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SOME CHARACTERIZATIONS OF THE DARBOUX CONTINUITY OF REAL FUNCTIONS

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1. Introduction

In recent years a number of articles appeared which deal with the limits of sequences of Darboux functions (we consider real-valued Darboux functions defined on the real line). It is known that the limit function of a sequence of Darboux functions may fail to be Darboux though the sequence converges uniformly (see the expository paper [1] of Bruckner and Ceder). The following problem has been stated by S. Marcus (see [1]): What is the „natural“ type of convergence „⇒“ for Darboux functions, i. e. what type of convergence „⇒“ has the property that if \(\{f_n\}_{n=1}^{\infty}\) is a sequence of Darboux functions converging pointwise to \(f\) then \(f\) is Darboux if and only if \(f_n \Rightarrow f\) (i. e. when \(f_n\) converges to \(f\) in the sense of „⇒“). It is very difficult to describe such a type of convergence in general but in the present paper a „characteristic“ type of convergence for uniformly converging sequences of Darboux functions is given (see Theorem 2 below). It is shown that the real-valued Darboux functions defined on the real line can be characterized as the continuous functions from one topological space to another topological space (Theorem 1 below). There are also given some types of convergence which preserve the Darboux continuity (see Theorems 3 and 4 below).

In the sequel, the set of real numbers is denoted as \(\mathbb{R}\) while the set \(\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}\) of extended real numbers as \(\mathbb{R}\). \(\mathcal{D}\) stands for the class of Darboux functions. The fact that \(f\) is a function with a domain \(A\) and a range \(B\) is written as \(f: A \rightarrow B\).

2. Preliminary Constructions

Let \(\mathcal{S}\) be the cartesian product \(\mathcal{S} = \mathbb{R} \times \{-, +\}\) of the set \(\mathbb{R}\) of real numbers ordered by the usual order-relation, and the set \(\{-, +\}\) whose only elements are the symbols — and + ordered by \(- < +\). If \((a, \alpha)\) is an element of \(\mathcal{S}\), then \(a \in \mathbb{R}\) is called the real part of \((a, \alpha)\), and \(\alpha \in \{-, +\}\)
the characteristic of \((a, x)\). Assume \(S\) to be ordered by the lexicographic relation defined as follows: If \((a, x)\) and \((b, \beta)\) are two elements of \(S\) then \((a, x) < (b, \beta)\) if and only if \(a < b\), or \(a = b\) and \(x < \beta\). Let \(\mathcal{T}\) be the order topology for \(S\) generated by this ordering.

It is easy to verify that \((S, \mathcal{T})\) is a first countable topological space (i.e. the neighbourhood system of every its point has a countable base). The space \((S, \mathcal{T})\) is also separable and does not satisfy the second axiom of countability. Hence \((S, \mathcal{T})\) fails to be a metric space (see Kelley [3]).

The following lemmas give more information on the structure of the topological space \((S, \mathcal{T})\).

**Lemma 1.** Each non-empty bounded subset \(M\) of \(S\) has the least upper bound.

**Proof:** Assign to each element \(x\) of \(M\) its real part \(x'\). The set \(M'\) of all this elements \(x'\) has the (real) least upper bound \(y'\). Now let \(y = (y', -)\) if \((y', +) \notin M\) and \(y = (y', +)\) if \((y', +) \in M\). It is easy to verify that \(y\) is the least upper bound of \(M\), q.e.d.

**Lemma 2.** Every closed bounded subinterval \(I\) of \(S\) is a compact.

**Proof:** Let \(I\) be some closed bounded subinterval of \(S\) with the end-points \(a, b\), \(a < b\), and let \(\mathcal{G} \subset \mathcal{T}\) be an open cover of \(I\). We may assume without loss of generality that the characteristic of \(a\) is +, and the characteristic of \(b\) is -. We wish to show that there is a finite subfamily of \(\mathcal{G}\) which covers the interval \(I\).

