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ON COMPLETE IDEALS IN SEMIGROUPS

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1.

R. Croisot introduced in [1] the following condition: An element $a$ of a semigroup $S$ satisfies the Condition $(m, n)$ if there exists an element $x \in S$ such that

$$a = a^m x a^n.$$ 

Here $m, n$ are non-negative integers, and $a^0$ means the void symbol. The set of all elements, satisfying the Condition $(m, n)$ is called a class of regularity and will be denoted by $R_S(m, n)$. (See [2]). By means of this notion some properties of semigroups have been studied. In this paper we show how these classes of regularity are connected with so-called complete ideals. For relations, which hold between the classes of regularity see [2] (p. p. 111—112). If all elements of a semigroup $S$ satisfy the Condition $(m, n)$ we shall write $S = R_S(m, n)$.

2.

**Definition 1.** We shall say that a left (right, two-sided) ideal $L(R, M)$ of a semigroup $S$ is complete if $SL = L (RS = R, SM = MS = M)$.

In the following we shall treat only left complete ideals. The case of right complete ideals is analogous.

**Remark 1.** Evidently: A left ideal $L$ of a semigroup $S$ is complete if for any $a \in L$ there exist $x \in S, b \in L$ such that

$$xb = a.$$ (1)

**Theorem 1.** The set union of two complete left ideals of a semigroup $S$ is a complete left ideal of $S$.

**Proof.** Let $L_1, L_2$ be two complete left ideals of $S$. Then $SL_1 = L_1, SL_2 = L_2$. Hence $S(L_1 \cup L_2) = SL_1 \cup SL_2 = L_1 \cup L_2$, which proves our assertion.
The question arises, whether the intersection of two left complete ideals is a left complete ideal. The next example gives a negative answer.

**Example 1.** Let \( S = \{a, b, c, d\} \) be a semigroup with the multiplication table.

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\( L_1 = \{a, b, c\}, L_2 = \{a, b, d\} \) are complete left ideals of \( S \), but \( L_1 \cap L_2 = L_3 = \{a, b\} \) is not a complete left ideal of \( S \).

**Example 2.** A left ideal \( I \) of a semigroup \( S \) is called minimal if there exists no left ideal of \( S \) properly contained in \( I \). Evidently, every minimal left ideal of a semigroup \( S \) is a complete left ideal of \( S \).

**Theorem 2.** Every left ideal of a semigroup \( S \) is a complete left ideal of \( S \) if and only if \( S = \mathbb{R}_s(0, 1) \).\(^{(1)}\)

**Proof.** (a) Let \( L = \bigcup_{a \in L} a \) be a left ideal of \( S \). Then \( SL \supseteq \{ \bigcup_{a \in L} x_a \} \cdot \{ \bigcup_{a \in L} a \} \supseteq \bigcup_{a \in L} a = \bigcup_{a \in L} a = L \). On the other hand since \( L \) is a left ideal, \( SL \subseteq L \). Hence \( SL = L \).

(b) Let every left ideal of \( S \) be complete. Let \( a \in S \) be any element of \( S \). The left ideal \( a \cup Sa \) satisfies \( S(a \cup Sa) = a \cup Sa \), i.e. \( Sa \cup S^2a = a \cup Sa \), hence \( Sa = a \cup Sa \). Therefore \( a \in Sa \), which proves that \( S = \mathbb{R}_s(0, 1) \).

**Remark 2.** Clearly the following assertions hold.

(a) If \( S \) contains a left unit, then every left ideal is complete.

(b) \( S = \mathbb{R}_s(0, 1) = \mathbb{R}_s(1, 0) \), if and only if every left, right and two-sided ideal of \( S \) is complete.

(c) If \( S = \mathbb{R}_s(1, 1) \), then every ideal of \( S \) is complete.

(d) If all left ideals of \( S \) are complete, then \( S^2 = S \).

The next example of a semigroup shows that the converse of the assertion (d) need not hold.

**Example 3.** Let \( S \) be an additive semigroup of positive numbers. Then \( S^2 = S \). Let \( L = \langle a, \infty \rangle \) with \( a > 0 \). Then \( SL = \langle a, \infty \rangle \subset \langle a, \infty \rangle \), so that \( L \) is not complete.

