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A NOTE ON THE STRUCTURE OF SOME TYPES OF SEMIGROUPS

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The purpose of the presented paper is to study the structure of semigroups of following types: 1. semigroups, each subsemigroup of which possesses a left identity; 2. semigroups, each left ideal of which possesses a left identity; 3. semigroups, each left ideal of which possesses a right identity. The main part of our discussion deals with the construction and with the properties of ideals (and F-classes). It can be shown that types 1 and 2 are special cases of the so called „product of semigroups over a given semigroup" which has been introduced in [4]. The construction of semigroup of type 3 is here not given. Many of the results of the present paper are contained in the paper [1] which I have read after having prepared my results for publication. I mention them here, because they have been obtained in a different, quite simple manner (similarly as in [3], [4]).

Let S be a semigroup. The set of all elements which generate the same principal ideal (left $(x)_L$, right $(x)_R$, two-sided (x)) is called the F-class (left $F_L(x)$, right $F_R(x)$, two-sided $F(x)$). An element $e \in S$ is called a left (right) identity iff $ex = x$ $(xe = x)$ for each $x \in S$. The set of idempotents of S will be denoted by $I(S)$; the elements of $I(S)$ will be denoted by e (with indices if necessary).

We shall introduce in $I(S)$ the relation R and L as follows:

Definition 1. *$e_i R e_k$ iff $e_i = e_k e_i$ (i.e. $(e_i)_R \subseteqq (e_k)_R$).*

Lemma 1. *The relation R is a quasiordering of the set $I(S)$ (in the sense of* [5]).

Proof. It is evident that $e_i R e_i$; further $e_i R e_k$, $e_k R e_n$ imply $e_i R e_n$.

The set of all elements e_k for which $e_i R e_k$, $e_k R e_i$ simultaneously hold will be denoted by $E_R(e_i)$.

Definition 2. *$e_i L e_k$ iff $e_k = e_i e_k$ (this means $(e_i)_L \subseteqq (e_k)_L$).*

We evidently have

Lemma 2. *The relation L is a quasiordering of the set $I(S)$.*

The set of all elements e_k for which e_iLe_k, e_kLe_i simultaneously hold will be denoted by $E_L(e_i)$.

Now we shall introduce the relation $\leqq$ in the set of F_L- (F_R-) classes:

Definition 3. $F_L(x) \leqq F_L(y)$ $(F_R(x) \leqq F_R(y))$ *iff* $(x)_L \subseteqq (y)_L$ $((x)_R \subseteqq (y)_R)$.

1. SEMIGROUPS, EACH SUBSEMIGROUP OF WHICH POSSESSES A LEFT IDENTITY

Definition. *The semigroup S will be said to have the property U iff each subsemigroup of S possesses at least one left identity.*

In what follows we mention some properties of these semigroups obtained in [3].

Theorem 1. *The necessary and sufficient condition for a semigroup S to have the property U is*: 1. *S is the union of disjoint periodic groups*; 2. *$I(S)$ is a subsemigroup of S and has the property U.*

Proof. (Analogously as in [3]). a) Let S have the property U. 1. Let $s \in S$; we consider the semigroup $S_n = \{s, s^2, \ldots\}$; by the assumption S_n possesses a left identity, which is evidently an identity of S_n. This means, s has a finite order, hence according to Theorem 7 [2] S is a union of disjoint periodic groups. 2. is evident since $I(S)$ is a subsemigroup of S (see Theorem 4 of [3]). b) Let S have the properties 1, 2. Let H be a subsemigroup of S; let $h \in H$. Then by 1. there exists a positive integer n such that $h^n = e_h$, where e_h is an idempotent and $e_h h = h$. Hence $I(H) \neq \emptyset$. According to 2., $I(H)$ is a subsemigroup of $I(S)$, hence $I(H)$ possesses a left identity e_H. Then $e_H h = e_H(e_h h) = (e_H e_h)h = = e_h h = h$ and so e_H is a left identity of H.

In this section S is always a semigroup having the property U. The groups in the decomposition of S in the sense of Theorem 1 will be denoted by G_i; e_i will denote the identity of G_i. The group with the identity $e_i e_k$ will be denoted by G_{ik}. The elements of G_i will be denoted by g_i (with indices if necessary).

