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ON POWERS OF NON-NEGATIVE MATRICES

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Let $A$ be a $n \times n$ matrix with non-negative entries. One of the main problems in studying such matrices is to study the distribution of zeros and ,,non-zeros“ in the sequence

$$A, A^2, A^3, \ldots$$

In the paper [2] I have shown that there is a simple semigroup treatment of this problem which leads to a series of results without any mention of such notions as characteristic values, characteristic vectors, etc.

This semigroup treatment leads to some pertinent questions which will be partly solved in this paper.

For convenience of the reader I briefly recall the necessary notions introduced in [2].

Let $N = \{1, 2, \ldots, n\}$. Consider the set of ,,n × n matrix-units”, i.e. the set $S$ of symbols $\{e_{ij} | i, j \in N\}$ together with a zero $\theta$ adjoined: $S = \{e_{ij} | i, j \in N\} \cup \{\theta\}$.

Define in $S$ a multiplication by

$$e_{ij}e_{ml} = \begin{cases} \theta & \text{for } j \neq m, \\ e_{il} & \text{for } j = m, \end{cases}$$

the zero $\theta$ having the usual properties of a multiplicative zero. The set $S = S_n$ with this multiplication is a $\theta$-simple semigroup. It contains exactly $n$ non-zero idempotents, namely the elements $e_{11}, e_{22}, \ldots, e_{nn}$.

Let $A = (a_{ij})$ be a non-negative $n \times n$ matrix. By the support of $A$ we shall mean the subset of $S$ containing $\theta$ and all those elements $e_{ij} \in S$ for which $a_{ij} > 0$.

The support of $A$ will be denoted by $C_A$. For typographical reasons we shall write occasionally $C_A = C(A)$.

For any two $n \times n$ non-negative matrices we clearly have $C_{A+B} = C_A \cup C_B$.

Consider further the set $\mathcal{S} = \mathcal{S}_n$ of all subsets of $S = S_n$ and define a multi-
plication in $S$ as the multiplication of complexes in $S$, i.e. if $C', C'' \in S$, then $C'C'' = \{c_1c_2| c_1 \in C', c_2 \in C''\}$. Then $S$ is again a finite semigroup containing exactly $2^n$ different elements.\(^{(1)}\)

If $A$, $B$ are two non-negative matrices it is easy to see that $C_{AB} = C_A \cdot C_B$.

In particular, the supports of the elements of the sequence of (1) are given by the sequence

$$(2) \quad C_A, C_A^2, C_A^3, \ldots$$

Though (1) may contain an infinity of different elements, the sequence (2) contains only a finite number of different elements. The correspondence $A \rightarrow C_A$ is a homomorphosing mapping of the semigroup of all non-negative matrices onto the semigroup $S$. [If we consider the union of sets as the second binary operation in $S$, we have even a homomorphosing mapping of the semiring of all non-negative $n \times n$ matrices onto the semiring $S$.]

The following facts easily follow from the elements of the theory of finite semigroups.

Let $A$ be a fixed $n \times n$ matrix. Let $k$ be the least integer such that $C_A^k = C_A^l$ for some $l > k$. Let further $l = k + d$ ($d \geq 1$) be the least integer satisfying this relation. Then the sequence (2) is of the form

$$C_A, \ldots, C_A^{k-1}, C_A^k, \ldots, C_A^{k+d-1}, C_A^k, \ldots, C_A^{k+d-1}, \ldots$$

Denote by $S_A$ the subsemigroup of $S$ generated by $C_A$. Then $S_A$ has exactly $k + d - 1$ different elements and we have

$$(3) \quad S_A = \{C_A, \ldots, C_A^{k-1}, C_A^k, \ldots, C_A^{k+d-1}\}.$$

For any $\alpha \geq k$ and every $\beta \geq 0$ we clearly have

$$(4) \quad C_A^\alpha = C_A^{\alpha + \beta d}.$$

It is well known that $S_A = \{C_A^k, \ldots, C_A^{k+d-1}\}$ is a cyclic group of order $d$ (subgroup of $S_A$). The unit element of the group $S_A$ is $C_A^0$ with a suitably chosen $\varrho$ satisfying $k \leq \varrho \leq k + d - 1$. Let $\tau$ be the uniquely determined integer such that $k \leq \tau d \leq k + d - 1$. Then $\varrho = \tau d$. To show this it is sufficient to show that $C_A^{\tau d}$ is an idempotent. In fact we have (by (4) with $\alpha = \tau d$, $\beta = \tau$) $C_A^{\tau d} = C_A^{\tau d + \tau d} = C_A^{\tau d}$.

In the following we shall consequently write $\varrho = \tau d$, so that $C_A^\varrho$ is the (unique) idempotent $\in S_A$. Clearly, we also have $S_A = \{C_A^\varrho, C_A^{\varrho + 1}, \ldots, C_A^{\varrho + d - 1}\}$.

Note explicitly that to every non-negative matrix $A$ we have associated three integers $k = k(A)$, $\varrho = \varrho(A)$ and $d = d(A)$ satisfying $k \leq \tau d = \varrho \leq \tau d + d - 1$.\(^{(1)}\)

---

\(^{(1)}\) $S$ may be considered — of course — also as the Boolean algebra of $n \times n$ square matrices with elements 0 and 1 and the usual binary operations.
\( \leq q + d - 1 \), which depend only on the distribution of the zeros and non-zeros in \( A \).

