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SUFFICIENT CONDITION FOR THE NON-OSCILLATION OF THE NON-HOMOGENEOUS LINEAR nTH ORDER DIFFERENTIAL EQUATION

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M. Švec deals in paper [1] with the following second order differential equations

(a) $$z'' + p(x)z = f(x),$$
(b) $$y'' + p(x)y = 0,$$

where $p, f \in C(-\infty, \infty)$. He proves that if the function $f(x)$ has a constant sign for all the value x and if the differential equation (b) is non-oscillating, the differential equation (a) is also non-oscillating (see the definition in [1]). In this paper this result will be generalized for the n-th order differential equations, where the presupposition about the function $f(x)$ will be weaker.

The following differential equations will be dealt with:

(1) $$L_n(z) = z^{(n)} + a_1 z^{(n-1)} + \ldots + a_n z = f,$$
(2) $$L_n(y) = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0,$$

where $a_1, a_2, \ldots, a_n, f \in C_0(a, b)$.

If $a_k \in C_{n-k}(a, b)$ $(k = 1, 2, \ldots, n)$, then the adjoint differential equation to the differential equation (2) is denoted by

$$L_n(u) = (-1)^n u^{(n)} + (-1)^{n-1} (a_1 u)^{(n-1)} + \ldots + a_n u = 0.$$

The function $F(x) \not\equiv 0$ is said to be oscillating in an interval (a, b) if at least one of the points a, b is the limit point of the zeros (belonging to the interval (a, b)) of $F(x)$. By the non-oscillating function in the interval (a, b) we mean the function which is not oscillating in this interval.

The differential equation (1) (resp.(2)) is said to be in the interval (a, b):

(I) non-oscillating if all its solutions are non-oscillating in (a, b)

(II) n-non-oscillating if every of its solution has at most $n - 1$ zero points in (a, b).

By the symbol $w_i = w(u_1, u_2, \ldots, u_i)$ we mean the Wronskian of the functions $u_1, u_2, \ldots u_i \in C_{i-1}$. The sequence $w_0 = 1$, $w_i = w(u_1, u_2, \ldots, u_i)$ $(i = 1, 2, \ldots, m;\ u_1, u_2, \ldots, u_m \in C_{m-1})$ is called the complete chain of the

Wronskians of the functions $u_1, u_2, \ldots, u_m$, analogously as in paper [2]. If all numbers of this sequence are different from zero in the interval (a, b), then this sequence is called the complete chain of the Wronskians without zero in (a, b).

First several lemmas will be proved.

Lemma 1. *Let $y_1, y_2, \ldots, y_{n-1}$ be the solutions of the differential equation* (1). *Then the Wronskian $W = w(z, y_1, y_2, \ldots, y_{n-1})$ is the solution of the differential equation*

$$Y' + a_1 Y = (-1)^{n+1} f w(y_1, y_2, \ldots, y_{n-1}). \tag{3}$$

Proof.

$$W' = (-1)^{n+1} (-a_1 z^{(n-1)} - \ldots - a_n z + f) \, w(y_1, y_2, \ldots, y_{n-1}) +$$
$$+ (-1)^{n+2} (-a_1 y_1^{(n-1)} - \ldots - a_n y_1) \, w(z, y_2, \ldots, y_{n-1}) + \ldots +$$
$$+ (-1)^{2n} (-a_1 y_1^{(n-1)} - \ldots - a_n y_{n-1}) \, w(z, y_1, \ldots, y_{n-2}) =$$
$$= (-1)^{n+1} f w(y_1, y_2, \ldots, y_{n-1}) - a_1 W - a_2 W_2 - \ldots - a_n W_n,$$

where

$$W_i = \begin{vmatrix} z & y_1 & \ldots & y_{n-1} \\ z' & y_1' & \ldots & y_{n-1}' \\ \ldots & \ldots & \ldots & \ldots \\ z^{(n-2)} & y_1^{(n-2)} & \ldots & y_{n-1}^{(n-2)} \\ z^{(n-i)} & y_1^{(n-i)} & \ldots & y_{n-1}^{(n-i)} \end{vmatrix} = 0 \; (i = 2, 3, \ldots, n).$$

From this the assertion of the lemma follows.

Lemma 2. *Let $p, f \in C_0(a, b)$. Then if the function g is non-oscillating in the interval (a, b), the differential equation*

$$v' + pv = g \tag{4}$$

is non-oscillating in (a, b).

