Tibor Šalát

Remarks on the Ergodic Theory of the Continued Fractions

Matematický časopis, Vol. 17 (1967), No. 2, 121--130

Persistent URL: `http://dml.cz/dmlcz/126697`

Matematický časopis 17 (1967), No. 2

REMARKS ON THE ERGODIC THEORY OF THE CONTINUED FRACTIONS

TIBOR ŠALÁT, Bratislava

The applications of the ergodic theory to the metric theory of the continued fractions are based on the following theorem of C. Ryll—Nardzewski.

Theorem I. *For each* $x \in (0, 1)$ *let* $\delta(x) = \frac{1}{x} - \left[\frac{1}{x}\right]$ ($[u]$ *denotes the integral part of the number* $[u]$). *Let* f *be a Lebesgue integrable function on the interval* (0, 1). *Then for almost all* $x \in (0, 1)$ *(in the sense of the Lebesgue measure) the following holds*:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\delta^i(x)) = \frac{1}{\log 2} \int_0^1 \frac{f(t)}{1+t} \, \mathrm{d}t.\text{[1]}$$

(See [1]).

Several applications of the above theorem to the metric theory of the continued fractions may be found in [1] and also in [2]. In [2] it is proved by means of Theorem I — the result, which will be used in what follows.

Theorem II. *If* f *is a measurable, non-negative and non-integrable function on* (0, 1), *then for almost all* $x \in (0, 1)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\delta^i(x)) = +\infty.$$

We shall study in this paper from the metric point of view the behaviour of the sequences

$$\left\{\frac{c_k^\alpha(x)}{c_{k+1}^\alpha(x)}\right\}_{k=1}^{\infty}; \quad \alpha \text{ real}; \quad \{|c_k(x) - c_{k+1}(x)|^\alpha\}_{k=1}^{\infty}, \quad \alpha \geqq 0$$

[1] $\delta^2(x) = \delta(\delta(x))$, $\delta^3(x) = (\delta(\delta^2 x))$, ...

(the notation see in what follows) and we shall give a new proof of a certain well-known result of the metric theory of continued fractions (see Theorem 1).

DEFINITIONS AND NOTATIONS

1. The expansion of the number $x \in (0, 1)$ into the continued fraction (the continued fraction of the number x) will be denoted in this paper as follows

$$x = \cfrac{1}{c_1(x) + \cfrac{1}{c_2(x) + \ddots + \cfrac{1}{c_k(x) + \ddots}}}, \tag{1}$$

$c_k(x)$ $(k = 1, 2, 3, \ldots)$ are natural numbers (so called quotients of the continued fraction of x). If the above expansion is finite and $c_k(x)$ is the last quotient of the continued fraction of x, then $c_k(x) > 1$. Further if (1) has more than one quotient (or in other words if $x \neq 1/p$, $p = 2, 3, 4, \ldots$), then evidently

$$\delta(x) = \cfrac{1}{c_2(x) + \cfrac{1}{c_3(x) + \ddots + \cfrac{1}{c_k(x) + \ddots}}}$$

(as to the meaning of $\delta(x)$ see Theorem I).

2. If A is (any) set of natural numbers, we put for a natural n $A(n) = \sum_{a \leqq n, a \in A} 1$. The number $h(A) = \lim_{n\to\infty} \frac{A(n)}{n}$, if it exists, is called the asymptotic density of the set A.

3. The sequence of numbers $\{a_n\}_{n=1}^{\infty}$ is said to be summable by the method $(C, 1)$ to the number $a \in (-\infty, +\infty)$ if $\lim_{n\to\infty} \frac{a_1 + a_2 + \ldots + a_n}{n} = a$. If the limit of the sequence $\left\{\frac{a_1 + a_2 + \ldots + a_n}{n}\right\}_{n=1}^{\infty}$ is improper or if it does not exist, then we say that $\{a_n\}_{n=1}^{\infty}$ is not summable by the method $(C, 1)$. The sequence of functions $\{g_n\}_{n=1}^{\infty}$ is said to be almost everywhere in $(0, 1)$ (for almost all $x \in (0, 1)$) summable by the method $(C, 1)$, if there exists a set $M \subset (0, 1)$ of measure 1 such that for each $x \in M$ the sequence $\{g_n(x)\}_{n=1}^{\infty}$

