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ON THE NILPOTENCY IN SEMIGROUPS

ROBERT ŠULKA, Bratislava

This paper is an extension of the results of papers [3] and [5]. The first three theorems of this paper are extensions of the first three theorems of the paper [3] and the fourth theorem of this paper is an extension of the theorem of paper [5]. We introduce three kinds of nilpotency and consider instead of ideals (which are as a rule used to define various kinds of radicals) more generally subsets or subsemigroups of a given semigroup \( S \).

Throughout the paper the empty subset \( \emptyset \) is considered as a subsemigroup of the semigroup \( S \). By an ideal we mean a two-sided ideal. If we consider a left ideal or a right ideal we shall indicate it explicitly. The \( \cap \)-semilattice of all left (right) [two-sided] ideals of the semigroup \( S \) contains always the empty set \( \emptyset \).

We now introduce three kinds of nilpotency and shall study the relations among them.

**Definition 1.** Let \( S \) be a semigroup and \( M \) a subset of \( S \).

a) Let \( x \) be such an element of \( S \) for which there exists a positive integer \( N(x) \) such that for each positive integer \( n \geq N(x) \) (for almost all \( n \)) \( x^n \in M \) holds. Then \( x \) will be called strongly nilpotent with respect to \( M \).

b) Let \( x \) be an element of \( S \) such that for infinitely many positive integers \( n \), \( x^n \in M \) holds. Then \( x \) will be called weakly nilpotent with respect to \( M \).

c) Let \( x \) be an element of \( S \) such that at least one power \( x^n \) is in \( M \). Then \( x \) will be called almost nilpotent with respect to \( M \).

The set of all strongly nilpotent elements with respect to \( M \) will be denoted by \( N_1(M) \), the set of all weakly nilpotent elements with respect to \( M \) will be denoted by \( N_2(M) \) and the set of all almost nilpotent elements with respect to \( M \) will be denoted by \( N_3(M) \).

**Remark 1.** From Definition 1 it is clear that every strongly nilpotent element with respect to \( M \) is weakly nilpotent with respect to \( M \) and every weakly nilpotent element with respect to \( M \) is almost nilpotent with respect to \( M \). Therefore we have \( N_1(M) \subseteq N_2(M) \subseteq N_3(M) \).

The following example shows that \( N_1(M) \neq N_2(M) \) and \( N_2(M) \neq N_3(M) \) can take place even if \( S \) is a commutative semigroup.
Example 1. Let $S = \langle 0, 1 \rangle$ with the ordinary multiplication as an operation in $S$. The element $x = \frac{1}{2}$ is almost nilpotent with respect to $M = \{\frac{1}{2}\}$ but it is not weakly nilpotent with respect to $M$.

In the same semigroup let us take $M = \left\{\frac{1}{2^k} \mid k = 1, 2, \ldots\right\}$. $M$ is a subsemigroup of $S$, the element $x = \frac{1}{2}$ is weakly nilpotent with respect to $M$, but it is not strongly nilpotent with respect to $M$.

The following lemmas are evident.

**Lemma 1.** Let $S$ be a semigroup and $M$ a subsemigroup of $S$. Then every almost nilpotent element with respect to $M$ is weakly nilpotent with respect to $M$, i.e. if $M$ is a subsemigroup then $N_3(M) = N_2(M)$.

**Lemma 2.** Let $S$ be a semigroup and $M$ a left (right) [two-sided] ideal of $S$. Then every weakly nilpotent element with respect to $M$ is also strongly nilpotent with respect to $M$. Thus we have $N_3(M) = N_2(M) = N_1(M)$.

**Lemma 3.** Let $S$ be a semigroup and let $M_1$ and $M_2$ be subsets of $S$. Then $N_1(M_1 \cap M_2) = N_1(M_1) \cap N_1(M_2)$.