Lemma 4. *Let e_iRe_k. Then $G_kG_i \subseteqq G_i$.*

Proof. First we shall prove that $g_ke_i \in G_i$. Let $g_ke_i \in G_n$. this means that for any positive integer n $(g_ke_i)_n = e_n$ holds, thus $e_ne_i = e_n$. By Lemma 3 for the couple e_i, e_n at least one of the relations e_iRe_n, e_nRe_i holds. Let e_iRe_n, i.e. $e_i = e_ne_i$. By the foregoing we have $e_i = e_n$. Let e_nRe_i, i.e. $e_ie_n = e_n$. Since $g_ke_i \in G_n$, we have $g_ke_i = g_ke_ie_n = g_ke_n = e_ng_ke_i$. Since for some integers m, n we have $(g_ke_i)^n = e_i$, $g_k^m = e_k$, we obtain $e_n = (g_ke_i)^{mn} = = (g_ke_i)^{mn-1}g_ke_i = (g_ke_i)^{mn-2}g_ke_ng_ke_i = (g_ke_i)g_kg_ke_i$ and repeating this proceeding we obtain after mn $-$ 1 steps $e_n = g_k^{mn}e_i = e_ke_i = e_i$, therefore $g_ke_i \in G_i$. Hence $g_kg_i = g_k(e_ig_i) = (g_ke_i)g_i \in G_i$, q.e.d.

Lemma 5. *$P_i = \bigcup \{G_k / e_k \in E_R / e_i)\}$ is a subsemigroup of S. Here G_k are isomorphic groups and the partition of P_i into the groups G_k yields a congruence relation on P_i.*

Proof. From Lemma 4 it follows $e_i R e_k$ implies $G_k G_i \subseteqq G_i$. Similarly $G_i G_k \subseteqq \subseteqq G_k$. Therefore P_i is a subsemigroup of S and the partition of P_i into $G_k(e_k \in E_R(e_i))$ yields a congruence relation on P_i.

Clearly the mapping $g_i \to g_i e_k$ is a homomorphism of G_i into G_k. We show that each element $g_k \in G_k$ is the image of some element of G_i. Since $g_k e_i \in G_i$, we have $(g_k e_i) e_k = g_k(e_i e_k) = g_k e_k = g_k$, thus g_k is the image of $g_k e_i$. Further let $g_{i1} e_k = g_{i2} e_k$, then $g_{i1} e_k e_i = g_{i2} e_k e_i$, whence $g_{i1} = g_{i2}$ (since $e_k e_i = e_i$, $g_{i1} e_i = g_{i1}$, $g_{i2} e_i = g_{i2}$). This shows that G_i and G_k are isomorphic groups.

Lemma 6. *Let $e_i R e_k$. Then $G_i g_k \subseteqq G_n$, where $e_n \in E_R(e_i)$.*

Proof. First we prove that $e_i g_k \in G_n$, where $e_n \in E_R(e_i)$. Suppose that $(e_i g_k)^n = e_n$, $g_k^m = e_k$ for some positive integers m, n. Therefore evidently $e_i e_n = e_n$, thus $e_n R e_i$. Further $e_n e_i = (e_i g_k)^{mn} e_i$, and by Lemma 4 $g_k e_i \in G_i$. Hence we obtain $e_i g_k e_i = g_k e_i$. Similarly as in the proof of Lemma 4 we get $e_n e_i = e_k e_i = e_i$, hence $e_i R e_n$. Together with $e_n R e_i$ we obtain $e_n \in E_R(e_i)$. With respect to Lemma 5 we have $g_i g_k = g_i(e_i g_k) \in G_n$. Hence $G_i g_k \subseteqq G_n$.

Lemma 7. *Let $e_i R e_k$. Then the following holds:*

a) *Let $e_i g_k^m \in G_m$, $e_i g_k^n \in G_n$, $(n < m)$, $e_m R e_n$; then $e_n g_k^{m-n} \in G_m$.*

b) *Let $e_i g_k^n \in G_m$, $e_i g_k^m \in G_m$ $(n < m)$, where if $g_k^{m+s} = e_k$, then $(m - n)/s$. We then have $e_m = e_i e_k$.*

c) *Let* b) *hold where at least two of the integers m, n, s are relatively prime. Then $G_i g_k^v \subseteqq G_{ik}$ for each $v = 1, 2, 3, \ldots$*

Proof. a) $(e_i g_k^n)^z = e_n$ for some z. Hence $e_n = (e_i g_k^n)^{z-1}(e_i g_k^n)$ and therefore $e_n g_k^{m-n} = (e_i g_k^n)^{z-1}(e_i g_k^{n+m-n}) = (e_i g_k^n)^{z-1}(e_i g_k^m) \in G_m$,

b) Let $m - n/s$, this means $s = k(m - n)$ for some k. According to a) we have $e_i e_k = e_i g_k^{m+s} = e_i g_k^m g_k^{k(m-n)} = e_i g_k^m e_m g_k^{m-n} g_k^{(k-1)(m-n)} = e_i g_k^m e_m g_k^{m-n} e_m g_k^{(k-1)(m-n)}$. Repeating this proceeding we obtain after $k - 1$ steps $e_i e_k = e_i g_k^m (e_m g_k^{(n-m)})^k \in \in G_m$. Thus $e_m = e_i e_k$.

