \geqq \frac{\varkappa_{(e)(e)(e)}}{\varkappa_{(e)}} + \frac{\varkappa_{(b)(b)(e)}}{\varkappa_{(b)}},$$

where the notations are chosen such that $\varkappa_{(e)(e)(e)} = 1$, $\varkappa_{(e)} = 1$, $\varkappa_{(b)(b)(e)} > 0$, $\varkappa_{(b)} > 0$ and, since G is locally normal, $\varkappa_{(b)}$ is finite. Thus

$$\sum_{(a)} \frac{\varkappa_{(a)(a)(e)}}{\varkappa_{(a)}} > 1\,,$$

contrary to what we supposed.

Remark 2.3. The next task would be to generalize the theorems of Section 2. for any prime number π. This applies specifically to Theorem 2.4., which has been introduced by the preceding assertions. It can be seen that the generalization of this theorem would read as follows: the order of every element of a locally normal group G is relatively prime to π if and only if for any class (g)

$$\sum_{(a)} \frac{\varkappa_{(a)^{\pi}(g)}}{\varkappa_{(a)}} \leqq 1\,.$$

As a counter-example consider the dihedral group D_3, where for different classes (g) and $\pi = 5$ the above sum becomes 6, 5,5 and 27.

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