Proof. a) If $A \subseteq B$, then $N_1(A) \subseteq N_1(B)$. Hence $N_1(M_1 \cap M_2) \subseteq N_1(M_1) \cap N_1(M_2)$.

b) Let $x \in N_1(M_1) \cap N_1(M_2)$. Then almost all powers of the element $x$ are in $M_1$ and almost all powers of the element $x$ are in $M_2$, i.e. almost all powers of the element $x$ are in $M_1 \cap M_2$. Therefore $x \in N_1(M_1 \cap M_2)$ and $N_1(M_1) \cap N_1(M_2) \subseteq N_1(M_1 \cap M_2)$. Together with a) we have $N_1(M_1 \cap M_2) = N_1(M_1) \cap N_1(M_2)$.

**Lemma 4.** Let $S$ be a semigroup and let $M_1$ and $M_2$ be subsemigroups of $S$. Then $N_3(M_1 \cap M_2) = N_3(M_1) \cap N_3(M_2)$ [hence $N_2(M_1 \cap M_2) = N_2(M_1) \cap N_2(M_2)$].

Proof. a) The relation $N_3(M_1 \cap M_2) \subseteq N_3(M_1) \cap N_3(M_2)$ can be proved in the same manner as in lemma 3.

b) If $x \in N_3(M_1) \cap N_3(M_2)$, then at least one power $x^n$ is in $M_1$ and at least one power $x^{n_2}$ is in $M_2$. But then $x^{n_1 \cdot n_2}$ is in $M_1 \cap M_2$ and $x \in N_3(M_1 \cap M_2)$. Thus $N_3(M_1) \cap N_3(M_2) \subseteq N_3(M_1 \cap M_2)$ and this together with a) implies $N_3(M_1 \cap M_2) = N_3(M_1) \cap N_3(M_2)$.

If $M_1, M_2$ and $M_1 \cap M_2$ are merely subsets of $S$ (not subsemigroups), then neither $N_2(M_1 \cap M_2) = N_2(M_1) \cap N_2(M_2)$ nor $N_3(M_1 \cap M_2) = N_3(M_1) \cap N_3(M_2)$ necessarily holds. This will be shown on the following examples.

**Example 2.** Let $S = \left\{\frac{1}{2^k} \mid k = 0, 1, 2, \ldots\right\}$ with the ordinary multiplica-
tion as operation. Let $M_1 = \{1, \frac{1}{2}\}$ and $M_2 = \{1, \frac{1}{4}\}$. Then $N_3(M_1) = \{1, \frac{1}{2}\}$, $N_3(M_2) = \{1, \frac{1}{4}, \frac{1}{2}\}$ and $N_3(M_1) \cap N_3(M_2) = \{1, \frac{1}{2}\} \cap \{1, \frac{1}{4}, \frac{1}{2}\} = M_1 \cap M_2 = N_3(M_1 \cap M_2)$.

Example 3. Let $S$ be the semigroup from Example 2. Let $N_2(M_1) = \{1\} \cup \left\{ \frac{1}{2k} \mid k = 1, 2, \ldots \right\}$ and $N_2(M_2) = \{1\} \cup \left\{ \frac{1}{2k-1} \mid k = 1, 2, \ldots \right\}$. Then $N_3(M_1) = S, N_2(M_2) = M_2$, but $N_2(M_1) \cap N_2(M_2) = \{1\} \cup \left\{ \frac{1}{2k-1} \mid k = 1, 2, \ldots \right\}$, $M_1 \cap M_2 = \{1\}$ and $N_2(M_1 \cap M_2) = \{1\} \neq N_2(M_1) \cap N_2(M_2)$.

Lemma 5. Let $S$ be a semigroup and $M_x, x \in K$, subsets of $S$. Then $\bigcup_{x \in K} N_3(M_x) = N_3(\bigcup_{x \in K} M_x)$.