Denote by \(A\) the set of all elements \(x \in I\) such that the closed interval \(\langle a, x \rangle = = \{y \in I; a \leq y \leq x\}\) has a finite subcover. Clearly \(a \in A \neq \emptyset\). Let \(s\) be the least upper bound of \(A\), and let \(s'\) be the real part of \(s\). Then the interval \(\langle a, (s', -) \rangle\) has a finite subcover. To see it we may assume that \(a < (s', -)\).

The point \((s', -)\) is in some open set \(G \in \mathcal{G}\), hence \(G\) contains some open interval \(\langle (s' - \varepsilon, +), (s', -) \rangle\), where \(\varepsilon > 0\) is sufficiently small. Since \((s' - - \varepsilon, +) < s\), the interval \(\langle a, (s' - \varepsilon, +) \rangle\) has a finite subcover and hence \(\langle a, (s', -) \rangle = \langle a, (s' - \varepsilon, +) \rangle \cup \langle (s' - \varepsilon, +), (s', -) \rangle\) has also a finite subcover. Now if \(s < b\), then the point \((s', +)\) is in some \(G' \in \mathcal{G}\), hence \(G'\) contains an open interval \(\langle (s', +), (s' + \varepsilon', -, -) \rangle\) with a sufficiently small \(\varepsilon' > 0\).

Since \(\langle a, (s', -) \rangle\) has a finite subcover the interval \(\langle a, (s' + \varepsilon', -) \rangle = \langle a, (s', -) \rangle \cup \langle (s', +), (s' + \varepsilon', -, -) \rangle\) has also a finite subcover contrary to the fact that \(s < (s' + \varepsilon', -)\). Lemma 2 is proved.

**Lemma 3.** Each non-empty closed subset \(P\) of \(S\) is a second category set in itself.

**Proof:** Let \(X = \bigcup_{i=1}^{\infty} P_i\), where \(P_i\) are nowhere dense in \(P\). We wish to show that \(P - X \neq \emptyset\). Since \(P_1\) is nowhere dense in \(P\) there is an open interval \(I\).
such that $I \cap P \neq \emptyset$ and $I \cap \overline{P} = \emptyset$ ($A$ denotes the closure of $A$). It is easy to verify that $I$ contains a closed bounded interval $J_1$ such that $(\text{int } J_1) \cap P \neq \emptyset$. Assume by induction that the closed intervals $J_k$, $1 \leq k < n$, have been constructed such that

$$J_1 \supset J_2 \supset \ldots \supset J_{n-1}, \ (\text{int } J_{n-1}) \cap P \neq \emptyset, \ \text{and} \ J_k \cap \overline{P}_k = \emptyset,$$

for every $k$, $1 \leq k < n$. Since $P_n$ is nowhere dense in $P$, the set $\text{int } J_{n-1}$ contains some closed interval $J_n$ such that $(\text{int } J_n) \cap P \neq \emptyset$ and $J_n \cap \overline{P}_n = \emptyset$. Now, by Lemma 2, the interval $J_1$ is a compact, and $\{J_n \cap P\}_{n=1}^\infty$ is a family of closed subsets of $J_1$ which have the finite intersection property, hence (see Kelley [3], p. 136) the set $\bigcap_{n=1}^\infty (J_n \cap P) = (\bigcap_{n=1}^\infty J_n) \cap P$ is non-empty and $\bigcap_{n=1}^\infty J_n \cap P \subseteq P - X$, q. e. d.

Next consider another topological space. Let $\mathcal{F}$ be the family of closed subintervals of $R = \mathbb{R}_0 \cup \{-\infty\} \cup \{+\infty\}$, and let $\mathcal{T}_1$ be a topology for $\mathcal{F}$ with the following base $\mathcal{B}: G \in \mathcal{B}$ if and only if there is an open set $G_1$ in $R$ such that $G = \{A \in \mathcal{F} ; A \subset G_1\}$. Clearly $(\mathcal{F}, \mathcal{T}_1)$ is a compact.

Let $f: R_0 \to R_0$ be a function. The left range $R_f(x, -)$ of $f$ in $x$, and the right range $R_f(x, +)$ of $f$ in $x$ are the sets

$$R_f(x, -) = \bigcap_{n=1}^\infty f\left(\left(x - \frac{1}{n}, x\right)\right),$$

and

$$R_f(x, +) = \bigcap_{n=1}^\infty f\left(\left(x, x + \frac{1}{n}\right)\right),$$

respectively. Clearly $f(x) \in R_f(x, -) \cap R_f(x, +)$.