For further purposes we mention also the following facts proved in [2]. If \( A \) is any \( n \times n \) non-negative matrix, then

\[
C_{A}^{n+1} \subseteq C_{A} \cup C_{A}^{2} \cup \ldots \cup C_{A}^{m}.
\]

Hence the set \( C_{A} \cup C_{A}^{2} \cup \ldots \cup C_{A}^{m} \) is always a subsemigroup of \( S = S_{n} \).

A non-negative matrix \( A \) is called reducible if there is a permutation matrix \( P \) such that \( P^{-1}AP \) is of the form

\[
P^{-1}AP = \begin{pmatrix} A_{1} & 0 \\ B & A_{2} \end{pmatrix}.
\]

Otherwise it is called irreducible. An \( n \times n \) non-negative matrix \( A \) is irreducible if and only if

\[
C_{A} \cup C_{A}^{2} \cup \ldots \cup C_{A}^{m} = S_{n}.
\]

It should be mentioned in advance that in this paper the emphasis is rather on the reducible case.

Consider now the semigroup \( \mathcal{S}_{A} \) as given in (3). The elements of \( \mathcal{S}_{A} \) are subsets of \( S \). At least one of the elements \( e \in \mathcal{S}_{A} \) (namely \( C_{A}^{s} \)) is itself a subsemigroup of \( S \). The first problem treated in this paper concerns the following question. Under what conditions concerning \( A \) and \( s \) may it happen that the set \( C_{A}^{s} \) is a subsemigroup of \( S \). The second problem is to find a “good” characterization of the number \( d = \text{card} \ \mathcal{S}_{A} \). It will turn out that both questions are intimately connected.

I.

**Lemma 1.** Let \( A \) be any \( n \times n \) non-negative matrix. Suppose that \( C_{A}^{s} \) is a subsemigroup of \( S = S_{n} \). Then

a) \( C_{A}^{s} \subseteq C_{A}^{s} \);

b) \( C_{A}^{s} \) contains all idempotents \( e \in S \) contained in the union \( C_{A} \cup C_{A}^{2} \cup \ldots \cup C_{A}^{m} \).

**Proof.** a) The sequence

\[
C_{A}^{s}, C_{A}^{2s}, C_{A}^{3s}, \ldots
\]

contains a unique idempotent \( C_{A}^{s} \). Hence there is an integer \( v \) such that \( C_{A}^{sv} = C_{A}^{s} \). Since \( C_{A}^{s} \) is a semigroup, we have \( C_{A}^{s} \supseteq C_{A}^{2sv} \), which implies

\[
C_{A}^{s} \supseteq C_{A}^{2sv} \supseteq C_{A}^{3sv} \supseteq \ldots \supseteq C_{A}^{nsv} = C_{A}^{s}.
\]

b) Let \( E_{A} = \{ e_{x} \} \) running through a subset of \( N \) be the set of all non-zero idempotents \( e \in S \) contained in \( C_{A} \cup C_{A}^{2} \cup \ldots \cup C_{A}^{m} \). If \( e_{x} \in C_{A}^{h} \) (\( 1 \leq h \leq n \)),
then \( e_{ax} \in C^h_A \) for any integer \( t \geq 1 \). Since some power of \( C^h_A \) is \( C^n_A \), we have \( e_{ax} \in C^n_A \), hence \( E_A \subset C^n_A \subset C^3_A \).

**Theorem 1.** The group \( \mathcal{G}_A = \{ C^1_A, \ldots, C^{k+1}_A \} \) contains exactly one element which is itself a subsemigroup of \( S \).

**Remark.** This is — of course — the idempotent \( C^n_A \).

**Proof.** Suppose that \( C^s_A, k \leq s \leq k + d - 1 \) is a semigroup. By Lemma 1 we have \( C^n_A \subset C^s_A \). Multiplying by \( C^n_A \) we have \( C^n_A . C^s_A \subset C^{2s}_A \subset C^s_A \). But since \( C^n_A \) is the unit element in \( \mathcal{G}_A \), \( C^n_A . C^s_A = C^s_A \). Now \( C^n_A \subset C^{2s}_A \subset C^s_A \) implies \( C^n_A = C^s_A \), i.e. \( C^n_A \) is an idempotent contained in \( \mathcal{G}_A \). Hence \( C^n_A = C^n_A \), q.e.d.

**Remark.** If \( k > 1 \), the set \( \{ C^1_A, C^2_A, \ldots, C^{k-1}_A \} \) may contain subsemigroups of \( S \). Let \( f.i. \) \( A \) be a non-negative \( 3 \times 3 \) matrix with the support (in an obvious notation(2))

\[
C_A = \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 1 & 0
\end{pmatrix}
\]

Then

\[
C^2_A = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

and \( C^3_A = \{ \emptyset \} \). Hence all elements \( C_A, C^2_A, C^3_A \) are subsemigroups of \( S_3 \).

**Theorem 2.** Let \( A \) be a non-negative \( n \times n \) matrix for which \( C_A \cup C^2_A \cup \ldots \cup C^n_A \) contains all non-zero idempotents \( e \in S \), i.e. the set \( E_A = \{ e_{11}, e_{22}, \ldots, e_{nn} \} \). Then \( \mathcal{G}_A \) contains exactly one element that is itself a subsemigroup of \( S \).

**Proof.** Let be \( 1 \leq s \leq k + d - 1 \) and \( C^s_A \) a subsemigroup of \( S \). By Lemma 1 we have \( \{ e_{11}, \ldots, e_{nn} \} \subset C^s_A \). If \( A \) is any subset of \( S \) we always have \( A \{ e_{11}, \ldots, e_{nn} \} = A \). In particular (in our case) we have

\[
C^s_A = C^s_A \{ e_{11}, \ldots, e_{nn} \} \subset C^{2s}_A.
\]

The "inequalities" \( C^s_A \subset C^{2s}_A \) and \( C^{2s}_A \subset C^{s}_A \) (describing the semigroup property of \( C^s_A \)) imply \( C^s_A = C^{2s}_A \). Since there is a unique idempotent \( e \in \mathcal{G}_A \) we have \( C^{s}_A = C^{n}_A \), q.e.d.