Proof. a) For every $x \in K$ we have $M_x \subseteq \bigcup_{x \in K} M_x$ and therefore $N_3(M_x) \subseteq \bigcup_{x \in K} N_3(M_x)$.

b) Let $x \in N_3(\bigcup_{x \in K} M_x)$. Then at least one power $x^n$ is in $\bigcup_{x \in K} M_x$. Thus there exists a $x_0 \in K$ such that $x^n \in M_{x_0}$, i.e. $x \in N_3(M_{x_0}) \subseteq \bigcup_{x \in K} N_3(M_x)$. Therefore we have $N_3(\bigcup_{x \in K} M_x) \subseteq \bigcup_{x \in K} N_3(M_x)$ and this together with a) implies $\bigcup_{x \in K} N_3(M_x) = N_3(\bigcup_{x \in K} M_x)$.

Lemma 6. Let $S$ be a semigroup and $M_1$ and $M_2$ subsets of $S$. Then $N_2(M_1) \cup N_2(M_2) \subseteq N_2(M_1 \cup M_2)$.

Proof. a) The relation $N_2(M_1) \cup N_2(M_2) \subseteq N_2(M_1 \cup M_2)$ is evident.

b) Let $x \in N_2(M_1 \cup M_2)$. Then infinitely many powers $x^n$ are in $M_1 \cup M_2$. Thus infinitely many powers $x^n$ are either in $M_1$ or in $M_2$. Therefore $x$ is either in $N_2(M_1)$ or in $N_2(M_2)$ and $N_2(M_1 \cup M_2) \subseteq N_2(M_1) \cup N_2(M_2)$. This together with a) implies $N_2(M_1) \cup N_2(M_2) = N_2(M_1 \cup M_2)$.

Lemma 6 cannot be extended to the case of infinitely many subsets $M_x, x \in K$. This is clear from the following example.

Example 4. Let $S$ be the set of all positive integers with ordinary addition as operation. Let $M_n = \{2n + 1\}$, where $n = 1, 2, 3, \ldots$. Then $\bigcup_{n=1}^{\infty} M_n = \{2n + 1 \mid n = 1, 2, 3, \ldots\}$ and $1 \in N_2(\bigcup_{n=1}^{\infty} M_n)$. On the other hand $N_2(M_n) = \emptyset$ for $n = 1, 2, 3, \ldots$ and therefore also $\bigcup_{n=1}^{\infty} N_2(M_n) = \emptyset$. This implies that $N_2(\bigcup_{n=1}^{\infty} M_n) \neq \bigcup_{n=1}^{\infty} N_2(M_n)$. 150
The next example shows that $N_1(M_1 \cup M_2) = N_1(M_1) \cup N_1(M_2)$ need not hold.

Example 5. Let $S$ be the set of all positive integers with the ordinary addition as operation. Let $M_1 = \{2k| k = 1, 2, \ldots \}$ and $M_2 = \{2k + 1| k = 1, 2, \ldots \}$. Then $1 \in N_1(M_1 \cup M_2)$ but $1 \notin N_1(M_1) \cup N_1(M_2)$.

Lemma 7. Let $S$ be a semigroup and $M_1$ and $M_2$ subsemigroups of $S$. Then

\[
N_1(M_1 \cup M_2) = N_1(M_1) \cup N_1(M_2)
\]

Proof. a) It follows from $M_1 \subseteq M_1 \cup M_2$ and $M_2 \subseteq M_1 \cup M_2$ that $N_1(M_1) \cup N_1(M_2) \subseteq N_1(M_1 \cup M_2)$.

b) Let $x \in N_1(M_1 \cup M_2)$. Then there exists a positive integer $N$ such that for every integer $n \geq N$ there exists $n \in M_1 \cup M_2$. Let $X = \{x^n| n \geq N\}$. Note that $X \cap M_1, X \cap M_2$ are semigroups and $(X \cap M_1) \cup (X \cap M_2) = X$.

We now show that at least one of the semigroups $M_1$ and $M_2$ contains two consecutive powers of the element $x$. If it were not so, then one of the semigroups $M_1$ and $M_2$ would contain all even and the other all odd powers $x^n$ of the element $x$ for $n \geq N$. If for example $X \cap M_1$ were the set of all even powers $x^n, n \geq N$, then $X \cap M_2$ would be the set of all odd powers $x^n, n \geq N$. This contradicts the fact that $X \cap M_2$ is a semigroup.