c) First we shall prove that $g_k^{m+s} = e_k$ implies $e_i g_k^s \in G_{ik}$. Suppose $e_i g_k^m \in G_t$, which means $e_i e_k e_t = e_t$, hence $e_t R e_i e_k$. Then according to Lemma 4 and with respect to the fact that by the assumption and b) $e_i g_k^m \in G_{ik}$ holds, we obtain $e_i g_k^m e_i g_k^s \in G_i$. Now $e_i g_k^m e_i g_k^s = (e_i g_k^m e_i e_k) g_k^s = e_i g_k^{m+s} = e_i e_k$. Thus $e_t = e_i e_k$. Suppose that at least two of the integers m, n, s be relatively prime. We denote them by x, y. We then have $1 = kx + ty$ for some integers k, t. Since $e_i g_k^m$, $e_i g_k^n$, $e_i g_k^s \in G_{ik}$, we obtain $e_i g_n^{kx} e_i g_k^{ty} \in G_{ik}$, whence $e_i g_k^{kx} e_i g_k^{ty} = e_i g_k^{kx+ty} = = e_i g_k \in G_{ik}$. Hence evidently $e_i g_k^v \in G_{ik}$ for each $v = 1, 2, 3, \ldots$

Lemma 4 and 6 lead immediately to

Theorem 2. *The partition of S into semigroups P_i* (see Lemma 5) *yields a congruence relation on S.*

The following two Theorems can be easily proved:

Theorem 3. *The set E consisting of all $E_R(e_i)$ (for $e_i \in I(S)$) is a dually well-ordered chain with respect to the relation $\bar{R}$ given as follows: $E_R(e_n)\bar{R}E_R(e_i)$ iff e_nRe_i.*

Theorem 4. *Let I be an idempotent semigroup having the property U. Then: $I = \cup E_R(e_i)$ where the elements $E_R(e_i)$ form a dually well-ordered chain with respect to the relation $\bar{R}$. At the same time $E_R(e_i)\bar{R}E_R(e_k)$ implies $E_R(e_i)E_R(e_k) \leqq \leqq E_R(e_i)$; $E_R(e_k)E_R(e_i) \leqq E_R(e_i)$. Further $e_ke_t = e_t$ for $e_t \in E_R(e_i)$, $e_k \in E_R(e_k)$ (e_k are left identities for $E_R(e_i)$).*

Lemma 8. *Let e_iRe_k. Then the mapping $g_k \to g_ke_i$ is a homomorphism of G_k into G_i.*

The proof follows from Lemmas 4,5 and 6.

As a consequence of the foregoing results we obtain the construction of any semigroup having the property U:

Theorem 5. *Let I be an idempotent semigroup having the property U. To every $e_n \in E_R(e_i)$ we associate a group G_n all isomorphic to G_i. Denote $P_i = \cup \{G_n/e_n \in \in E_R(e_i)\}$ and define a multiplication in P_i by the following rule: $g_ig_n = (\psi^i_n g_i)g_n$, where ψ^i_n is a homomorphism of G_i into G_n.*

Let $\mathfrak{H}$ be a set of homomorphisms such that for each $E_R(e_i)RE_R(e_k)$ there exists in $\mathfrak{H}$, a homomorphism of P_k into P_i (denoted by φ^k_i), where φ^i_i is the identical mapping and $\varphi^n_k\varphi^i_n = \varphi^i_k$. Denote $P = \cup \{P_i/E_R(e_i) \subseteqq I\}$ and define in P a multiplication as follows: let $E_R(e_i)\bar{R}E_R(e_k)$ in I and let $g_i \in P_i$, $g_k \in P_k$, then $g_ig_k = g_i(\varphi^k_i g_k)$, $g_kg_i = (\varphi^k_i g_k)g_i$.

The semigroup P has the property U and any semigroup having the property U can be constructed in this manner by choosing suitably I and $\mathfrak{H}$.

Remark 1. In [4] the semigroup P constructed in the manner described in Theorem 5 is called a product of semigroups P_i over the semigroup I. [4] deals with the structure of such semigroups.