If \( S \) is irreducible, then \( C_A \cup C^2_A \cup \ldots \cup C^n_A = S \), so that the suppositions of Theorem 2 are satisfied and we obtain:

**Corollary 1.** If \( A \) is irreducible, then \( C^n_A \) is the unique element \( e \in \mathcal{G}_A \) which is itself a subsemigroup of \( S \).

(2) We shall occasionally use this obvious notation by putting 1 on those places \((i, k)\) for which \( e_{ik} \in C_A \). F.i. in our example the "Boolean matrices" \( C_A, C^2_A, C^3_A \) denote \( C_A = \{ \emptyset, e_{21}, e_{31}, e_{32} \}, C_{A}^{2} = \{ \emptyset, e_{21} \}, C_{A}^{3} = \{ \emptyset \} \).
**Corollary 2.** If \( A \) is any \( n \times n \) non-negative matrix and \( C_A^s \) is a semigroup containing \( \{e_{11}, \ldots, e_{nn}\} \), then \( C_A^s = C_A^0 \).

Proof. By supposition \( C_A^s = C_A^s(e_{11}, \ldots, e_{nn}) \subset C_A^{2s} \). On the other hand \( C_A^{2s} \subset C_A^s \), hence \( C_A^s = C_A^{2s} \); therefore \( C_A^s = C_A^0 \), q.e.d.

Remark. In Corollary 2 the supposition that \( C_A^s \) is a semigroup cannot be omitted. Let f.i. \( A \) be a \( 3 \times 3 \) matrix with

\[
C_A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.
\]

Then

\[
C_A^3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.
\]

contains \( \{e_{11}, e_{22}, e_{33}\} \), but \( C_A^3 \) is not the idempotent \( e \in \mathcal{G}_A \). (The idempotent \( e \in \mathcal{G}_A \) is \( C_A^0 \).)

The next two Lemmas will enable us to locate, so to say, the semigroups in the sequence (2) and to find at the same time a new characterization of the number \( d \).

**Lemma 2.** Let \( s \) be an integer such that \( C_A^s \) is a subsemigroup of \( S \). We then have:

a) \( C_A^0 = C_A^{s+1} \);

b) \( d \mid s \);

c) \( C_A^s \subset C_A^{s+td} \) for any integer \( t \geq 0 \).

Proof. a) We have \( C_A^s = C_A^s \in \mathcal{G}_A \). Further \( C_A^{s+1} \) is a subsemigroup of \( S \) since

\[
C_A^{2(s+1)} = C_A^s \cdot C_A^{2s} = C_A^s \cdot C_A^s \subset C_A^s \cdot C_A^s = C_A^{s+1}.
\]

Hence by Theorem 1 \( C_A^{s+1} = C_A^s \).

b) Suppose that \( d \nmid s \) and write \( s = ax + \beta \), where \( a \geq 0 \) is an integer and \( 0 < \beta < d \). Since for any integer \( a \) we have \( C_A^{s+1} = C_A^s \) the relation \( C_A^s = C_A^{s+1} \) implies

\[
C_A^s = C_A^{s+1} = C_A^{s+ad} C_A^\beta = C_A^a \cdot C_A^\beta = C_A^{s+\beta}.
\]

The relation \( C_A^s = C_A^{s+\beta} \) contradicts to the fact that the group \( \mathcal{G}_A = \{C_A^s, C_A^{s+1}, \ldots, C_A^{s+d-1}\} \) is of order \( d \).

c) By Lemma 1, we have \( C_A^s = C_A^s \), hence \( C_A^{s+td} \subset C_A^{s+td} \) and since \( C_A^{s+td} = C_A^s \), we obtain \( C_A^s \subset C_A^{s+td} \). This proves our Lemma.

**Lemma 3.** If \( C_A^s \) is a semigroup, then none of the sets \( C_A^{s+1}, C_A^{s+2}, \ldots, C_A^{s+d-1} \) can be a semigroup.

Proof. If \( C_A^{s+\lambda}, 1 \leq \lambda \leq d - 1 \), were a semigroup, then Lemma 3b) would imply that \( d \mid s \) and \( d \mid s + \lambda \), which is impossible.
Let \( s_0 \) be the least integer \( s \) such that \( C_s^A \) is a semigroup. Then \( s_0 \leq \varrho \) and we may arrange the set of powers in the following way:

\[
\begin{align*}
C_s^A, C_{s+1}^A, \ldots, C_{s+d-1}^A, C_{s+d}^A, \ldots, C_{s+2d-1}^A, \\
C_{s+2d}^A, C_{s+2d+1}^A, \ldots, C_{s+3d-1}^A, \\
C_{s+t}^A, \ldots, C_{s+d}^A.
\end{align*}
\]

Since \( d \mid \varrho \) and \( d \mid s_0 \) there is necessarily an integer \( t \) such that \( \varrho = s_0 + td \). We get exactly \( t + 1 \) rows. The last of them contains at least one element \( \in \mathcal{G}_A \) which does not occur in the foregoing row. (This means: It may happen that to obtain all different elements \( \in \mathcal{G}_A \) it is not necessary to consider the whole last row, but certainly at least the first element contained in it.)