Suppose that $M_1$ contains two consecutive powers of the element $x$. Then it can be easily verified that $M_1$ contains all powers $x^n$ for $n \geq N$. Therefore $x \in N_1(M_1)$ and hence $x \in N_1(M_1) \cup N_1(M_2)$.

We proved that $N_1(M_1 \cup M_2) \subseteq N_1(M_1) \cup N_1(M_2)$. This together with a) implies $N_1(M_1 \cup M_2) = N_1(M_1) \cup N_1(M_2)$.

The results we obtained can be arranged into two tables (see Table 1 and 2)

<table>
<thead>
<tr>
<th>$\cap$</th>
<th>$M_1$ and $M_2$ are:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>subsets</td>
</tr>
<tr>
<td>$N_1(M_1 \cap M_2) = N_1(M_1) \cap N_1(M_2)$</td>
<td>+ (L3)</td>
</tr>
<tr>
<td>$N_2(M_1 \cap M_2) = N_2(M_1) \cap N_2(M_2)$</td>
<td>+ (L4)</td>
</tr>
<tr>
<td>$N_3(M_1 \cap M_2) = N_3(M_1) \cap N_3(M_2)$</td>
<td>+ (L5)</td>
</tr>
<tr>
<td>$N_4(M_1 \cap M_2) = N_4(M_1) \cap N_4(M_2)$</td>
<td>+ (L6)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\cup$</th>
<th>$M_1$ and $M_2$ are:</th>
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<tbody>
<tr>
<td></td>
<td>subsets</td>
</tr>
<tr>
<td>$N_1(M_1 \cup M_2) = N_1(M_1) \cup N_1(M_2)$</td>
<td>+ (L7)</td>
</tr>
<tr>
<td>$N_2(M_1 \cup M_2) = N_2(M_1) \cup N_2(M_2)$</td>
<td>+ (L8)</td>
</tr>
<tr>
<td>$N_3(M_1 \cup M_2) = N_3(M_1) \cup N_3(M_2)$</td>
<td>+ (L9)</td>
</tr>
</tbody>
</table>

Table 1

Table 2

Table 1 and 2
in which the signs + and − have an apparent meaning. In parentheses a reference to the corresponding Lemma or Example is given.

The above results imply:

**Theorem 1.** Let $S$ be a semigroup. Then the mapping $M \to N_1(M)$ is:

a) a homomorphism of the lattice of all left (right) [two-sided] ideals of $S$ into the lattice of all subsets of $S$,

b) a homomorphism of the $\cap$-semilattice of all subsemigroups of $S$ into the $\cap$-semilattice of all subsets of $S$,

c) an endomorphism of the $\cap$-semilattice of all subsets of $S$.

The mapping $M \to N_2(M)$ is:

a) a homomorphism of the $\cap$-semilattice of all subsemigroups of $S$ into the $\cap$-semilattice of all subsets of $S$,

b) an endomorphism of the $\cup$-semilattice of all subsets of $S$.

The mapping $M \to N_3(M)$ is an endomorphism of the $\cup$-semilattice of all subsets of $S$.

We now introduce some further notions which are generalizations of the notions of Clifford's, Schwarz's and Ševrin's radicals from the papers [3] and [5].

**Definition 2.** Let $S$ be a semigroup and $M$ a subset of $S$. An ideal $I$, each element of which is strongly nilpotent with respect to $M$, is called a strong nilideal with respect to $M$.

An ideal $I$, each element of which is weakly nilpotent with respect to $M$, is called a weak nilideal with respect to $M$.

The union of all strong nilideals with respect to $M$ will be denoted by $B^*(M)$.

The union of all weak nilideals with respect to $M$ will be denoted by $B^*_w(M)$.

**Definition 3.** Let $S$ be a semigroup and $M$ a subset of $S$. An ideal (a subsemigroup) $I$, for which there exists a positive integer $N$ such that for all integers $n \geq N$ (for almost all $n$) $I^n \subseteq M$ holds, will be called a nilpotent ideal (a nilpotent subsemigroup) with respect to $M$.