We have the following special case:

Theorem 6. *Let I be an idempotent semigroup each subsemigroup of which possesses a unique left identity (I is a chain). To each $e_i \in I$ we assign a periodic group G_i. Let $\mathfrak{H}$ be a set of homomorphisms such that if $e_ie_k = e_i$, then there exists a homomorphism of G_k into G_i (denoted by φ^k_i) with φ^i_i as the identical mapping and $\varphi^n_k\varphi^i_n = \varphi^i_k$. Let $P = \cup \{G_i/e_i \in I\}$. Define a multiplication in P as follows: Let $e_ie_k = e_i$, then $g_ig_k = g_i(\varphi^k_i g_k)$, $g_kg_i = (\varphi^k_i g_k)g_i$. Then each subsemigroup of P possesses a unique left identity. Conversely every semigroup P*

each subsemigroup of which possesses a unique left identity can be constructed in this manner.

Remark 2a. The statement that I is a chain follows from Theorem 3 and Theorem 4 by which each $E_R(e_k)$ possesses a unique element.

Remark 2b. In [4] the semigroup constructed by the construction given in Theorem 6 is called a product of groups G_i over the semigroup I. [4] deals with the structure of such semigroups.

Evidently the subsemigroup $I(S)$ is isomorphic to I (see Theorem 5). Accordingly we use the same symbols in J as in $I(S)$.

From the foregoing we evidently have:

Theorem 7. *Let the semigroup S have the property U. Then:*

a) *In J we have $(e_i)_R = \cup E_R(e_n)$ for $E_R(e_n)\bar{R}E_R(e_i)$; further $F_R(e_i) = E_R(e_i)$.*

b) *In S we have $(e_i)_R = \cup G_k$ for $e_k \in (e_i)_R$ in J; further $F_R(e_i) = \cup G_k$ for $e_k \in E_R(e_i)$.*

In both cases the elements of $E_R(e_i)$ are left identities of the ideals $(e_i)_R$ in J as well as in S.

Theorem 8. *Let the semigroup S have the property U. Then:*

a) *In J we have $F_L(e_i) = \{e_i\}$; $(e_i)_L \cap E_R(e_i) = \{e_i\}$;*

b) *In S we have $F_L(e_i) = G_i$, $(e_i)_L = \cup G_k$ for $e_k \in (e_i)_L$ in J.*

c) *$(e_i)_L$ in J and in S possesses an identity e_i.*

d) *Let $e_k \in E_k(e_i)$, $e_k \neq e_i$. Then $(e_i)_L \subsetneqq (e_k)_L$ does not hold.*

Remark 3. $(e_i)_L \cap E_R(e_n)$ in J for $E_R(e_n)\bar{R}E_R(e_i)$, $n \neq i$ can contain more than one element of $E_R(e_n)$.

Example. Let S be a semigroup given by the following multiplication table:

	a_1	a_2	a_3	a_{21}	a_{32}	a_{321}	a_{31}
a_1	a_1	a_2	a_3	a_{21}	a_{32}	a_{321}	a_{31}
a_2	a_{21}	a_2	a_3	a_{21}	a_{32}	a_{321}	a_{31}
a_3	a_{31}	a_{32}	a_3	a_{321}	a_{32}	a_{321}	a_{31}
a_{21}	a_{21}	a_2	a_3	a_{21}	a_{32}	a_{321}	a_{31}
a_{32}	a_{321}	a_{32}	a_3	a_{321}	a_{32}	a_{321}	a_{31}
a_{321}	a_{321}	a_{32}	a_3	a_{321}	a_{32}	a_{321}	a_{31}
a_{31}	a_{31}	a_{32}	a_3	a_{321}	a_{32}	a_{321}	a_{31}

Each subsemigroup of S possesses at least one left identity. S is an idempotent semigroup. We can obtain a graphical representation of S as follows: Small circles are drawn to represent the elements of S. An oriented segment is then drawn from a_i to a_k whenever a_iRa_k. (Fig. 1.) We have $(a_1)_L = \{a_1, a_{21}, a_{321}, a_{31}\}$, $E_R(a_3) = \{a_3, a_{32}, a_{321}, a_{31}\}$. Hence $E_R(e_3) \cap (a_1)_L = \{a_{321}, a_{31}\}$.

Remark 4. Considering a left ideal L in J (not necessarily principal), it is evident that there exists such an $E_R(e_i)$ that $L \cap E_R(e_i) \neq \emptyset$ and we have $E_R(e_n)\bar{R}E_R(e_i)$ for all $E_R(e_n)$ with $L \cap E_R(e_n) \neq \emptyset$. Then all elements of $E_R(e_i)$ are left identities in L.