The idempotent \( C_{s_0}^A \) is necessarily contained in the column \( \{C_{s_0}^A, C_{s_0+d}^A, \ldots\} \) and (by Lemma 2c) \( C_{s_0}^A \) is a subset of each element of this column.

Also (by Lemma 2b) all elements \( \in \mathcal{G}_A \) which are themselves subsemigroups of \( S \) are located in the column \( \{C_{s_0}^A, C_{s_0+d}^A, C_{s_0+2d}^A, \ldots, C_{s_0}^A\} \). Hence the semigroups contained in the sequence (2) are some of the powers

\[
C_{s_0}^A, C_{s_0+d}^A, \ldots, C_{s_0+(t-1)d}^A
\]

and all the following

\[
C_{s_0}^A = C_{s_0+d}^A = C_{s_0+(t+1)d}^A = C_{s_0+(t+2)d}^A = \ldots
\]

Now since \( d \mid s_0 \), the number \( d \) is the greatest common divisor of the sequence of integers

\[
s_0, s_0 + d, s_0 + 2d, \ldots
\]

We have proved:

**Theorem 3.** The number \( d = \text{card} \mathcal{G}_A \) is the greatest common divisor of all such integers \( s \) for which \( C_s^A \) is a semigroup (subsemigroup of \( S \)).

We make some supplementary remarks to the "tableau" (5).

**Remark 1.** None of the sets \( C_{s_0}^A, \ldots, C_{s_0+(t-1)d}^A \) is contained as a proper subset in another, i.e. \( C_{s_0}^A \subseteq C_{s_0+v}^A \) implies \( C_{s_0}^A = C_{s_0+v}^A \).

**Proof.** We first prove that \( C_s^A \subseteq C_{s+u}^A, 0 \leq u \leq d - 1 \), implies \( C_s^A = C_{s+u}^A \).

Note that by Lemma 2 a \( C_s^A = C_{s+\lambda s_0}^A \) for any integer \( \lambda \geq 0 \). The relation \( C_s^A \subseteq C_{s+u}^A \) implies

\[
C_s^A \subseteq C_{s+u}^A \subseteq C_{s+2u}^A \subseteq \ldots \subseteq C_{s+\lambda s_0}^A = C_s^A.
\]

Hence \( C_s^A = C_{s+u}^A \). Suppose now

(6)

\[
C_{s+u}^A \subseteq C_{s+v}^A
\]

for some \( u, v \geq 0 \). Since \( C_{s+u}^A \in \mathcal{G}_A \), there is a \( C_{s+w}^A \) such that \( C_{s+u}^A \cdot C_{s+w}^A = C_s^A \). Here \( u + w \equiv 0 \) (mod \( d \)). Multiplying (6) by \( C_{s+w}^A \) we have \( C_s^A \subseteq C_{s+w}^A \)
\[ C_A^{v+u'}, \text{ hence } C_A^n = C_A^{v+u'}, \text{ so that } v + u' \equiv 0 \pmod{d}. \] Therefore \( u - v \equiv 0 \pmod{d} \) and \( C_A^{v+u} = C_A^{v+u}, \text{ q.e.d.} \)

Remark 2. The statement just proved implies that none of the elements \( C_A^n, C_A^{n+1}, \ldots, C_A^{n+d-1} \) can be contained (as a proper subset) in another. For \( C_A^{n+i} \subseteq C_A^{n+l}, \ 0 \leq i, l \leq d - 1, \ i \neq l \) multiplied by \( C_A^d \) would imply \( C_A^{n+i} \subseteq C_A^{n+l} \), i.e. \( C_A^{n+i} \subseteq C_A^{n+l} \), hence \( C_A^{n+i} = C_A^{n+l} \), which is not true. An analogous statement holds for the remaining rows.

Remark 3. In [2] we have proved that for an irreducible matrix the intersection \( T _ A = C_A \cap C_A^n \cap \ldots \cap C_A^{n+d-1} \) is \( \{0\} \). [Even the intersection of any two of these sets is \( \{0\} \).] This is not necessarily true in the case of a reducible matrix. Consider f.i. a \( 3 \times 3 \) matrix \( A \) with \( C_A = \{e_{12}, e_{21}, e_{33}, 0\} \). Then \( C_A^2 = \{e_{11}, e_{22}, e_{33}, 0\} \) and \( G_A = \{C_A, C_A^2\} \). Here \( T_A = C_A \cap C_A^2 = \{e_{33}, 0\} \).

But it is easy to show that \( T_A \) is always a subsemigroup of \( S \). For let be \( a \in T_A, b \in T_A \). Then \( a \in C_A^{n+k} \) for any \( k = 0, 1, \ldots, d - 1 \) and \( b \in C_A^{n+l} \) for any \( l = 0, 1, \ldots, d - 1 \). Hence \( ab \in C_A^{n+k+l} \). If \( k, l \) run through a residue system \( \pmod{d} \) so does \( k + l \) so that \( ab \in \bigcap_{m=0}^{d-1} C_A^{n+m} \); hence \( ab \in T_A \), q.e.d.

Remark 4. For an irreducible matrix \( A \) we have \( s_0 = q \) and we always have \( C_A^d \subseteq C_A^n \). Again this is not necessarily true for a reducible matrix. This is shown on the following example. Let \( A \) be a matrix with

\[
C_A = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1
\end{pmatrix}.
\]

Here \( d = 1 \) and \( G_A \) is the one-point group \( G_A = \{C_A^2\} \), where

\[
C_A^2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1
\end{pmatrix}.
\]

We have \( s_0 = 2 \) and \( C_A \subset C_A^2 \) does not hold.