**Remark 3.** If not every left ideal of a semigroup \( S \) is complete, then essentially less can be said about this semigroup. This statement holds:

If \( L \subseteq \mathbb{R}_s(0, 1) \), where \( L \) is a left ideal of a semigroup \( S \), then \( L \) is a complete left ideal of \( S \).

\(^{(1)}\) Similar questions are studied in [4].
Proof. The statement follows from the assumption and from part (a) of the proof of Theorem 2.

The semigroup in example 3 shows that the converse is not true. It is sufficient to take $L = (a, \infty)$. It can be easily shown that $L$ is complete but $L \subset \mathcal{R}_d(0, 1)$ does not hold.

3.

Let $\{S_i\}, i \in I$ be an arbitrary system of semigroups. Denote by $S$ the set of all functions $\xi$, defined on $I$ such that $\xi(i) \in S_i$. Introduce in $S$ a multiplication in this way: If $\alpha, \beta \in S$ are arbitrary elements of $S$, then the product $\gamma = \alpha \cdot \beta$ is given by $\gamma(i) = \alpha(i) \cdot \beta(i)$ (for every $i \in I$). The set $S$ with this multiplication is a semigroup, which is called a direct product of semigroups $\{S_i\}, i \in I$, and is denoted by $S = \prod_{i \in I} S_i$.

If $L_i$ is a left ideal of the semigroup $S_i, i \in I$, then $L = \prod_{i \in I} L_i$ is a left ideal of the semigroup $S = \prod_{i \in I} S_i$. (See [3]).

Let us put the question, whether the completeness of left ideals $L_i$ in $S_i, i \in I$, implies the completeness of a left ideal $L = \prod_{i \in I} L_i$ in $S = \prod_{i \in I} S_i$.

**Theorem 3.** Let $L_i$ be for every $i \in I$ a complete left ideal of the semigroup $S_i$. Then $L = \prod_{i \in I} L_i$ is a complete left ideal of $S = \prod_{i \in I} S_i$.

**Proof.** Let $L_i$ be a complete left ideal of a semigroup $S_i$, hence $S_i L_i = L_i$. We have to prove that for any $\mu \in L$, there exist $v \in L$ and $\alpha \in S$ such that $\alpha \cdot v = \mu$.

Since $L_i$ is a complete left ideal of $S_i$, there exist for every $\mu(i) = a_i \in L_i$ two elements $b_i \in L_i$ and $x_i \in S_i$ such that $x_i b_i = a_i$.

The functions $v, \alpha$ defined by $v(i) = b_i$, $\alpha(i) = x_i$ satisfy

$\alpha \cdot v = \mu$.

This proves our statement.

Let $N \subseteq S = \prod_{i \in I} S_i$. The set of all elements $x_i \in S_i$ for which there exists at least one element $\xi \in N$ such that $\xi(i) = x_i$ will be denoted by $P_t(N)$ and called the projection of the set $N$ into the semigroup $S_t$.

**Theorem 4.** Let $L$ be a complete left ideal of a semigroup $S = \prod_{i \in I} S_i$. Then
(a) $P_i(L)$ is a complete left ideal of $S_i$.
(b) $\prod_{i \in I} P_i(L)$ is a complete left ideal of $S$.

Proof. (a) Let $L$ be a complete left ideal of $S = \prod_{i \in I} S_i$. The fact that the $P_i(L)$ is a left ideal of $S_i$ is known from [3]. It is only necessary to prove that it is complete. Let $a_t \in P_i(L)$. To prove that $P_i(L)$ is a complete left ideal, it is sufficient to show that there exist $b_t \in P_i(L)$, and $x_t \in S_i$ such that
\[ x_t b_t = a_t. \]
Since $a_t \in P_i(L)$, it follows that there exists an element $\mu \in L$ such that $\mu(i) = a_t$. Since $L$ is a complete left ideal of $S = \prod_{i \in I} S_i$, there exist elements $v \in L$, and $\alpha \in L$, and $\alpha \in S$ such that
\[ \alpha . v = \mu. \]
This means that for every $i \in I$, we have $\alpha(i) . v(i) = \mu(i)$, where $\mu(i) = a_t$, $v(i) = b_t \in P_i(L)$ and $\alpha(i) = x_t \in S_i$. Therefore, we have
\[ x_t b_t = a_t. \]
This proves (a).

(b) The statement (b) follows from (a) and Theorem 3.