The union of all nilpotent ideals with respect to $M$ will be denoted by $R(M)$.

**Definition 4.** Let $S$ be a semigroup and $M$ a subset of $S$. An ideal $I$, every subsemigroup of which generated by a finite number of elements is nilpotent with respect to $M$, will be called a locally nilpotent ideal with respect to $M$.

The union of all locally nilpotent ideals with respect to $M$ will be denoted by $L(M)$.

**Lemma 8.** An ideal $I$ is a weak nilideal with respect to $M$ if and only if every element $x \in I$ is almost nilpotent with respect to $M$.

Proof. a) If the ideal $I$ is a weak nilideal with respect to $M$, then clearly
each element $x \in I$ is an almost nilpotent element with respect to $M$.

b) Let every element $x$ of the ideal $I$ be an almost nilpotent element with respect to $M$. Then $x \in I$ implies that $\{x, x^2, x^3, \ldots, x^n, \ldots\} \subseteq I$. In addition to this for some power $x^{n_1}$ we have $x^{n_1} \in M$. But since $x^{n_1} \in I$, there exists again a positive integer $n_2 > n_1$ for which $x^{n_2} \in M$. Thus there exists a sequence $x^{n_1}, x^{n_2}, \ldots, x^{n_k}, \ldots$, $n_1 < n_2 < n_3 < \ldots < n_k < \ldots$ of powers of the element $x$, the members of which are in $M$. This means that $x$ is a weakly nilpotent element with respect to $M$. Since $x$ is any element of $I$, $I$ is a weak nilideal with respect to $M$.

The following example shows that $R_1^*(M)$ and $R_2^*(M)$ may be distinct even if $M$ is a subsemigroup of $S$.

Example 6. Let $S$ be the set of all positive integers with the ordinary addition as operation. Let $M$ be the subsemigroup of all even integers. Every odd positive integer is weakly nilpotent with respect to $M$ but it is not strongly nilpotent with respect to $M$. Every even positive integer is strongly nilpotent with respect to $M$. Note that every ideal contains together with each integer $a > 0$ all integers $\geq a$. Hence $R_1^*(M) = \varnothing = \alpha = R_2^*(M)$.

Lemma 9. Let $S$ be a semigroup, $M$ a subset and $A$ a subsemigroup of $S$. Then the following three statements are equivalent:

a) The subsemigroup $A$ is nilpotent with respect to $M$.

b) There exist infinitely many positive integers $n$ such that $A^n \subseteq M$.

c) There exists a positive integer $n$ such that $A^n \subseteq M$.

Proof. It is clear from definition 3 that a) implies b) and b) implies c). It remains only to prove that c) implies a). Let $n$ be a positive integer such that $A^n \subseteq M$. Since $A$ is a subsemigroup we have $A^{n+1} \subseteq A^n \subseteq M$, $A^{n+2} \subseteq \subseteq A^n \subseteq M$, ... and therefore $A$ is a nilpotent subsemigroup with respect to $M$.

Remark 2. Lemma 9 evidently holds also in the case where $A$ is a left (right) [two-sided] ideal.

Lemma 10. Let $S$ be a semigroup and let $M_1$ and $M_2$ be subsets of $S$. Then

$R_1^*(M_1 \cap M_2) = R_1^*(M_1) \cap R_1^*(M_2)$.

Proof. a) Evidently $R_1^*(M_1 \cap M_2) \subseteq R_1^*(M_1) \cap R_1^*(M_2)$.

b) Let $x \in R_1^*(M_1) \cap R_1^*(M_2)$. Then $x \in R_1^*(M_1)$ and $x \in R_1^*(M_2)$, i.e. $x \in I_1$ and $x \in I_2$, where $I_1$ is a strong nilideal with respect to $M_1$ and $I_2$ is a strong nilideal with respect to $M_2$. We show that $I_1 \cap I_2$ is a strong nilideal with respect to $M_1 \cap M_2$. Let $y \in I_1 \cap I_2$. Then $y \in I_1$, $y \in I_2$, i.e. there exists a positive integer $N$ such that for every integer $n \geq N$ we have $y^n \in M_1$ and $y^n \in M_2$. Hence for all integers $n \geq N$ we have $y^n \in M_1 \cap M_2$. This means that $I_1 \cap I_2$ is a strong nilideal with respect to $M_1 \cap M_2$.