Example. We conclude this section with a simple example of a matrix with \( \operatorname{card} G_A > 1 \) and \( s_0 < q \). Let \( A \) be a matrix with

\[
C_A = \begin{pmatrix}
0 & 1 \\
1 & 0 \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
\hline
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0
\end{pmatrix}.
\]

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Then

\[
C^2_A = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 1 \\
1 & 0 \\
1 & 1
\end{pmatrix}, \quad C^3_A = \begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0
\end{pmatrix},
\]

\[
C^4_A = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{pmatrix}, \quad C^5_A = \begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}.
\]

Here \( \mathfrak{S}_A \) has 5 different elements, \( \mathfrak{S}_A = \{C^4_A, C^5_A\} \), \( d = 2, \ s_0 = 2 \), while \( \ell = 4 \).

II.

The result of Theorem 2 may be formulated in a somewhat other way by introducing the notion of the normal form of a non-negative matrix \( M \).

Let \( M \) be a non-negative matrix (of order \( n \)). It is well known that there is a permutation matrix \( P \) (of order \( n \)) such that \( PMP^{-1} = A \) is of the form

\[
A = \begin{pmatrix}
A_{11}, & 0, & \ldots, & 0 \\
A_{21}, & A_{22}, & \ldots, & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_{r1}, & A_{r2}, & \ldots, & A_{rr}
\end{pmatrix},
\]

where \( A_{ii} \) (\( 1 \leq i \leq r \)) are irreducible matrices (including the case that some of the \( A_{ii} \)'s may be zero matrices of order 1).

Consider the sequences

\[
C_M, C_M^2, C_M^3, \ldots
\]

(8)

\[
C_A, C_A^2, C_A^3, \ldots
\]

(9)

The semigroups \( \mathfrak{S}_A \) and \( \mathfrak{S}_M \) are clearly isomorphic. If \( C_M^e \) is a semigroup, then so is \( C_A^e \) since

\[
C_A^{2e} = C_FC_M^eC_{P^{-1}}, \quad C_FC_M^eC_{P^{-1}} = C_FC_M^eC_{P^{-1}} \subset C_FC_M^eC_{P^{-1}} = C_A^e,
\]

and conversely. In particular, if \( C_M^e \) is the idempotent \( \in \mathfrak{S}_M \), then \( C_FC_M^eC_{P^{-1}} \) is the idempotent \( C_A^e \in \mathfrak{S}_A \), so that \( \varrho(A) = \varrho(M) \). Hence instead of studying the sequence (8) we may restrict ourselves to the study of the sequence (9).

We shall use the following notations. \( d_\ell \) will denote the order of the group \( \mathfrak{G}_{A_\ell} \), \( \ell_1 \) will denote the least integer for which \( C_{A_\ell}^{\ell_1} \) is an idempotent \( \in \mathfrak{S}_{A_\ell} \).

If \( C_{A_\ell}^{\ell_1} \) is the idempotent \( \in \mathfrak{S}_{A_\ell} \), then \( C_{A_\ell}^{\ell_1} \) is necessarily the idempotent \( \in \mathfrak{S}_{A_\ell} \).
If $\varrho = \varrho(A)$ has the meaning introduced from the beginning (i.e. the smallest integer for which $C_eA$ is an idempotent $e\varrho A$), then $\varrho$ is necessarily of the form $Q = Q_1 + \ldots + d_r = 0$ with suitably chosen non-negative integers $x_1, x_2, \ldots, x_r$. Since $q_i = \tau d_i$, we have $\varrho = d_i(\tau_1 + x_i)$, $i = 1, 2, \ldots, r$. Denote $d* = [d_1, d_2, \ldots, d_r]$ the least common multiple of the integers $d_1, \ldots, d_r$. The relation $d_i | \varrho$ implies $d* | \varrho$. We have proved: there is an integer $\tau*$ such that $\varrho(A) = \tau*d*$.

In what follows it is often of decisive importance whether in the normal form (7) there is among the $A_{ii}'s$ a zero matrix (of order 1) or not. If none of the $A_{ii}'s$ is a zero matrix, then

$$C_v^\circ = C_v^*d* \subseteq C_A \cup C_A^2 \cup \ldots \cup C_A^n$$

contains $\{e_{11}, e_{22}, \ldots, e_{nn}\}$. With respect to Theorem 2 we have

**Theorem 4.** If a matrix $A$ written in the normal form (7) has no zero matrix in the main diagonal, then $C_v^\circ$ is the unique semigroup contained in the sequence (9).

The condition mentioned in this Theorem is not necessary. There are classes of non-negative matrices with zeros in the main diagonal having the same property. We prove f.i.:

**Theorem 5.** Let

$$A = \begin{pmatrix} A_1 & 0 \\ R & 0 \end{pmatrix},$$

where $A_1$ is irreducible and not the zero matrix of order 1. Then $C_A^s$ is a semigroup if and only if it is the idempotent $e\varrho A$.