Now to each function $f: R_0 \to R_0$ assign three functions

$$f_*: S \to R, \quad f^*: S \to R, \quad \text{and} \ f: S \to \mathcal{F}$$

defined as follows: If $I = \langle a, b \rangle$ is the closure of the connected component of a set $R_f(x, -)$ (resp. $R_f(x, +)$), which contains the point $f(x)$, then

$$f_*(x, -) = a, \quad f^*(x, -) = b, \quad \text{and} \ f(x, -) = I$$

(resp. $f_*(x, +) = a, \ f^*(x, +) = b, \ \text{and} \ f(x, +) = I$).

The functions $f_*$ play an essential role in the next sections.

3. A Characterization Theorem for Darboux Functions

The following two lemmas show that if $f: R_0 \to R_0$ is a Darboux function, then the functions $f_*$ and $f^*$ have characteristic properties.
Lemma 4. For each Darboux function $f: R_0 \to R_0$, $f_*$ is a lower semi-continuous function, and $f^*$ is an upper semi-continuous function.

Proof: We prove the Lemma for $f_*$ (for $f^*$ the proof is similar). Let $z \in [f_* > \lambda]$. Since the construction is symmetric we may assume the characteristic of $z$ to be $-$, i.e. $z = (z', -)$. Hence $f_*(z) > \lambda$. Choose a $\lambda'$ such that $f_*(z) > \lambda' > \lambda$. Since $f$ is a Darboux function, the set $R_f(z) = R_f(z', -)$ is connected (see Bruckner and Ceder [1]) and hence $f_*(z) = f_*(z', -) = \inf_{\xi} R_f(\xi, -)$; thus $\lambda' < \xi$ for every $\xi \in R_f(z', -) = \bigcap_{n=1}^{\infty} f((z' - 1/n, z'))$, and since every set $f((z' - 1/n, z'))$ is connected, there is some $n_0$ such that $\lambda' < \xi$ for every $\xi \in f\left((z' - \frac{1}{n_0}, z')\right)$. Now for each $y \in (z' - \frac{1}{n_0}, z')$, $R_f(y, +) < f\left((z' - \frac{1}{n_0}, z')\right)$ and $R_f(y, -) < f\left((z' - \frac{1}{n_0}, z')\right)$ hence, for each such $y$ we have

$$\inf R_f(y, +) = f_*(y, +) \geq \lambda' > \lambda$$

and

$$\inf R_f(y, -) = f_*(y, -) \geq \lambda' > \lambda.$$ 

Thus the set $[f_* > \lambda]$ contains an open neighbourhood $((z' - 1/n_0, +), (z', -))$ of $z = (z', -)$ which proves the set $[f_* > \lambda]$ to be open, q.e.d.

Lemma 5. For each function $f: R_0 \to R_0$, if $f_*$ is lower semi-continuous, and $f^*$ upper semi-continuous, then $f$ is a Darboux function.

Proof: Let $f_*$ be lower semi-continuous and $f^*$ upper semi-continuous. Assume that contrary to what we wish to show there are numbers $x_1 < x_2$ and $c$ such that $f(x_1) < c < f(x_2)$ and $f(\xi) \neq c$ for every $\xi \in (x_1, x_2)$ (for $f(x_1) > f(x_2)$ the proof is similar). Let $A = [f > c] \cap (x_1, x_2)$ and $B = [f < c] \cap (x_1, x_2)$. Both the sets $A$ and $B$ are bilaterally dense in itself. To see it assume that there is a point $x_0 \in A$, and some $\varepsilon > 0$ such that $A \cap (x_0, x_0 + + \varepsilon) = \emptyset$. In this case we have $f_*(x_0, +) = f(x_0) > c$ but $f_*(z) < c$ for each $z \in ((x_0, +), (x_0 + \varepsilon, -))$, hence $(x_0, +)$ cannot be an interior point of $[f_* > c]$ and consequently $f_*$ fails to be lower semicontinuous. Thus we have proved that every point of $A$ is a cluster point of every its right-hand neighbourhood. In other cases the proof is similar.