is summable by the method $(C, 1)$ to any finite number $s = s(x)$. The sequence $\{g_n\}_{n=1}^{\infty}$ is said to be almost everywhere in $(0, 1)$ (for almost all $x \in (0, 1)$) non-summable by the method $(C, 1)$ if there exists a set $P \subset (0, 1)$ of measure 1 such that for each $x \in P$ the sequence $\{g_n(x)\}_{n=1}^{\infty}$ is non-summable by the method $(C, 1)$.

By means of Theorem I we shall easily prove the following result of A. Chinčin (see [4]) which we shall use.

Lemma 1. *Let $\alpha < 1$. Then for almost all $x \in (0, 1)$ the following holds: The sequence $\{c_k^{\alpha}(x)\}_{k=1}^{\infty}$ is summable by the method $(C, 1)$ to the number (which does not depend on x):*

$$\frac{1}{\log 2} \int_0^1 \frac{c_1^{\alpha}(t)}{1 + t}\, dt = \frac{1}{\log 2} \sum_{p=1}^{\infty} p^{\alpha} \log \frac{(p + 1)^2}{p(p + 2)} .$$

Remark 1. It is proved in [2] that for almost all $x \in (0, 1)$ the sequence $\{c_k(x)\}_{k=1}^{\infty}$ is not summable by the method $(C, 1)$.

The proof of the lemma. Let $\alpha < 1$. Put in Theorem I $f(t) = c_1^{\alpha}(t) > 0$. Since $c_1(t) = \left[\frac{1}{t}\right]$, the above defined function has in the interval $(0, 1)$ at most a countable number of points of discontinuity (in the case of $\alpha \neq 0$ these points of discontinuity are of the foım $1/p$, $p = 2, 3, 4, \ldots$). From the fundamental properties of the Lebesgue integral we have

$$\int_0^1 f(t)\, dt = \int_0^1 c_1^{\alpha}(t)\, dt = \sum_{p=1}^{\infty} \int_{\frac{1}{p+1}}^{\frac{1}{p}} c_1^{\alpha}(t)\, dt.$$

For $\frac{1}{p + 1} < t < \frac{1}{p}$ $c_1(t) = \left[\frac{1}{t}\right] = p$ holds, so we have

$$\int_0^1 f(t)\, dt = \sum_{p=1}^{\infty} \frac{p^{\alpha}}{p(p + 1)} < +\infty.$$

Thus f is integrable on $(0, 1)$ and in view of Theorem I for almost all $x \in (0, 1)$ the following holds:

$$\lim_{n\to\infty} \frac{1}{n} (c_1^{\alpha}(x) + \ldots + c_n^{\alpha}(x)) = \frac{1}{\log 2} \int_0^1 \frac{c_1{}^{\alpha}(t)}{1 + t}\, dt = \frac{1}{\log 2} \sum_{p=1}^{\infty} \int_{\frac{1}{p+1}}^{\frac{1}{p}} \frac{c_1^{\alpha}(t)}{1 + t}\, dt =$$

$$= \frac{1}{\log 2} \sum_{p=1}^{\infty} p^{\alpha} \log \frac{(p+1)^2}{p(p+2)}.$$

Lemma 2. *Let* $a_n \geqq 0$, $t_n \to +\infty$, $\sup_n \frac{\sum_{k=1}^{n} a_k}{n} < +\infty$. *Put* $A = \{n;\ a_n \geqq t_n\}$. *Then* $h(A) = 0$.