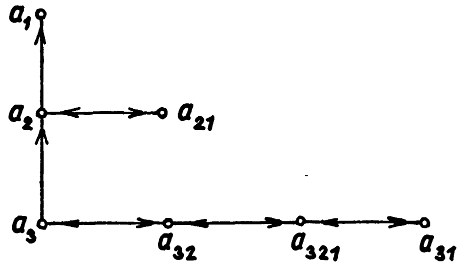

Fig. 1.

2. SEMIGROUPS EACH LEFT IDEAL OF WHICH POSSESSES A LEFT IDENTITY

Definition. *The semigroup S is said to have the property L iff each left ideal of S possesses at least one left identity.*

In this section we shall consider the semigroup S having the property L.

Lemma 9. *e is an identity of $(e)_L$.*

Proof. Evidently e is a right identity of $(e)_L$. Further, let e' be a left identity of $(e)_L$, hence $e'e = e$. Since $e' \in (e)_L$, we have $e'e = e'$. Thus $e = e'$, hence e is a left identity. This implies that e is an identity of $(e)_L$.

Lemma 10. *For each $e_i, e_k \in I(S)$ at least one of the relations e_iRe_k, e_kRe_i holds.*

Proof. Consider the left ideal of S: $N = (e_1)_L \cup (e_2)_L$. Let e be the left identity of N. Then either $e \in (e_1)_L$, or $e \in (e_2)_L$. Let $e \in (e_1)_L$, then by Lemma 9 we have $e = e_1$. Thus $e_1e_2 = e_2$, whence e_2Re_1. In the case that $e \in (e_2)_L$ we prove analogously that e_1Re_2 holds.

With respect to the property L we evidently have:

Lemma 11. *The set E consisting of the subsets $E_R(e_i)$ is a dually well-ordered chain with respect to the relation $\bar{R}$ defined in Theorem 3.*

Lemma 12. *Let e_1Re_2, then $e_1e_2 \in E_R(e_1)$.*

Proof. $(e_1e_2)e_1 = e_1(e_2e_1) = e_1$, hence $e_1R(e_1e_2)$. Further $e_1(e_1e_2) = e_1e_2$, hence $(e_1e_2)Re_1$. Together we have $e_1e_2 \in E_R(e_1)$.

Theorem 9. *$I(S)$ is a subsemigroup of S.*

Proof. Let for $e_1, e_2 \in I(S)$ e_1Re_2 holds. Then $e_2e_1 = e_1 \in I(S)$, further $(e_1e_2)(e_1e_2) = e_1(e_2e_1)e_2 = e_1e_2 \in I(S)$, q.e.d.

Theorem 10. *Each element $x \in S$ belongs to some $F_L(e)$-class.*

Proof. We have to prove that $(x)_L = (e)_L$ for some $e \in I(S)$. Let e be a left identity of $(x)_L$. Then $e = sx$ for some $s \in S$. Let e' be a left identity of $(s)_L$. Then $e = e'sx$, hence $e'e = e$. For some $z \in S$ we have $e' = zs$, whence $ee' = sxe' = sxzs$. But $ee'x = x$ (since $ex = x$, $e = e's$), whence $ee'(ex) = ex = x$, hence $ee'x = x$. Since $ee' = sxzs$, we obtain $x = ee'x = sxzsx = eze$, thus $x \in (e)_L$. This means that $(x)_L \subseteqq (e)_L$. Since $e \in (x)_L$, we have $(e)_L \subseteqq (x)_L$; this, together with $(x)_L \subseteqq (e)_L$ proves that $(x)_L = (e)_L$ as required.

Theorem 11. *S is a union of groups $F_L(e)$ $(e \in I(S))$.*

Proof. The following holds: Let $(x)_L = (y)_L = (e)_L$, then $(xy)_L = (ey)_L = (y)_L = (e)_L$; further $(yx)_L = (ex)_L = (x)_L = (e)_L$. Hence $F_L(e)$ is a semigroup. We have to prove that $F_L(e)$ is a group. It follows from Lemma 9 that e is an identity of $F_L(e)$. We shall show that for any $x \in F_L(e)$ there exists an $y \in F_L(e)$ such that $yx = e$. We have already seen that $e = sx = s(ex) = (se)x$ for some $s \in S$. We shall show that $se \in F_L(e)$. Evidently $se \in (e)_L$, hence $(se)_L \subseteqq (e)_L$. Let e' be a left identity of $(s)_L$, hence $e' = zs$ for some $z \in S$. From $e = sx$ we obtain $e = e'sx$, hence $e'e = e$. Therefore $e = e'e = (zs)e = z(se)$, thus $e \in (se)_L$ or $(e)_L \subseteqq (se)_L$. This, together with $(se)_L \subseteqq (e)_L$ proves $(e)_L = (se)_L$. To accomplish our proof it is sufficient to put $y = se$ According to Lemma 9, each $F_L(e)$-class of S consists of a unique group, thus the $F_L(e)$-class is a group. According to Theorem 10 S is a union of groups.