Thus the connected components of the sets $A$, $B$ are closed intervals. Let $M$ be the set of components of $A$ and $B$ which contain more than one point, i.e. of components of the form $K = (x, y)$, $x < y$. To every such component $K$
assign the set \( K' = \{(x, y)\} \times \{-, +\} \cup \{x\} \times \{+\} \cup \{y\} \times \{-\}. \) Clearly, \( K' \) is an open set (in \((S, \mathcal{T})\)). Now put

\[
P = \{(x_1, x_2)\} \times \{-, +\} - \bigcap_{K' \in K'} K'.
\]

The interval \((x_1, x_2)\) cannot be written as the union of a (at most countable) family of pairwise disjoint closed nontrivial intervals (Sierpiński [4], p. 220—221), hence there are components of \( A \) or \( B \) which contain exactly one point. From this it follows that \( P \) is non-empty. The set \( P \) is also closed. Now let \( P = P_1 \cup P_2 \), where \( P_1 \) is the set of \( z \in P \) with real part in \( A \), and \( P_2 \) the set of \( z \in P \) with real part in \( B \). Both the sets \( P_1 \) and \( P_2 \) are dense in \( P \), i.e.

\[(1) \quad \bar{P}_1 = \bar{P}_2 = P.\]

Indeed, let \( z \in P \) and assume \( z = (z', -) \), where \( z' \in A \) (in other cases the proof is similar). Since \( z' \in A \), we have \( z \in \bar{P}_1 \). On the other hand \( z \in P \), hence the point \( z' \) cannot be the right-hand end-point of any non-trivial component of the set \( A \); thus in every left-hand neighbourhood of \( z' \) there is a point of \( B \). But in this case every left-hand neighbourhood of \( z = (z', -) \) contains some point of \( P_2 \), hence \( z \in \bar{P}_2 \).

Since \( P \) is closed \( f_* \) is lower semi-continuous, and \( f^* \) is upper semi-continuous, each of the sets

\[
\left[ f_* \leq c - \frac{1}{n} \right] \cap P, \quad \left[ f^* \geq c + \frac{1}{n} \right] \cap P, \quad n = 1, 2, \ldots,
\]

is closed. There is also

\[(2) \quad \left[ f_* \leq c - \frac{1}{n} \right] \cap P \subset P_2 \quad \text{and} \quad (3) \quad \left[ f^* \geq c + \frac{1}{n} \right] \cap P \subset P_1;\]

indeed, if \( f_*(z) \leq c - \frac{1}{n} \) and (say) \( z = (z', +) \), then \( f(z') < c \) hence \( z \in P_2 \) (similarly for \( f^* \)). Now from (1) it follows that each of the sets (2) and (3) is nowhere dense in \( P \). But

\[
P = \left( \bigcup_{n=1}^{\infty} \left[ f_* \leq c - \frac{1}{n} \right] \cap P \right) \cup \left( \bigcup_{n=1}^{\infty} \left[ f^* \geq c + \frac{1}{n} \right] \cap P \right) \]
hence $P$ is a set of the first category in itself contrary to the fact that $P$ is closed and non-empty (see Lemma 3). Thus Lemma 5 is proved.

The next theorem is a consequence of Lemmas 4 and 5 and gives a characterization of Darboux functions using the notion of continuity.

**Theorem 1.** Let $f : R_0 \to R_0$. Then $f$ is Darboux if and only if $\tilde{f}$ is continuous.

**Proof:** It is easy to see that $\tilde{f}$ is continuous if and only if $f^\ast$ is lower semi-continuous and $f^\ast$ upper semi-continuous. From this and from Lemmas 4 and 5 the theorem follows.