**Proof.** Let $A_1$ be a $m \times m$ matrix (so that $R$ is a $(n-m) \times m$ rectangular matrix). Denote $E = \{e_{11}, e_{22}, \ldots, e_{mm}\}$. The support of

$$A^s = \begin{pmatrix} A_1^s & 0 \\ RA_1^{s-1} & 0 \end{pmatrix}$$

is a semigroup if and only if

\begin{equation}
C_{A_1}^{2s} \subseteq C_{A_1}^s, \quad C(RA_1^2) \subseteq C(RA_1^{s-1}).
\end{equation}

Now $C_{A_1}^s$ is a semigroup if and only if $C_A^s = C_{A_1}^{2s}$ is the idempotent $e\varrho A_1$ and $C_{A_1}^s$ contains then $E$. Hence we have

$$C_R = C_R \cdot \{e_{11}, e_{22}, \ldots, e_{mm}\} \subseteq C(RA_1^1).$$

Now if $C_A^s$ is a semigroup, (10) implies

$$C(RA_1^{s-1}) \supseteq C(RA_1^{2s-1}) = C(RA_1)C(A_1^{s-1}) \supseteq C(R)C(A_1^{s-1}) = C(RA_1^{s-1}).$$

Hence $C(RA_1^{s-1}) = C(RA_1^{2s-1})$. Therefore $C_A^s = C_A^{2s}$, q.e.d.

Theorem 5 may be generalized as follows:
Theorem 6. Let

\[ A = \begin{pmatrix} A_1 & 0 \\ R & A_2 \end{pmatrix}, \]

with \( A_1 \) irreducible and not the zero matrix of order 1. If \( C_A^s \) is a semigroup, then \( C_A^s \) is the idempotent \( e \in \mathcal{S}_A \) if and only if \( C_A^{2s} = C_A^{2s} \).

Proof. Denote

\[ A^s = \begin{pmatrix} A_1^s & 0 \\ R^s & A_2^s \end{pmatrix} \]

and \( R_1 = R \). Then

\[ A^{2s} = \begin{pmatrix} A_1^{2s} & 0 \\ R^s A_1^s + A_2^s R^s & A_2^{2s} \end{pmatrix}. \]

The set \( C_A^s \) is a semigroup if and only if

\[ C_A^{2s} \subset C_A^s, \quad C_A^{2s} \subset C_A^s, \]

\[ C(R^s A_1^s) \cup C(A_2^s R^s) \subset C(R^s). \]

Since \( A_1 \) is irreducible, we conclude \( C_A^{2s} = C_A^s \), and the diagonal of \( C_A^s \) is positive, i.e. if \( A_1 \) is a \( m \times m \) matrix, we have \( \{e_{11}, e_{22}, \ldots e_{mm}\} \subset C_A^s \), so that \( C(R^s) = C(R^s)\{e_{11}, \ldots, e_{mm}\} \subset C(R^s)C(A_1^s) \). The relation

\[ C(R^s A_1^s) \cup C(A_2^s R^s) \subset C(R^s) \subset C(R^s A_1^s) \]

implies

\[ C(R^s A_1^s) \cup C(A_2^s R^s) = C(R^s) = C(R^s A_1^s). \]

Therefore \( C(A^s) = C(A^{2s}) \) if and only if \( C(A_2^s) = C(A_2^{2s}) \), q.e.d.

III.

In this last section we shall deal with some special types of matrices for which \( \text{card} \mathfrak{G}_A = 1 \).

Let \( A \) be the matrix of the form (7). The question arises what can be said about \( \text{card} \mathfrak{G}_A \) by knowing \( \text{card} \mathfrak{G}_A = d_1 \).

The following Lemma holds.

Lemma 4. If \( d^* = [d_1, \ldots, d_r] \), then \( \text{card} \mathfrak{G}_A = d^* \).

The proof of this Lemma (which has been known to the author for some time) is given in the recent paper of Ю. И. Лубич (Ju. I. Ljubič) [see [1], Lemma 2, p. 344].

A non-negative irreducible matrix \( A \) is called primitive if some power of \( A \) is positive. This is the case if and only if \( d(A) = 1 \). In this case \( \mathfrak{G}_A \) is a one-point group, namely the idempotent \( e \in \mathcal{S}_A \).
If \( A \) is reducible of the form (7) then Lemma 4 implies \( \text{card } \mathfrak{G}_A = 1 \) if and only if \( d_1 = d_2 = \ldots = d_r = 1 \). Hence:

**Theorem 7.** If \( A \) is of the form (7), then \( \mathfrak{G}_A \) is a one point group if and only if the matrices \( A_n \) are either primitive or zero matrices of order 1.

**Remark.** There are some special cases in which we may decide that \( \mathfrak{G}_A \) is a one-point group without reference to the normal form (7).

**Assertion 1.** If \( C_A \) is a semigroup, then \( \text{card } \mathfrak{G}_A = 1 \).

**Proof.** By Lemma 2 \( d = d(A) \) divides every \( s \) for which \( C_A^s \) is a semigroup. Since in our case we may put \( s = 1 \), we conclude \( d = 1 \).

**Assertion 2.** If \( A \) is any non-negative \( n \times n \) matrix and \( C_A \) contains \( E = \{ e_{11}, \ldots, e_{nn} \} \), then \( \text{card } \mathfrak{G}_A = 1 \).

**Proof.** By supposition \( C_A = C_A \cdot E \subset C_A C_A = C_A^2 \). Hence \( C_A \subset C_A^2 \subset C_A \subset \ldots \subset C_A^n \subset C_A^{n+1} \). On the other hand we always have \( C_A^{n+1} \subset C_A \cup C_A^2 \cup \ldots \cup C_A^n \), i.e. \( C_A^{n+1} \subset C_A^n \). Hence \( C_A^n = C_A^{n+1} \). This implies that \( C_A^n \) is the idempotent \( \in \mathfrak{G}_A \) and, moreover, \( \text{card } \mathfrak{G}_A = 1 \).