Proof. A can be supposed to be infinite. Put $V_n = \sum t_k$, where the summation is taken over all $k \in A$ for which $k \leqq n$. The last sum has $A(n)$ summands, from which fact it easily follows that $\frac{V_n}{A(n)} \to +\infty$. If we put $s_n = \sum_{k=1}^{n} a_k$, then clearly $s_n \geqq V_n$ and consequently $\frac{A(n)}{n} \leqq \frac{A(n)}{V_n} \frac{s_n}{n} \to 0$.

Now we shall give a new proof of the following result belonging to the fundamental results of the metric theory of the continued fractions. The original proof of this result is based on Lévy's well—known theorem on the frequency of quotients in the continued fraction expansions of the numbers $x \in (0, 1)$ (see [2]). According to Lévy's theorem, for almost all $x \in (0, 1)$ the following holds: each of the numbers p ($p = 1, 2, 3, \ldots$) appears in the sequence $\{c_k(x)\}_{k=1}^{\infty}$ with the frequency $\frac{1}{\log 2} \log \frac{(p+1)^2}{p(p+2)}$ (see [3] p. 110). Note that the frequency of the number p in the sequence $\{c_k(x)\}_{k=1}^{\infty}$ means the asymptotic density of the set of all such k for which $c_k(x) = p$.

The proof of the following theorem is based on Lemma 2. We shall illustrate the usefulness of Lemma 2 also in the proofs of Theorems 3, 5. But note that these theorems follow also easily from Theorem 1.

Theorem 1. *Let* $\tau_n \to +\infty$. *Then for almost all* $x \in (0, 1)$ $h(\{n;\ c_n(x) \geqq \tau_n\}) = 0$ *holds.*

Proof. We can already suppose that $\tau_n \geqq 0$ ($n = 1, 2, 3, \ldots$). Put $t_n = \sqrt{\tau_n}$ ($n = 1, 2, 3, \ldots$) and further let $g_n(x) = \sqrt{c_n(x)}$ for all irrational x, $x \in (0, 1)$. In view of Lemma 1, the sequence of functions $\{g_n\}_{n=1}^{\infty}$ is almost everywhere summable by the method $(C, 1)$. There exists a set $M \subset (0, 1)$ of measure 1 such that for $x \in M$ $\sup_n \frac{\sum_{k=1}^{n} g_k(x)}{n} < +\infty$. From Lemma 2 it follows that for $x \in M$ $h(\{n;\ g_n(x) \geqq t_n\}) = 0$ holds, consequently $h(\{n;\ c_n(x) \geqq \tau_n\}) = 0$.

In [2] S. Hartman studies the question of summability (by the method $(C, 1)$) of the sequences

$$\left\{\frac{c_k(x)}{c_{k+1}(x)}\right\}_{k=1}^{\infty}, \quad \left\{\frac{c_{k+1}(x)}{c_k(x)}\right\}_{k=1}^{\infty}$$

defined for each irrational $x \in (0, 1)$. He shows by means of Theorem II that for almost all $x \in (0, 1)$ the above mentioned sequences are non-summable by the method $(C, 1)$. In what follows an analogical question concerning the sequences

$$\left\{\frac{c_k^\alpha(x)}{c_{k+1}^\alpha(x)}\right\}_{k=1}^{\infty} \quad (\alpha \text{ real number})$$

will be solved and a result (see Theorem 3) similar to the one in Theorem 1 will be proved.

Theorem 2. *If* $|\alpha| < 1$ *then for almost all* $x \in (0, 1)$ *the following holds: The sequence* $\left\{\frac{c_k^\alpha(x)}{c_{k+1}^\alpha(x)}\right\}_{k=1}^{\infty}$ *is summable by the method* $(C, 1)$ *to the number (which does not depend on* x*):*

$$\frac{1}{\log 2}\int_0^1 \frac{\left[\frac{1}{t}\right]^\alpha \left[\left(\frac{1}{t} - \left[\frac{1}{t}\right]\right)^{-1}\right]^{-\alpha}}{1+t}\, \mathrm{d}t.$$

If $|\alpha| \geqq 1$, *then for almost all* $x \in (0, 1)$

$$\lim_{n\to\infty} \frac{1}{n}\left(\frac{c_1^\alpha(x)}{c_2^\alpha(x)} + \ldots + \frac{c_n^\alpha(x)}{c_{n+1}^\alpha(x)}\right) = +\infty$$

holds.