Lemma 13. *Let $e_i R e_k$, then $F_L(e_k)F_L(e_i) \subseteqq F_L(e_i)$.*

Proof. $e_i R e_k$ implies $e_k e_i = e_i$. Let $x \in F_L(e_i)$, $y \in F_L(e_k)$. There exists an element $z \in F_L(e_k)$ such that $zy = e_k$, hence $zye_i = e_k e_i = e_i$ and $e_i = (ye_i)_L$; this, together with the evident statement $ye_i \in (e_i)_L$ proves that $(e_i)_L = (ye_i)_L$. This means that $ye_i \in F_L(e_i)$. Now $yx = y(e_i x) = (ye_i)x \in F_L(e_i)$ as required.

Theorem 12. *$P_i = \cup \{F_L(e_n)/e_n \in E_R(e_i)\}$ is a subsemigroup of S. Here $F_L(e_n)$ are isomorphic groups. The partition of P_i into the union of $F_L(e_n)$ yields a congruence relation on P_i.*

Proof. According to Lemma 13 for $F_L(e_n)$, $F_L(e_k) \subseteqq P_i$ we have $F_L(e_n)F_L(e_k) \subseteqq F_L(e_k)$. Hence P_i is a subsemigroup of S and the partition of P_i into $F_L(e_n)$ yields a congruence relation on P_i. The assertion stating that $F_L(e_n)$ are isomorphic groups can be proved similarly as the same assertion in Lemma 5.

From Lemma 13 it is evident:

Remark 5. Let e_i be a left identity of the left ideal N. Then all $e_k \in E_R(e_i)$ are exactly all left identities of N.

Theorem 13. $F_R(e_k) = \cup \{F_L(e_i)/e_i \in E_R(e_k)\}$.

Proof. The definitions of the relation R and of the set $E_R(e_k)$ implies $(e_i)_R = (e_k)_R$. Evidently $\cup \{F_L(e_i)/e_i \in E_R(e_k)\} \subseteqq F_R(e_k)$, since all elements of a group generate the same right principal ideal. We show that $\cup F_L(e_i)$ is equal to the whole class $F_L(e_k)$. Let $(e_m)_R = (e_k)_R$; this means $e_m R e_k$, $e_k R e_m$, hence $e_m \in E_R(e_k)$.

Lemma 14. *Let* $e_i R e_k$. *Then:* a) $F_L(e_i)e_k \subseteqq F_L(e_i e_k)$; b) $F_L(e_i)F_L(e_k) \subseteqq F_L(e_n)$, *where* $e_n \in E_R(e_i)$.

Proof. a) Let $x \in F_L(e_i)$. Clearly $xe_k = xe_ie_k$, hence $xe_k \in (e_ie_k)_L$. Let $e_i = sx$ for $s \in F_L(e_i)$; then $e_ie_k = sxe_k$, consequently $e_ie_k \in (xe_k)_L$. This, together with $xe_k \in (e_ie_k)_L$ implies $(e_ie_k)_L = (xe_k)_L$; in other words $xe_k \in F_L(e_ie_k)$.

b) Let $x \in F_L(e_i)$, $y \in F_L(e_k)$. Hence $(e_k)_R = (y)_R$ (since $F_L(e_k)$ is a group), whence $(e_ie_k)_R = (e_iy)_R$. By Theorem 14 we obtain $e_iy \in \cup \{F_L(e_n)/e_n \in E_R(e_i)\}$. Further $xy = (xe_i)y = x(e_iy)$, whence, by Theorem 12 $xy \in \cup \{F_L(e_n)/e_n \in E_R(e_i)\}$.

Clearly we have

Lemma 15. *Let* $e_i R e_k$, $y \in F_L(e_k)$. *Then the mapping* $y \to ye_i$ *is a homomorphism of* $F_L(e_k)$ *into* $F_L(e_i)$.

Lemma 11 implies:

Theorem 15. *Let J be an idempotent semigroup having the property L. Then:* $J = \cup E_R(e_i)$, *where the set* $\{E_R(e_i)\}$ *is a dually well-ordered chain with respect to the relation* $\bar{R}$ *given as follows:* $E_R(e_i)\bar{R}E_R(e_n)$ *iff* $e_i R e_k$.