A special class of matrices with \( d(A) = 1 \) is the class of lower triangular non-negative matrices, i.e. matrices of the following form:

\[
A = \begin{pmatrix}
  a_{11}, & 0, & 0, & \ldots, & 0 \\
  a_{21}, & a_{22}, & 0, & \ldots, & 0 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  a_{n1}, & a_{n2}, & a_{n3}, & \ldots, & a_{nn}
\end{pmatrix}
\]

where \( a_{ik} \) (for \( i \geq k \)) are non-negative elements, while all elements above the main diagonal are zeros.

**Theorem 8.** For a lower triangular non-negative matrix \( A \) of order \( n \) the set \( C_A^n \) is the idempotent \( \in \mathfrak{G}_A \).

**Proof.** a) We first prove that \( C_A^n \subset C_A^{n+1} \). Any element \( \alpha \in C_A^n \) is the product of \( n \) elements \( e_{i_1} e_{i_2} \ldots e_{i_n} \). Such a product is certainly zero if the subscripts do not follow in the following order

\[
(\text{12}) \quad (i_1, i_2), (i_2, i_3), \ldots, (i_n, i_{n+1}).
\]

Suppose \( \alpha \neq 0 \). Then by supposition we have \( i_1 \geq i_2 \geq \ldots \geq i_n \geq i_{n+1} \). The integers \( i_1, i_2, \ldots, i_{n+1} \) cannot be all different. There is therefore a couple, say \( i_j, i_{j+1} \), such that \( i_j = i_{j+1} \). The sequence (12) is of the form

\[
(\text{12}) \quad (i_1, i_2), (i_{j-1}, i_j), (i_j, i_j), (i_j, i_{j+2}) \ldots (i_n, i_{n+1})
\]

and \( \alpha \) may be written as the product

\[
\alpha = e_{i_2} \ldots e_{i_{j-1}} e_{i_j} e_{i_{j+1}} \ldots e_{i_{n+1}}
\]

But then we may write also
\[ \alpha = e_{i_1,i_1} \cdots e_{i_n,i_n} e_{i_{n+1},i_{n+1}}, \]

so that \( \alpha \in C^{n+1}_A \). Hence \( C^n_A \subset C^{n+1}_A \).

b) On the other hand if \( \alpha \in C^m_A \) and \( \alpha \neq 0 \), \( \alpha \) is of the form (13) and we may omit \( e_{i,j} \) in \( \alpha \) (without changing the value of \( \alpha \)) so that

\[ \alpha = e_{i_1,i_1} \cdots e_{i_{n+1},i_{n+1}}. \]

Hence \( C^n_A \subset C^{m-1}_A \).

The last relation implies \( C^{m+1}_A \subset C^m_A \). Both “inequalities” \( C^m_A \subset C^{m+1}_A \subset C^m_A \) imply \( C^n_A = C^{m+1}_A \) and \( C^m_A = C^{m+1}_A = \cdots = C^{2n}_A \), q.e.d.

Remark 1. The exponent \( n \) is sharp since for a matrix with \( n \) zeros along the main diagonal and all elements below the main diagonal equal to 1 we have \( C^{n-1}_A \neq 0 \), but \( C^n_A = 0 \).

Remark 2. Also the exponent \( n \) in the relation \( C^n_A \subset C^{m-1}_A \) (proved in b) cannot be in general replaced by a smaller one. Take f.i. the matrix \( A \) with

\[ C_A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \]

Then \( C^3_A \subset C^2_A \), but it is not true that \( C^2_A \subset C_A \), since \( C_A \not\subset C^2_A = \{e_{11}, e_{21}, e_{22}, e_{31}, e_{32}, 0\} \) holds.

**Theorem 9.** For a lower triangular matrix of the type (11) and \( n \geq 2 \) there is always a number \( s \leq n - 1 \) such that \( C^s_A \) is a semigroup.

**Proof.** In Theorem 8 we have proved \( C^{n-1}_A \subset C^n_A = C^{n+1}_A \). Since for \( n \geq 2 \) we have \( 2n - 2 \geq n \), we conclude \( C^{n-1}_A \subset C^{2(n-1)}_A \).

We now give a non-trivial generalization of Theorem 8 concerning a larger class of matrices with \( d(A) = 1 \).

**Theorem 10.** Let \( A \) be a matrix of the form

\[ A = \begin{pmatrix} A_{11}, 0, \ldots, 0 \\ A_{21}, A_{22}, \ldots, 0 \\ \vdots \\ A_{r1}, A_{r2}, \ldots, A_{rr} \end{pmatrix}, \]

where \( A_{ii} \) is either a positive square matrix or a zero matrix of order 1. Then \( C^{2r-1}_A \) is the idempotent \( \in \mathbb{S}_A \).

**Proof.** Denote — for typographical reasons — \( C(A_{ij}) \) by \( C_{ij} \).

We first prove that \( C_{q_0} \subset C_{r_0} = 0 \) for \( \sigma \neq \tau \). Let \( n_1 \) be the order of \( A_{ij} \). Then, if \( e_{q_0} e_{q_0} \in C_{q_0} \), we have \( n_1 + \cdots + n_{k-1} < \sigma_0 \leq n_1 + \cdots + n_{\sigma} \). If \( e_{r_0} e_{r_0} \in C_{r_0} \), we have \( n_1 + \cdots + n_{r-1} < \tau_0 \leq n_1 + \cdots + n_{\tau} \). If \( \sigma > \tau \), then \( \tau_0 \leq n_1 + \cdots + n_{\tau} \leq n_1 + \cdots + n_{\sigma-1} < \sigma_0 \), hence \( \sigma_0 \neq \tau_0 \), and \( e_{q_0} e_{r_0} = 0 \). If \( \sigma < \tau \),
then \( \sigma_0 \leq n_1 + \ldots + n_{\sigma} \leq n_1 + \ldots + n_{\tau - 1} < \tau_0 \), hence \( \sigma_0 \neq \tau_0 \), and \( e_{\theta, \sigma} e_{\tau, \lambda} = \theta \). Therefore the product \( C_{\theta} C_{\tau} \) can be different from zero only if it is of the form \( C_{\theta} C_{\sigma, \lambda} \) (and of course \( \varrho \geq \sigma \geq \lambda \)).