Since $I_1 \cap I_2$ is a strong nilideal with respect to $M_1 \cap M_2$ and $x \in I_1 \cap I_2$,
we have $x \in R^*(M_1 \cap M_2)$. Thus $R^*(M_1) \cap R^*(M_2) \subseteq R^*(M_1 \cap M_2)$ and this together with a) proves $R^*(M_1 \cap M_2) = R^*(M_1) \cap R^*(M_2)$.

Lemma 11. Let $S$ be a semigroup and let $M_1$ and $M_2$ be subsets of $S$. Then $R(M_1 \cap M_2) = R(M_1) \cap R(M_2)$.

Proof. a) Evidently $R(M_1 \cap M_2) \subseteq R(M_1) \cap R(M_2)$.

b) Let $x \in R(M_1) \cap R(M_2)$. Then $x \in R(M_1)$ and $x \in R(M_2)$, i.e. $x \in I_1$ and $x \in I_2$, where $I_1$ is a nilpotent ideal with respect to $M_1$ and $I_2$ is a nilpotent ideal with respect to $M_2$. We show that $I_1 \cap I_2$ is a nilpotent ideal with respect to $M_1 \cap M_2$. As a matter of fact for almost all $n$ we have $I_1^n \subseteq M_1$ and $I_2^n \subseteq M_2$, thus $(I_1 \cap I_2)^n \subseteq M_1 \cap M_2$. Since $x \in I_1 \cap I_2$, we obtain $R(M_1) \cap R(M_2) \subseteq R(M_1 \cap M_2)$ and this together with a) proves $R(M_1) \cap R(M_2) = R(M_1 \cap M_2)$.

Lemma 12. Let $S$ be a semigroup and let $M_1$ and $M_2$ be subsets of $S$. Then $L(M_1 \cap M_2) = L(M_1) \cap L(M_2)$.

Proof. a) Evidently $L(M_1 \cap M_2) \subseteq L(M_1) \cap L(M_2)$.

b) Let $x \in L(M_1) \cap L(M_2)$. Then $x \in L(M_1)$ and $x \in L(M_2)$, i.e. $x \in I_1$, where $I_1$ is a locally nilpotent ideal with respect to $M_1$ and $x \in I_2$, where $I_2$ is a locally nilpotent ideal with respect to $M_2$. We show that $I_1 \cap I_2$ is a locally nilpotent ideal with respect to $M_1 \cap M_2$.

Let $A$ be a subsemigroup generated by a finite number of elements of $I_1 \cap I_2$. Since $A \subseteq I_1$ and $A \subseteq I_2$ for almost all positive integers $n$, $A^n \subseteq M_1$ and $A^n \subseteq M_2$ holds. Thus $A^n \subseteq M_1 \cap M_2$ and $I_1 \cap I_2$ is a locally nilpotent ideal with respect to $M_1 \cap M_2$.

As $x \in I_1 \cap I_2$, we obtain $x \in L(M_1 \cap M_2)$. Hence $L(M_1) \cap L(M_2) \subseteq L(M_1 \cap M_2)$ and this together with a) gives $L(M_1) \cap L(M_2) = L(M_1 \cap M_2)$.

Lemma 13. Let $S$ be a semigroup and $M_1$ and $M_2$ subsemigroups of $S$. Then $R^*(M_1 \cap M_2) = R^*(M_1) \cap R^*(M_2)$.

Proof. a) Evidently $R^*(M_1 \cap M_2) \subseteq R^*(M_1) \cap R^*(M_2)$.

b) Let $x \in R^*(M_1) \cap R^*(M_2)$. Then $x \in R^*(M_1)$ and $x \in R^*(M_2)$, i.e. $x \in I_1$, where $I_1$ is a weak nilideal with respect to $M_1$ and $x \in I_2$ where $I_2$ is a weak nilideal with respect to $M_2$. Therefore $x \in I_1 \cap I_2$.