Theorem 16. *Let J be an idempotent semigroup having the property L. To every* $e_n \in E_R(e_i)$ *we associate a group* G_n *all isomorphic to* G_i. *Denote* $P_i = \cup \{G_n/e_n \in E_R(e_i)\}$ *and define a multiplication in* P_i *by the following rule:* $g_ig_n = (\psi_n g_i)g_n$, *where* ψ_n^i *is a homomorphism of* G_i *to* G_n.

Let $\mathfrak{H}$ *be a set of homomorphisms, where for each* $E_R(e_i)\bar{R}E_R(e_k)$ *in* J *there exists in* $\mathfrak{H}$ *a homomorphism of* P_k *into* P_i *(denoted by* φ_i^k*), where* φ_i^i *is the identical mapping and* $\varphi_k^n\varphi_n^i = \varphi_k^i$. *Denote* $P = \cup \{P_i/E_R(e_i) \subseteqq J\}$ *and define in P a multiplication as follows: Let* $E_R(e_i)\bar{R}E_R(e_k)$ *in J and let* $g_i \in P_i$, $g_k \in P_k$, *then* $g_ig_k = g_i(\varphi_i^k g_k)$, $g_kg_i = (\varphi_i^k g_k)g_i$.

The semigroup P has the property L and any semigroup having the property L can be constructed in this manner by choosing suitably J and $\mathfrak{H}$.

It is easy to prove, that the foregoing construction gives a semigroup of required properties. In consequence of Lemmas 12—15 and Theorems 11 and 12 every semigroup having the property L can be constructed in this manner.

Remark 6. In case that each left ideal of S possesses a unique left identity, each $E_R(e_i)$ contains a unique element, hence P_i are groups. We can obtain a similar construction of S as in Theorem 6 (with the exception that G_i need not be periodic).

Remark 7. For a semigroup having the property L it is possible to give a construction of S as a product of groups G_i over an idempotent semigroup J having the property L, with the multiplication defined by homomorphisms (similarly as in Theorem 16): for $e_i R e_k$ let $g_k g_i = (\varphi_i^k g_k) g_i$, $g_i g_k = (\varphi_n^i g_i)(\varphi_n^k g_k)$, with similar conditions for n as in Lemma 14.

Remark 8. A semigroup having the property U has also the property L. Therefore all results proved for the semigroups having the property L hold for semigroups having the property U.

3. SEMIGROUPS, EACH LEFT IDEAL OF WHICH POSSESSES A RIGHT IDENTITY

Definition. *The semigroup S is said to have the property R iff each left idea of S possesses a right identity.*

In this section we suppose that the semigroup S has the property R.

Lemma 16. *For each e_i, $e_k \in I(S)$ at least one of the relations $e_i L e_k$, $e_k L e_i$ holds.*

Proof. Let $e_i \neq e_k$. Clearly e_i is a right identity of $(e_i)_L$, e_k is a right identity of $(e_k)_L$. Let e_n be a right identity of $(e_i)_L \cup (e_k)_L$. Then either $e_n \in (e_i)_L$, or $e_n \in (e_k)_L$. Let $e_n \in (e_i)_L$, this means that $e_n e_i = e_n$. Since $e_k = e_k e_n$, we have $e_k = e_k e_n e_i = e_k e_i$, hence $e_k L e_i$. In the case that $e_n \in (e_k)_L$, we show similarly that $e_i L e_k$.

Theorem 17. *$I(S)$ is a subsemigroup of S.*

Proof. Let $e_i L e_k$, this means that $e_i e_k = e_i$. Further $e_k e_i e_k = e_k e_i$, whence $e_k e_i e_k e_i = e_k e_i e_i = e_k e_i$; hence $e_k e_i \in I(S)$.

Theorem 18. *S is a regular semigroup.*

Proof. Let $x \in S$, let e be a right identity of $(x)_L$. Then $e = sx$ for some $s \in S$, thus $xe = xsx$. Since $xe = x$ hence $x = xsx$, which proves our assertion.

Theorem 19. *Each element $x \in S$ belongs to some $F_L(e)$-class.*

Proof. Let e be a right identity of $(x)_L$. Then $xe = x$, this means that $x \in (e)_L$, consequently $(x)_L \subseteqq (e)_L$. Since $e \in (x)_L$, we have $(e)_L \subseteqq (x)_L$, hence $(x)_L = (e)_L$.