**Theorem 5.** A semigroup $S = \prod_{i \in I} S_i$ satisfies the Condition $(m, n)$ if and only if each of the semigroups $S_i$ satisfies this Condition.

Proof. (a) Let us assume that every semigroup $S_i$ satisfies Condition $(m, n)$. Let $\alpha \in S$ be an arbitrary element. Then $\alpha(i) = a_t \in S_t$ for every $i \in I$. Since $S_t$ satisfies Condition $(m, n)$, there exists an $x_t \in S_t$ such that
\[(*) \quad a_t = a_t^m x_t a_t^n.\]
Define $\eta \in S$ by the requirement that $\eta(i) = x_t$, for every $i \in I$. The relation (*) can be written in the form $\alpha(i) = [\alpha(i)]^m . \eta(i) . [\alpha(i)]^n$, for every $i \in I$. This means
\[ \alpha = \alpha^m . \eta . \alpha^n. \]
But the last relation says that $S = \prod_{i \in I} S_t$ satisfies Condition $(m, n)$.

(b) Let $S = \prod_{i \in I} S_t$ satisfy Condition $(m, n)$. Let $a_t \in S_t$ be an arbitrary element. Then there exists at least one element $\alpha \in S$ such that $\alpha(i) = a_t$. Since $S$ satisfies Condition $(m, n)$, there exists an element $\eta \in S$ such that
\[ \alpha = \alpha^m . \eta . \alpha^n. \]
Hence for our $i$
\[ a_i = a_i^\mu x_i a_i^n. \]
This means that $S_i$ satisfies Condition $(m, n)$.

Theorems 2 and 5 imply:

**Corollary 1.** Every left ideal of the semigroup $S = \prod_{i \in I} S_i$ is complete if and only if every left ideal of the semigroup $S_i (i \in I)$ is complete.

**Corollary 2.** The following statements are equivalent:
(a) Each of the semigroups $S_i (i = I)$ satisfies Condition $(0,1)$.
(b) The semigroup $S = \prod_{i \in I} S_i$ satisfies Condition $(0,1)$.
(c) Every left ideal of $S_i (i \in I)$ is complete.
(d) Every left ideal of $S = \prod_{i \in I} S_i$ is complete.

**Proof.** (a) $\Rightarrow$ (b) according to Theorem 5. (b) $\Rightarrow$ (c) according to Theorem 5 and Theorem 2. (c) $\Rightarrow$ (d) according to Corollary 1. (d) $\Rightarrow$ (a) according to Corollary 1 and Theorem 2.

A left ideal $L$ of a semigroup $S$ is called semiprime if for every element $a \in S$ and an arbitrary integer $n$ the relation $a^n \in L$ implies $a \in L$.

In [2] (p. 241) it is proved that every left ideal of a semigroup $S$ is a semiprime ideal if and only if $S$ satisfies Condition $(0,2)$.

**Theorem 6.** Let $L_i$ be a left semiprime ideal of $S_i$ for every $i \in I$. Then $L_i \prod_{i \in I} L_i$ is a left semiprime ideal of $S = \prod_{i \in I} S_i$.

**Proof.** Let $\alpha \in S = \prod_{i \in I} S_i$ be an arbitrary element and let $\alpha^n \in L = \prod_{i \in I} L_i$. Then $[\alpha(i)]^n \in L_i$ for every $i \in I$. Since $L_i$ is a semiprime ideal of $S_i$, we have $\alpha(i) \in L_i$ for every $i \in I$. Hence $\alpha \in L = \prod_{i \in I} L_i$.

**Corollary.** Let every left ideal of a semigroup $S_i$ be a semiprime ideal of $S_i$ for every $i \in I$. Then:

(a) Every left ideal of $S = \prod_{i \in I} S_i$ is a semiprime ideal of $S$.
(b) Every left ideal of $S = \prod_{i \in I} S_i$ is a complete left ideal of $S$.

**Proof.** The statement (a) follows from Theorem 5 and the Remark preceding Theorem 6. The statement (b) follows from the relation: $\mathcal{A}_s(m_1, n_1) \subseteq \mathcal{A}_s(m_2, n_2)$ if $m_1 \geq m_2$, $n_1 \geq n_2$ (see pp. 111–112 in [2]) and from Theorem 5.
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