We now show that every element $y \in I_1 \cap I_2$ is weakly nilpotent with respect to $M_1 \cap M_2$, i.e. that $I_1 \cap I_2$ is a weak nilideal with respect to $M_1 \cap M_2$. Since $y \in I_1 \cap I_2$, there exist positive integers $n_1$ and $n_2$ such that $y^{n_1} \in M_1$ and $y^{n_2} \in M_2$. As $M_1$ and $M_2$ are subsemigroups of $S$ we have for the cyclic semigroups generated by the elements $y^{n_1}$ and $y^{n_2}$: $\{y^{n_1}, y^{2n_1}, \ldots\} \subseteq M_1$ and $\{y^{n_2}, y^{2n_2}, \ldots\} \subseteq M_2$. But then for the cyclic semigroup generated by the element $y^{n_1} \cdot y^{n_2}$ we have $\{y^{n_1} \cdot y^{n_2}, y^{2n_1} \cdot y^{2n_2}, \ldots\} \subseteq M_1 \cap M_2$. Hence $y$ is a weakly
nilpotent element with respect to $M_1 \cap M_2$, thus $I_1 \cap I_2$ is a weak nilideal with respect to $M_1 \cap M_2$.

Since $x \in I_1 \cap I_2$ we have $x \in R_2^*(M_1 \cap M_2)$. We proved that $R_2^*(M_1) \cap R_2^*(M_2) \subseteq R_2^*(M_1 \cap M_2)$ and this together with a) gives $R_2^*(M_1 \cap M_2) = R_2^*(M_1) \cap R_2^*(M_2)$.

The following example shows that $R_2^*(M_1) \cap R_2^*(M_2) = R_2^*(M_1 \cap M_2)$ need not hold.

Example 7. Let $S$ be the set of all positive integers with the ordinary addition as operation. Let $M_1$ contain the number 1 and those integers $n > 1$ whose factorization into primes has either an even number of factors equal to the number 2 or it has no factor equal to 2. Let $M_2$ contain the number 1 and those integers $n > 1$ whose factorization into primes has an odd number of factors equal to 2. Clearly $M_1 \cap M_2 = \{1\}$ and $R_2^*(M_1 \cap M_2) = \emptyset$. Further $R_2^*(M_1) = S$ and $R_2^*(M_2) = S$ and therefore $R_2^*(M_1) \cap R_2^*(M_2) = S \cap \emptyset = \emptyset = R_2^*(M_1 \cap M_2)$.

The results we obtained are arranged into tables. (See Tables 3, 4 and 5.

### Table 3

<table>
<thead>
<tr>
<th>$\cap$</th>
<th>$M_1$ and $M_2$ are:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>subsets</td>
</tr>
<tr>
<td>$R_1^<em>(M_1 \cap M_2) = R_1^</em>(M_1) \cap R_1^*(M_2)$</td>
<td>+ (L10)</td>
</tr>
<tr>
<td>$R_2^<em>(M_1 \cap M_2) = R_2^</em>(M_1) \cap R_2^*(M_2)$</td>
<td>+ (E6)</td>
</tr>
</tbody>
</table>

### Table 4

<table>
<thead>
<tr>
<th>$\cap$</th>
<th>$M_1$ and $M_2$ are:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>subsets</td>
</tr>
<tr>
<td>$R(M_1 \cap M_2) = R(M_1) \cap R(M_2)$</td>
<td>+ (L11)</td>
</tr>
</tbody>
</table>

### Table 5

<table>
<thead>
<tr>
<th>$\cap$</th>
<th>$M_1$ and $M_2$ are:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>subsets</td>
</tr>
<tr>
<td>$L(M_1 \cap M_2) = L(M_1) \cap L(M_2)$</td>
<td>+ (L12)</td>
</tr>
</tbody>
</table>