Theorem 20. *Each element $x \in S$ belongs to some $F_R(e)$-class.*

Proof. According to Theorem 18 S is regular, hence there is an s such that

$x = xsx$. Therefore $xs = xsxs$; thus xs is an idempotent, this means that $xs \in I(S)$. Evidently $x \in (xs)_R$, $xs \in (x)_R$, this implies $(x)_R = (xs)_R$.

Evidently we have:

Lemma 17. *Let $e_i L e_k$. Then either $e_i R e_k$, or e_i, e_k are incomparable.*

Theorem 21. *$F_L(e) \cap F_R(e)$ is a maximal group of S.*

Proof. Denote $F_L(e) \cap F_R(e) = T$. Let $x, y \in T$. Then we have $xe = ex = x$. This means that e is an identity of T. We have $(x)_L = (y)_L = (e)_L$, $(x)_R = (y)_R = (e)_R$. Hence $(x^2)_L = (yx)_L = (ex)_L = (x)_L = (e)_L$, $(x^2)_R = (xy)_R = (xe)_R = (x)_R = (e)_R$. This says $x^2 \in T$. Similarly we obtain $y^2 \in T$. At the same time we have $(xy)_R = (e)_R$, $(xy)_L = (e)_L$, hence $xy \in T$. In a similar way we obtain $yx \in T$, which says that T is a semigroup. We shall show that T is a group. We have $e = sx$ for some $s \in S$. Now $e = es(ex) = (ese)x$, hence ese is a left inverse for x. We shall show that $ese \in T$. Since $e = sx$, we obtain $e = es(ex) = (ese)x$, hence $e \in (ese)_R$; but clearly $ese \in (e)_R$. Summarily we have $(ese)_R = (e)_R$. Further we assert that $e = xese$. Namely $e = xz$ for some $z \in S$ (by the assumption $(e)_R = (x)_R$). Then $x(ese) = xes(xz) = xe(sx)z = xeez = xez = xz = e$, hence $e \in (ese)_L$. Evidently also $ese \in (e)_L$, hence $(ese)_L = (e)_L$. Consequently $ese \in T$. We proved that T is a group. It is evidently a maximal group, since all elements of a group generate the same left (right) principal ideal.

Theorem 22. *$F_R(e_i) \cap F_L(e_k)$ can possess at most one idempotent.*

Proof. Let $e_n, e_m \in F_R(e_i) \cap F_L(e_k)$. Then $(e_n)_L = (e_m)_L$, whence $e_n e_m = e_n$. At the same time $(e_n)_R = (e_m)_R$, thus $e_n e_m = e_m$. Hence $e_n = e_m$.

Lemma 18. *Let $x \in F_L(e_i)$, $y \in F_R(e_k)$ and let $e_i L e_k$. Then $xy \in F_R(e_i)$, $xy \in F_L(e_i y) \leqq F_L(y)$.*

Proof. Since $x \in (e_i)_L \subseteqq (e_k)_L$, we have $xe_k = x$. Since $(y)_R = (e_k)_R$, we have $e_k = yz$ for some $z \in S$. Hence $x = xe_k = xyz$, whence $x \in (xy)_R$; evidently $xy \in (x)_R$, thus $(x)_R = (xy)_R$. Further $(x)_L = (e_i)_L$ implies $(xy)_L = (e_i y)_L \subseteqq (y)_L$; this proves the second part of our assertion.

Theorem 23. *Let each left ideal of S possess a unique right identity. Then $I(S)$ is a commutative semigroup, which is a chain with respect to the relation L (R).*

Proof. Let $e_i \in (e_k)_L$; then $e_i = se_k$ for some $s \in S$, whence $e_i = se_k e_i$; thus $e_i \in (e_k e_i)_L$. This implies $(e_i)_L \subseteqq (e_k e_i)_L$. Evidently $e_k e_i \in (e_i)_L$, and $(e_k e_i)_L \subseteqq (e_i)_L$, hence $(e_i)_L = (e_k e_i)_L$. Further: e_i is a right identity of $(e_i)_L$, $e_k e_i$ a right identity of $(e_k e_i)_L$. With respect to the uniqueness of the identity we have $e_i = e_k e_i$. Further $e_i = se_k$ implies $e_i e_k = e_i$, hence $e_i = e_k e_i = e_i e_k$. By Lemma 16 $I(S)$ is a chain with respect to the relation $L(R)$.

Corollary. *In such semigroups $e_i L e_k$ implies $e_i R e_k$.*

Theorem 24. *Let each left ideal of S possess a unique right identity. Then each F_R- (F_L-) class possesses a unique idempotent.*

Proof. Let $(e_i)_R = (e_k)_R$. According to Theorem 22 we have $e_i = e_k$. Analogously for the F_L-classes.

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