Proof. For t irrational, $t \in (0, 1)$ let $\psi(t) = \frac{c_1(t)}{c_2(t)} > 0$. It follows from the construction of the continued fractions that

$$c_1(t) = \left[\frac{1}{t}\right], \quad c_2(t) = \left[\left(\frac{1}{t} - \left[\frac{1}{t}\right]\right)^{-1}\right],$$

hence if α is real and t irrational, $t \in (0, 1)$, we have

$$\psi^\alpha(t) = \left[\left(\frac{1}{t} - \left[\frac{1}{t}\right]\right)^{-1}\right]^{-\alpha}\left[\frac{1}{t}\right]^\alpha.$$

The function ψ^α is evidently measurable. Let us examine $\int_0^1 \psi^\alpha(t)\, \mathrm{d}t$. We get

$$\int_0^1 \psi^\alpha(t)\,dt = \sum_{p=1}^{\infty} I_p, \quad I_p = \int_{\frac{1}{p+1}}^{\frac{1}{p}} \psi^\alpha(t)\,dt.$$

Further, if t is irrational, $t \in \left(\frac{1}{p+1}, \frac{1}{p}\right)$, we have $\left[\frac{1}{t}\right] = p$, hence $I_p = = p^\alpha \int_{\frac{1}{p+1}}^{\frac{1}{p}} \left[\frac{t}{1-tp}\right]^{-\alpha} dt$. Since the interval $\left(\frac{1}{p+1}, \frac{1}{p}\right)$ is a union of the countable system of pairwise disjoint intervals

$\left(\frac{1}{p+\frac{1}{n}}, \frac{1}{p+\frac{1}{n+1}}\right)$ $(n = 1, 2, 3, \ldots)$, we get on the basis of the simple properties of the Lebesgue integral

$$I_p = \sum_{n=1}^{\infty} I_{pn}, I_{pn} = \int_{\frac{1}{p+\frac{1}{n}}}^{\frac{1}{p+\frac{1}{n+1}}} \left[\frac{t}{1-tp}\right]^{-\alpha} . dt.$$

By means of a simple computation we find that if t is irrational,

$t \in \left(\frac{1}{p+\frac{1}{n}}, \frac{1}{p+\frac{1}{n+1}}\right)$, then $\left[\frac{t}{1-tp}\right] = n$ holds, hence

$$I_{pn} = \frac{p^\alpha}{n^{\alpha+1} . (n+1)} \cdot \frac{1}{\left(p+\frac{1}{n}\right)\left(p+\frac{1}{n+1}\right)}.$$

From the last we get by means of a simple estimation

$$\frac{p^\alpha}{(p+1)^2} \cdot \frac{1}{n^{\alpha+1}(n+1)} \leqq I_{pn} \leqq \frac{1}{p^{2-\alpha}} \cdot \frac{1}{n^{\alpha+1}(n+1)}.$$

If $|\alpha| < 1$, then

$$0 < \sigma(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}(n+1)} < +\infty$$

and thus

$$\int_0^1 \psi^\alpha(t)\,\mathrm{d}t < \sigma(\alpha) \sum_{p=1}^{\infty} \frac{1}{p^{2-\alpha}} < +\infty.$$

With respect to Theorem I we get the correctness of our affirmation.

If $|\alpha| \geqq 1$, then two cases will be distinguished.

1. $\alpha \geqq 1$. Clearly $I_p \geqq I_{p1} \geqq \dfrac{p^\alpha}{2(p+1)^2}$, hence $\sum_{p=1}^{\infty} I_p = +\infty$ and thus $\int_0^1 \psi^\alpha(t)\,\mathrm{d}t = +\infty$.