Proof. Let v be an arbitrary solution of the differential equation (4) and let u be a solution of the corresponding homogeneous differential equation. The Wronskian $W(v, u) = -g\,u$ is a non-oscillating function in (a, b) and therefore from the Theorem about the separation of zeros for second order differential equations it follows that the function v is also non-oscillating in (a, b). Thus Lemma 2 is proved.

Lemma 3. *Let $a_k \in C_{n-k}(a, b)$ $(k = 1, 2, \ldots, n)$, $y, z \in C_n(a, b)$.*

If $L_n(y) = L_{b_n} L_{b_{n-1}} \ldots L_{b_1} y$, *where* $L_{b_i} = \dfrac{\mathrm{d}}{\mathrm{d}x} + b_i (i = 1, 2, \ldots, n)$, *then*

$\bar{L}_n(z) = \bar{L}_{b_1}\bar{L}_{b_2}\ldots\bar{L}_{b_n}z$, *where* $\bar{L}_{b_i} = -\dfrac{\mathrm{d}}{\mathrm{d}x} + b$ $(i = 1, 2, \ldots, n)$ *in the interval* (a, b).

Proof. The proof is accomplished by complete induction. For $n = 2$ the assertion of the Lemma is easy to prove.

Let $\overline{L_{b_{n-1}}L_{b_{n-2}}\ldots L_{b_1}}\, z = \bar{L}_{b_1}\bar{L}_{b_2}\ldots\bar{L}_{b_{n-1}}z$. It is sufficient to prove that

$$\overline{L_{b_{n-1}}\ldots L_{b_1}}\,\bar{L}_{b_n}z = \overline{L_{b_n}L_{b_{n-1}}\ldots L_{b_1}}z. \tag{5}$$

Let $L_{b_{n-1}}\ldots L_{b_1}z = z^{(n-1)} + c_1 z^{(n-2)} + \ldots + c_{n-1}z$. It is easy to prove the following assertion: $L_{b_n}L_{b_{n-1}}\ldots L_{b_1}z = z^{(n)} + P_1 z^{(n-1)} + \ldots + P_n z$, where $P_i = b_n c_{i-1}' + c_{i-1}' + c_i$, where $c_0 = 1$, $i = 1, 2, \ldots, n$ and $c_n = 0$ by the definition.

$$\begin{aligned} &\overline{L_{b_{n-1}}\ldots L_{b_1}}\bar{L}_{b_n}z = (-1)^n z^{(n)} + (-1)^{n-1}(c_1 z)^{(n-2)} + \ldots - c_{n-1}z' + \\ &\quad + (-1)^{n-1}(b_n z)^{(n-1)} + (-1)^{n-2}(c_1 b_n z)^{(n-2)} + \ldots + c_{n-1}b_n z. \\ &\overline{L_{b_n}L_{b_{n-1}}\ldots L_{b_1}}z = (-1)^n z^{(n)} + (-1)^{n-1}(b_n + c_1)z^{(n-1)} + \\ &\quad + (-1)^{n-2}(b_n c_1 + c_1' + c_2)z^{(n-2)} + \ldots + (b_n c_{n-1} + c_{n-1}')z. \end{aligned} \tag{6}$$

After the simple modification of the right side of the equality (6) it is possible to prove the equality (5).

Lemma 4. *Let* $w_0 = 1$, $w_1 = w(y_1)$, $w_2 = w(y_1, y_2)$, $\ldots$, $w_n = w(y_1, y_2, \ldots, y_n)$ *be the complete chain of the Wronskians without zero ponits in the interval* (a, b) *of the solutions of the differential equation* (2). *Then for every* y_i $(1 < i \leq n)$ *there exist such numbers* $d_1, d_2 \in (a, b)$ *that* $\bar{w}_0 = 1$, $\bar{w}_1 = w(y_i)$, $\bar{w}_2 = w(y_i, y_1)$, $\ldots$, $\bar{w}_{i+1} = w(y_i, y_1, \ldots, y_{i-1}, y_{i+1})$, $\ldots$,
$\bar{w}_n = w(y_i, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ *is the complete chain of the Wronskians without zero in the intervals* (a, d_1), (d_2, b).