We shall now study the behaviour of the powers of \( C_A = \bigcup_{i > j} C_{ij} \).

The set \( C_A \) is a union of products of the form \( C_{i_1, i_2} C_{j_1, j_2} \ldots C_{w_1, w_2} \). Such a product can be non-zero only if the subscripts follow in the order indicated in the product

\[
C_{i_1, i_2} C_{i_3, i_2} \ldots C_{i_r, i_1}.
\]

Suppose that this product is non-zero. Since \( i_1 \geq i_2 \geq \ldots \geq i_{r+1} \), there is necessarily a couple, say \( i_j, i_{j+1} \), such that \( i_j = i_{j+1} \), and each of the non-zero summands in the set \( C_{ij} \) is of the form

\[
C_{i_{ij}, i_{ij}} C_{i_{ij+1}, i_{ij+1}} \ldots C_{i_{ij+r}, i_{ij+r}}.
\]

But since \( C_{ij}^2 = C_{ij} \) (and \( C_{ij} \) is not zero) this is the same as

\[
C_{i_1, i_2} C_{i_3, i_2} \ldots C_{i_{r+1}, i_1},
\]

which belongs to the set \( C_{r+1} \). Hence \( C_{r+1} \subset C_{r+1} \).

We next show that \( C_{2r} \subset C_{2r-1} \). Each non-zero summand of \( C_{2r} \) is of the form

\[
C_{i_1, i_2} \ldots C_{i_{2r}, i_{2r}}.
\]

The non-increasing sequence of \( 2r + 1 \) integers

\[
i_1 \geq i_2 \geq \ldots \geq i_j \geq i_{j+1} \ldots \geq i_{2r+1}
\]

contains at most \( r \) integers different one from the other. Hence there must be at least one triple such that \( i_j = i_{j+1} = i_{j+2} \). (For if each of the \( r \) numbers appeared at most twice, the system would contain at most \( 2r \) members.) Hence any non-zero summand of \( C_{2r} \) may be written in the form

\[
C_{i_1, i_2} \ldots C_{i_{r+1}, i_{r+2}} C_{i_{r+3}, i_r} \ldots C_{i_{2r}, i_{2r+1}}.
\]

Now since \( C_{ij}^2 = C_{ij} \), this product is yet contained in \( C_{2r-1} \). Hence \( C_{2r} \subset C_{2r-1} \).

Now the relation \( C_{r} \subset C_{r+1} \) implies \( C_{2r-1} \subset C_{2r} \). This combined with \( C_{2r} \subset C_{2r-1} \) gives \( C_{2r-1} = C_{2r} \), which proves our Theorem. (By the way the last result proves again that \( G_A \) is a one-point group.)

Remark 1. In general the exponent \( 2r - 1 \) cannot be replaced by a smaller one. This is shown on the following example. Let \( A \) be a matrix with

\[
C_A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad C_A^2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.
\]
Here \( r = 2 \), \( C_A^2 \) is not an idempotent, while \( C_A^3 \) is the idempotent \( \in S_A \).

Remark 2. This example shows at the same time that it is in general not true that \( C_A^r \subset C_A^{r-1} \) as one could expect by analogy with the proof of Theorem 8. On the other hand we cannot prove \( C_A^{r-1} \subset C_A^r \) since, for instance, for the matrix \( A \) with \( C_A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) we have \( r = 2 \) and \( C_A \nsubseteq C_A^2 = \{0\} \).

The next theorem gives an information concerning the semigroups in the sequence
\[
C_A, C_A^2, C_A^3, \ldots ,
\]
with \( A \) given by (14).

**Theorem 11.** If \( C_A^r \) is not a semigroup, then the sequence (15) contains a unique subsemigroup of \( S_n \) (namely the idempotent \( C_A^3 \in S_A \)). If \( C_A^r \) is a semigroup, then it is at the same time the idempotent \( \in S_A \) and (15) contains at most \( r \) different elements.

$$
\text{Proof. Let } s_0 \text{ be the least integer for which } C_A^{s_0} \text{ is a semigroup.}
$$

(a) Let first \( s_0 > r \). Since \( C_A^r \subset C_A^{r+1} \), we have \( C_A^r \subset C_A^{r+1} \subset \ldots \subset C_A^{s_0} \subset C_A^{2s_0} \). The semigroup property implies \( C_A^{2s_0} \subset C_A^{s_0} \). Hence \( C_A^{s_0} = C_A^{2s_0} \) and the idem-

(b) Let \( s_0 \leq r \). Then \( C_A^{2s_0} \subset C_A^{s_0} \) implies (multiplied by \( C_A^{r-s_0} \)) \( C_A^{s_0+r} \subset C_A^r \). But \( C_A^r \subset C_A^{r+1} \) implies \( C_A^r \subset C_A^{r+s_0} \). Hence \( C_A^{s_0+r} = C_A^r \). Now a power of \( C_A \) which occurs in the sequence (15) more than once is contained in \( S_A \). Since \( S_A \) is a one-point group, we conclude that \( C_A^r \) is the idempotent \( \in S_A \). Moreover in this case the sequence (15) has at most \( r \) different members.

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Received June 24, 1964.