4. A Characteristic Type of Convergence for Uniformly Converging Sequences of Darboux Functions

The following Theorem 2 gives a characteristic type of convergence for uniformly converging sequences of Darboux functions. (For facts concerning the uniform closure of $\mathcal{D}$ see Bruckner, Ceder and Weiss [2]). In this section and in Section 5 we use this convention: If $x, y \in R$, and $\varepsilon \in R_0$, $\varepsilon > 0$, then $|x - y| < \varepsilon$ if and only if $x, y \in R_0$ and $|x - y| < \varepsilon$ in the usual sense, or $x = y = +\infty$, or $x = y = -\infty$. Cauchy sequences and uniformly converging sequences of functions with $R$ as domain must be interpreted similarly.

To prove the theorem the following three lemmas are necessary.

**Lemma 6.** Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence of Darboux functions $f_n : R_0 \to R_0$. Then both $\{f^\ast_n\}_{n=1}^\infty$ and $\{f^{\ast\ast}_n\}_{n=1}^\infty$ are Cauchy sequences.

**Proof:** Because of symmetry of the construction it suffices to prove that there is some $n_0$ such that $m > n_0$ implies $|f^\ast_n(z) - f^\ast_m(z)| < \varepsilon$ for arbitrary $z \in S$ with characteristic $+(z = (z', +))$ (for $\{f^{\ast\ast}_n\}_{n=1}^\infty$, and for $z = (z', -)$ the argument is similar).

Each $f_n$ is in $\mathcal{D}$, hence for every positive integers $n, k$, the set $f_n\left(\left\langle z', z' + \frac{1}{k} \right\rangle\right)$ is an interval and since $f_n\left(\left\langle z', z' + \frac{1}{k} \right\rangle\right) \supset f_n\left(\left\langle z', z' + \frac{1}{k + 1} \right\rangle\right)$ we have

$$f^\ast_n(z) = \sup R_{f_n}(z) = \sup \bigcap_{k=1}^\infty f_n\left(\left\langle z', z' + \frac{1}{k} \right\rangle\right) =$$

$$= \lim_{k \to \infty} \left(\sup f_n\left(\left\langle z', z' + \frac{1}{k} \right\rangle\right)\right),$$

for every $n$. Let $\varepsilon > 0$. There is some $n_0$ such that, for each $x \in R_0$, $|f_{n_0}(x) -
\[ -f_m(x) < \varepsilon \text{ whenever } m > n_0. \] For such \( m, n_0 \), from (4) it follows that

\[
 f_m^*(z) - \varepsilon = \lim_{k \to \infty} \left( \sup f_m \left( \left\langle z', z' + \frac{1}{k} \right\rangle \right) - \varepsilon \right) \leq
\]

\[
 \leq \lim_{k \to \infty} \left( \sup f_n \left( \left\langle z', z' + \frac{1}{k} \right\rangle \right) \right) = f_n^*(z) \leq
\]

\[
 \leq \lim_{k \to \infty} \left( \sup f_m \left( \left\langle z', z' + \frac{1}{k} \right\rangle \right) + \varepsilon \right) = f_m^*(z) + \varepsilon.
\]

Thus \( |f_n^*(z) - f_m^*(z)| < \varepsilon \), whenever \( m > n_0 \), q. e. d.

**Lemma 7.** Let \( \{f_n\}_{n=1}^\infty \) be a sequence of Darboux functions \( f_n : R_0 \to R_0 \) converging uniformly to a function \( f \). Then \( \lim_{n \to \infty} (f_n)_* \leq f_* \), and \( \lim f_n^* \geq f^* \).

**Proof:** We prove that \( \lim f_n^*(z) \geq f^*(z) \), where \( z = (z', +) \) (for \( \lim (f_n)_* \leq f_* \), and for \( z = (z', -) \) the argument is similar). Let \( \varepsilon > 0 \). There is some \( n_0 \) such that \( f_n + \varepsilon > f \), whenever \( n > n_0 \). For such \( n_0 \), using (4) we get

\[
 (5) \quad f_n^*(z) + \varepsilon = \lim_{k \to \infty} \left( \sup f_n \left( \left\langle z', z' + \frac{1}{k} \right\rangle \right) \right) + \varepsilon \geq \lim_{k \to \infty} \left( \sup f \left( \left\langle z', z' + \frac{1}{k} \right\rangle \right) \right).
\]

It is easy to verify that

\[
 f^*(z) \leq \sup_{k=1}^\infty f \left( \left\langle z', z' + \frac{1}{k} \right\rangle \right) \leq \lim_{k \to \infty} \left( \sup f \left( \left\langle z', z' + \frac{1}{k} \right\rangle \right) \right).
\]

From this and from (5) it follows that \( f_n^*(z) + \varepsilon \geq f^*(z) \), which proves the Lemma.