2. $\alpha \leqq -1$. Then $\int_0^1 \psi^\alpha(t)\,\mathrm{d}t \geqq I_1 = \sum_{n=1}^{\infty} I_{1n} \geqq \dfrac{1}{2^2} \sum_{n=1}^{\infty} \dfrac{1}{n^{\alpha+1}(n+1)} = +\infty$, hence $\int_0^1 \psi^\alpha(t)\,\mathrm{d}t = +\infty$.

According to Theorem II we get in both cases for almost all $x \in (0, 1)$

$$\lim_{n\to\infty} \frac{1}{n}\left(\frac{c_1^\alpha(x)}{c_2^\alpha(x)} + \ldots + \frac{c_n^\alpha(x)}{c_{n+1}^\alpha(x)}\right) = +\infty.$$

Theorem 3. *Let $\tau_n \to +\infty$. Then for almost all $x \in (0, 1)$*

$$h\left(\left\{n; \frac{c_n(x)}{c_{n+1}(x)} \geqq \tau_n\right\}\right) = 0,\ h\left(\left\{n; \frac{c_{n+1}(x)}{c_n(x)} \geqq \tau_n\right\}\right) = 0$$

holds.

Proof. We may already suppose that $\tau_n \geqq 0$ $(n = 1, 2, 3, \ldots)$. Put $t_n = \sqrt{\tau_n}$ $(n = 1, 2, 3, \ldots)$ and $g_n(x) = \sqrt{c_n(x)/c_{n+1}(x)}$ for each irrational $x \in (0, 1)$. With respect to Theorem 2, the sequence $\{g_n\}_{n=1}^{\infty}$ is almost everywhere summable by the method $(C, 1)$. Consequently, there exists a set $M \subset (0, 1)$ of measure 1 such that for $x \in M$ $\sup_n \dfrac{\sum_{k=1}^{n} g_k(x)}{n} < +\infty$ holds. It follows from Lemma 2 that for $x \in M$ $h(\{n;\ g_n(x) \geqq t_n\}) = 0$ takes place, hence

$$h\left(\left\{n;\ \frac{c_n(x)}{c_{n+1}(x)} \geqq \tau_n\right\}\right) = 0.$$

In a quite similar way the existence of $M' \subset (0, 1)$ of measure 1 may be proved such that for

$$x \in M' \quad h\left(\left\{n;\ \frac{c_{n+1}(x)}{c_n(x)} \geqq \tau_n\right\}\right) = 0$$

is true. The set $M \cap M'$ is of measure 1 and for $x \in M \cap M'$ the following holds simultaneously

$$h\left(\left\{n;\ \frac{c_n(x)}{c_{n+1}(x)} \geqq \tau_n\right\}\right) = 0, \quad h\left(\left\{n;\ \frac{c_{n+1}(x)}{c_n(x)} \geqq \tau_n\right\}\right) = 0.$$

This completes the proof.

In connection with Theorem 2 the problem arises to examine the behaviour of the differences of two subsequent quotients of the continued fraction of x. Such a question is discussed in Theorem 4, Theorem 5 is a consequence of Theorem 4 and Lemma 2.

Theorem 4. *Given* $0 \leqq \alpha < 1$ *then for almost all* $x \in (0, 1)$ *the following holds: The sequence* $\{|c_k(x) - c_{k+1}(x)|^\alpha\}_{k=1}^\infty$ *is summable by the method* $(C, 1)$ *to the finite number (which does not depend on* x*):*

$$\frac{1}{\log 2} \int_0^1 \frac{\left|\left[\frac{1}{t}\right] - \left[\left(\frac{1}{t} - \left[\frac{1}{t}\right]\right)^{-1}\right]\right|^\alpha}{1 + t}\, dt.$$

If $\alpha \geqq 1$, *then for almost all* $x \in (0, 1)$

$$\lim_{n\to\infty} \frac{1}{n} (|c_1(x) - c_2(x)|^\alpha + \ldots + |c_n(x) - c_{n+1}(x)|^\alpha) = +\infty$$

holds.