Proof. From assumption of the Lemma according to [2] it follows that there exist such real functions $b_1, b_2, \ldots b_n$, for which

$L_n(y) = \left(\dfrac{\mathrm{d}}{\mathrm{d}x} + b_n\right)\ldots\left(\dfrac{\mathrm{d}}{\mathrm{d}x} + b_1\right)y$ in (a, b), where y_i is the solution of the differential equations

$$(\mathrm{A}_s)\quad L_s(y) = \left(\frac{\mathrm{d}}{\mathrm{d}x} + b_s\right)\ldots\left(\frac{\mathrm{d}}{\mathrm{d}x} + b_1\right)y = 0 \text{ for } s = 1, 2, \ldots, i-1,\ i+1,$$

$\ldots, n$ and $f_s \not\equiv 0$ for $s = 1, 2, \ldots, i-1$. By complete induction with the help of Lemma 2 the functions $f_s (s = 1, 2, \ldots, i-1)$ are easily proved to be non-oscillating in (a, b).

Let $L_s(y) = y^{(s)} + a_{1s}y^{(s-1)} + \ldots + a_{ss}y$ $(s = 1, 2, \ldots, n)$. According to Lemma 1 $\bar{w}_j' + a_{1j}\bar{w}_j = g_j$, where $g_j = (-1)^{i+1} f_j w_{j-1} (j = 1, 2, \ldots, i-1)$,

$w_j = (-1)^{i-1}\ w_j$ $(j = i, i + 1, \ldots, n)$. From Lemma 2 it follows that the functions w_j $(j = 1, 2, \ldots, i - 1)$ are different from zero in (a, b). Evidently there exist such numbers $d_1, d_2 \in (a, b)$ that $\overline{w}_j \neq 0$ $(j = 0, 1, \ldots, n)$ in the intervals (a, d_1), (d_2, d). Thus the Lemma is proved.

Lemma 5. *Let the differential equation* (2) *be n-non-oscillating in the interval* (a, b). *Then for any solution* $u(x)$ *of the differential equation* (2) *there exist such numbers* d_1, $d_2 \in (a, b)$ *and such a complete chain of the Wronskians* (a, d_1), (d_2, b) *of the solutions of the differential equation* (2) *that* $w_1 = u(x)$.

Proof. The differential equation (2) is n-non oscillating in (a, b) and therefore according to [2] there exists a complete chain of the Wronskians $\overline{w}_0 = 1$, $\overline{w}_1 = w(y_1), \ldots, \overline{w}_n = w(y_1, y_2, \ldots, y_n)$ without zero in (a, b) of the solutions of the differential equation (2). Then $u(x) = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ in (a, b), where c_i $(i = 1, 2, \ldots, n)$ are constants. Because the differential equation (2) is n-non-oscillating in (a, b), there exist such numbers $\overline{a}_1, \overline{b}_1 \in (a, b)$ that $u(x) \neq 0$ in the intervals $(a, \overline{a}_1)$, $(\overline{b}_1, b)$. It follows that there exists such a $c_i \neq 0$ that $c_j = 0$ for $i < j \leqq n + 1$ (we define $c_{n+1} = 0$).

Let us construct the following sequence of the Wronskians:

(c) $\tilde{w}_0 = 1, \tilde{w}_1 = \overline{w}_1, \ldots, \tilde{w}_{i-1} = \overline{w}_{i-1}, \tilde{w}_i = w(y_1, \ldots, y_{i-1}, u)$

$$\overline{w}_{i+1} = w(y_1, \ldots, y_{i-1}, u, y_{i+1}), \ldots, \tilde{w}_n = w(y_1, \ldots, y_{i-1}, u, y_{i+1}, \ldots, y_n)$$

Without difficulties it is possible to prove that $\tilde{w}_j = c_j \overline{w}_j$ $(j = i, i + 1, \ldots, n)$. It follows that (c) is the complete chain of the Wronskians without zero in (a, b) of the solutions of the differential equation (2). According to Lemma 4 there exist such numbers $d_1, d_2 \in (a, b)$ that $w_0 = 1, w_1 = u(x), \ldots, w_n = w(u, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ is the complete chain of the Wronskians without zero in the intervals (a, d_1), (d_2, b). Q. e. d,

Now with the help of the preceding Lemmas the following Theorem will be proved.

Theorem. *Let* $a_k \in C_{n-k}(a, b)$, $(k = 1, 2, \ldots, n)$. *Let the differential equation* (2) *be n-non-oscillating in the intervals* (a, d_1), (d_2, b) $(d_1, d_2 \in (a, b))$. *Let* $u(x)$ *be the solution of the differential equation adjoint to the differential equation* (2) *such that* $u(x) \neq 0$ *in the intervals* (a, d_3), (d_4, d) $(d_1 \geqq d_3, d_2 \leqq d_4)$ *and the differential equation*

$$v' + \frac{u'}{u} v = f \tag{7}$$

is non-oscillating in (a, d_3), (d_4, d). *Then the differential equation* (1) *is non-oscillating in* (a, b).