Remark 3. For the unions the relations $R_1^*(M_1 \cup M_2) = R_1^*(M_1) \cup R_1^*(M_2)$, $R_2^*(M_1 \cup M_2) = R_2^*(M_1) \cup R_2^*(M_2)$, $R(M_1 \cup M_2) = R(M_1) \cup R(M_2)$ and $L(M_1 \cup M_2) = L(M_1) \cup L(M_2)$ need not hold. This follows from an example in paper [3], p. 213, even if $M_1$ and $M_2$ are two-sided ideals. (See also [5].)
Remark 4. Lemmas 3, 4, 6, 7, 10, 11, 12 and 13 can be extended by induction from two subsets \( M_1 \) and \( M_2 \) to any finite number of subsets \( M_x, x \in K \). But the following example shows that Lemmas 3, 4, 10, 11, 12 and 13 cannot be extended to an infinite number of subsets.

Example 8. The closed interval \( S = \langle 0, \frac{2}{3} \rangle \) with the ordinary multiplication as operation is a semigroup. The closed intervals \( J_n = \langle 0, \frac{1}{n} \rangle, n = 2, 3, \ldots \) are ideals of \( S \). \( N_1(J_n) = S \) for \( n = 2, 3, \ldots \) and therefore \( \bigcap_{n=2}^{\infty} N_1(J_n) = S \). But \( \bigcap_{n=2}^{\infty} J_n = \{0\} \) and \( N_1(\bigcap_{n=2}^{\infty} J_n) = \{0\} \neq S \).

Since \( S \) is a commutative semigroup and \( J_n, n = 2, 3, \ldots \) are ideals of \( S \), the foregoing sets of strongly nilpotent elements are at the same time sets of weakly nilpotent elements and also sets of almost nilpotent elements. By \([3]\) and \([1]\) they are clearly radicals with respect to these ideals.

The above lemmas imply the following theorems:

**Theorem 2.** Let \( S \) be a semigroup. Then the mapping \( M \rightarrow R_1^*(M) \) is:
- a) a homomorphism of the \( \cap \)-semilattice of all subsets of \( S \) into the \( \cap \)-semilattice of all (two-sided) ideals of \( S \),
- b) a homomorphism of the \( \cap \)-semilattice of all subsemigroups of \( S \) into the \( \cap \)-semilattice of all (two-sided) ideals of \( S \),
- c) a homomorphism of the \( \cap \)-semilattice of all left (right) [two-sided] ideals of \( S \) into the \( \cap \)-semilattice of all (two-sided) ideals of \( S \).

The mapping \( M \rightarrow R_2^*(M) \) is a homomorphism of the \( \cap \)-semilattice of all subsemigroups of \( S \) into the \( \cap \)-semilattice of all (two-sided) ideals of \( S \).

**Theorem 3.** Let \( S \) be a semigroup. Then the mapping \( M \rightarrow R(M) \) is:
- a) a homomorphism of the \( \cap \)-semilattice of all subsets of \( S \) into the \( \cap \)-semilattice of all (two-sided) ideals of \( S \),
- b) a homomorphism of the \( \cap \)-semilattice of all subsemigroups of \( S \) into the \( \cap \)-semilattice of all (two-sided) ideals of \( S \),
- c) a homomorphism of the \( \cap \)-semilattice of all left (right) [two-sided] ideals of \( S \) into the \( \cap \)-semilattice of all (two-sided) ideals of \( S \).

**Theorem 4.** Let \( S \) be a semigroup. Then the mapping \( M \rightarrow L(M) \) is:
- a) a homomorphism of the \( \cap \)-semilattice of all subsets of \( S \) into the \( \cap \)-semilattice of all (two-sided) ideals of \( S \),
- b) a homomorphism of the \( \cap \)-semilattice of all subsemigroups of \( S \) into the \( \cap \)-semilattice of all (two-sided) ideals of \( S \),
- c) a homomorphism of the \( \cap \)-semilattice of all left (right) [two-sided] ideals of \( S \) into the \( \cap \)-semilattice of all (two-sided) ideals of \( S \).
REFERENCES


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