Remark 2. For $\alpha < 0$ the sequence $\{|c_k(x) - c_{k+1}(x)|^\alpha\}_{k=1}^\infty$ is not defined on a set of positive measure. In fact it can be easily found out that for each irrational x belonging to the interval

$$(2) \qquad \left(\frac{1}{p + \frac{1}{p}},\ \frac{1}{p + \frac{1}{p+1}}\right)$$

$$c_1(x) = \left[\frac{1}{x}\right] = \left[\left(\frac{1}{x} - \left[\frac{1}{x}\right]\right)^{-1}\right] = c_2(x)$$

holds and the set of all the irrational numbers contained in the union of the intervals (2) is of positive measure.

Proof of Theorem 4. Put for t irrational, $t \in (0, 1)$ $f(t) = |c_1(t) - c_2(t)|$. Let us examine $\int_0^1 f^\alpha(t)\,dt$. Evidently

$$\int_0^1 f^\alpha(t)\,dt = \sum_{p=1}^{\infty} I_p, \quad I_p = \int_{\frac{1}{p+1}}^{\frac{1}{p}} f^\alpha(t)\,dt = \int_{\frac{1}{p+1}}^{\frac{1}{p}} \left| p - \left[\frac{t}{1 - tp}\right]\right|^\alpha dt.$$

Further

$$I_p = \sum_{n=1}^{\infty} I_{pn}, \quad I_{pn} = \int_{\frac{1}{p+\frac{1}{n}}}^{\frac{1}{p+\frac{1}{n+1}}} |p - n|^\alpha\,dt = \frac{|p - n|^\alpha}{n(n+1)\left(p + \frac{1}{n}\right)\left(p + \frac{1}{n+1}\right)}.$$

From the last, with $0 \leqq \alpha < 1$, we get by means of a simple estimation

$$I_p \leqq \frac{1}{p^2}\sum_{n=1}^{\infty} \frac{|p-n|^\alpha}{n(n+1)} = \frac{1}{p^2}\left(\sum_{n=1}^{p} + \sum_{n=p+1}^{\infty}\right) \leqq \frac{1}{p^2}\left\{(p-1)^\alpha \sum_{n=1}^{\infty}\frac{1}{n(n+1)} + \right.$$

$$\left. + \sum_{n=1}^{\infty}\frac{n^\alpha}{n(n+1)}\right\} = \frac{(p-1)^\alpha}{p^2} + \frac{\sigma(\alpha)}{p^2}, \quad \sigma(\alpha) = \sum_{n=1}^{\infty}\frac{n^\alpha}{n(n+1)} < +\infty.$$

Hence it is evident that $\int_0^1 f^\alpha(t)\,dt = \sum_{p=1}^{\infty} I_p < +\infty$.

In the case of $\alpha \geqq 1$ we have $I_p \geqq I_{p1} \geqq \frac{(p-1)^\alpha}{2(p+1)^2}$ and consequently

$$\int_0^1 f^\alpha(t)\,dt = \sum_{p=1}^{\infty} I_p = +\infty.$$

The correctness of the affirmation follows immediately from Theorems I, II.

Theorem 5. *Let* $\tau_n \to +\infty$. *Then for almost all* $x \in (0, 1)$ $h(\{n; |c_n(x) - c_{n+1}(x)| \geqq \tau_n\}) = 0$ *holds.*

The author wishes to express his thanks to J. Mařík for the valuable suggestions improving the original version of the paper.

REFERENCES

[1] Ryll-Nardzewski C., *On the ergodic theorems* II, Studia math. *12* (1951), 74—79.

[2] Hartman S., *Quelques propriétés ergodiques des fractions continues*, Studia math. *12* (1951), 271—278.

[3] Хинчин А., Цепные дроби, Москва 1961.

[4] Chinčin A., *Metrische Kettenbruchprobleme,* Compositio math. *1* (1935), 361—382.

Received January 10, 1966.

Katedra algebry a teórie čísel
Prírodovedeckej fakulty
Univerzity Komenského,
Bratislava