Proof. The differential equation (2) is n-non-oscillating in (a, d_1), (d_2, b) and therefore according to [2] there exist such real functions $b_1, b_2, \ldots, b_n$

that $L_n(y) = \left(\frac{d}{dx} + b_n\right) \dots \left(\frac{d}{dx} + b_1\right) y = 0$ in every of the intervals (a, d_1), (d_2, d). According to Lemma 3 $L_n(z) = \left(-\frac{d}{dx} + b_1\right) \dots \left(-\frac{d}{dx} + b_n\right) z$. According to [2] and to Lemma 5 there exist such numbers $\bar{a}_1 \in (a, d_1)$, $\bar{b}_1 \in \in (d_2, d)$ and such a complete chain of the Wronskians without zero in every of the intervals $(a, \bar{a}_1)$, $(\bar{b}_1, b)$, $w_0, w_1, \dots, w_n$ of the solutions of the adjoint differential equation adjoint to the differential equation (2) that $w_1 = u(x)$. According to [2]

$$L_n(u) = \left(\frac{d}{dx} - c_1\right) \dots \left(\frac{d}{dx} - c_n\right) u, \text{ where } c_i = D \lg \frac{w_{n-i+1}}{w_{n-i}} \ (i = 1, 2, \dots, n),$$

$c_n = D \lg \frac{w_1}{w_0} = \frac{u'}{u}$. According to Lemma 3 in every of the intervals $(a, \bar{a}_1)$, $(\bar{b}_1, b)$ we have

$$(1') \qquad L_n(z) = (-1)^n \left(\frac{d}{dx} + \frac{u'}{u}\right)\left(\frac{d}{dx} + c_{n-1}\right) \dots \left(\frac{d}{dx} + c_1\right) z = f.$$

Let us consider the following system of the differential equations:

$$
\begin{aligned}
&(B_1) && z' + c_1 z = u_1, \\
&(B_2) && u_1' + c_2 u_1 = u_2, \\
& && \dots\dots\dots\dots \\
&(B_{n-1}) && u_{n-2}' + c_{n-1} u_{n-2} = u_{n-1}, \\
&(B_n) && u_{n-1}' + \frac{u'}{u} u_{n-1} = (-1)^n f.
\end{aligned}
$$

Without difficulties it is possible to prove that the arbitrary solution of the differential equation (B_1) is also the solution of the differential equation (1') and if z is an arbitrary solution of the differential equation (1'), then there exist such functions $u_1, u_2, \dots, u_{n-1}$ ($u_k \in C_{n-k}$, $k = 1, 2, \dots, n$) that z is the solution of the differential equation (B_1) (in every of the intervals $(a, \bar{a}_1)$, $(\bar{b}_1, b)$. From the assumption regarding the differential equation (7) it follows that the differential equation (B_n) is non oscillating in $(a, \bar{a}_1)$, $(\bar{b}_1, b)$. By the complete induction it is possible to prove without difficulties that all the differential equations (B_1), (B_2), ..., (B_n) are non-oscillating in $(a, \bar{a}_1)$, $(\bar{b}_1, b)$. Because the differential equation (B_1) is non-oscillating in (a, a_1), (b_1, b) the differential equation (1) is also non-oscillating in $(a, \bar{a}_1)$, $(\bar{b}_1, b)$. Hence it follows directly that the differential equation (1) is non-oscillating in the interval (a, b).

Remark. Evidently Theorem 3 of [1] is the direct consequence of our

Theorem. The method of the proof of our Theorem is different from the method used in [1], which is applicable only in special cases.

REFERENCES

1] Švec M., *On various properties of the solutions of third and fourth order linear differential equations,* Differential Eguations and Their Applications (Proc. Conf., Prague, 1962), 187—198. Publ. House of the Czechoslovak Acad. Sci., Prague, Academic Press, New York, 1963.

[2] Mammana G., *Decomposizione delle espressioni differenziali lineari omogenee in prodotti di fattori simbolici e applicazione relativa allo statio delle equazioni differenziali lineari,* Math. Z. *33* (1931), 186—231.

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