In the proof of the next Lemma 8 we use this property of semi-continuous functions: The uniform limit of a sequence of lower semi-continuous functions defined on a first countable topological space \( X \) is lower semi-continuous (similarly with upper semi-continuity). Although this property must be known I have been unable to find a reference. The property follows simply from the fact that a function \( f \) on \( X \) is lower semicontinuous if and only if, for each \( x \in X \), and each sequence \( \{x_n\}_{n=1}^\infty \) of points in \( X \) which converges to \( x \),

\[
 (6) \quad \lim_{n \to \infty} \inf g(x_n) \geq g(x)
\]

(see Kelley [3], pp. 72 and 101).

**Lemma 8.** Let \( \{f_n\}_{n=1}^\infty \) be a sequence of Darboux functions \( f_n : R_0 \to R_0 \) converging uniformly to a function \( f \). Then \( f \) is Darboux if and only if both \( \lim f_n^* = f^* \), and \( \lim (f_n)_* = f_* \).
Proof: Let \( f \notin \mathcal{D} \). By Lemma 6, \( \{f_{n}^{*}\}_{n=1}^{\infty} \) converges uniformly to a function \( g : S \to \mathbb{R} \); since every \( f_{n}^{*} \) is upper semi-continuous the function \( g \) is also upper semi-continuous. Similarly the sequence \( \{f_{n}^{-}\}_{n=1}^{\infty} \) converges uniformly to a lower semi-continuous function \( h \). But \( f \notin \mathcal{D} \), hence by Lemma 5 either \( f^{*} \neq g \), or \( f^{-} \neq h \), which proves the first implication.

Now let \( f \in \mathcal{D} \). Clearly, it suffices to prove that \( \lim_{n \to \infty} f_{n}^{*}(z) = f^{*}(z) \) for some \( z = (z', +) \in S \) whose characteristic is \( + \) (in other cases the proof is similar). Let \( \varepsilon > 0 \). Using (4) we get, for sufficiently large \( n \),

\[
 f_{n}^{*}(z) + \varepsilon = \lim_{k \to \infty} \left( \sup_{f_{n}} \left( (z', z' + \frac{1}{k}) \right) + \varepsilon \right) \geq \lim_{k \to \infty} \sup_{f_{n}} \left( (z', z' + \frac{1}{k}) \right) = f^{*}(z) \geq \lim_{k \to \infty} \left( \sup_{f_{n}} \left( (z', z' + \frac{1}{k}) \right) \right) - \varepsilon = f_{n}^{*}(z) - \varepsilon, \ q. \ e. \ d.
\]

Now we are able to prove the following

**Theorem 2.** Let \( \{f_{n}\}_{n=1}^{\infty} \) be a sequence of Darboux functions \( f_{n} : \mathbb{R}_{0} \to \mathbb{R}_{0} \) converging uniformly to a function \( f \). Then \( f \) is Darboux if and only if \( \lim_{n \to \infty} f_{n} = f \)

(in the topology \( \mathcal{T}_{1} \)).

Proof: Let \( f \in \mathcal{D} \). Let \( z \in \mathbb{S} \) and let \( G \) be an open neighbourhood (in \( \mathcal{T}_{1} \)) of \( f(z) \). There is an open interval \( J \subset \mathbb{R} \) such that \( f(z) = f_{*}(z) = (f_{n}(z), f_{*}(z)) \subset J \), and every closed subinterval of \( J \) is in \( G \). By Lemma 8, \( \lim_{n \to \infty} f_{n}^{*} = f^{*} \), and \( \lim_{n \to \infty} f_{n}^{-} = f_{-}^{*} \); hence \( f_{n}(z) = (f_{n}(z), f_{n}(z)) \subset J \) and hence \( f_{n}(z) \in G \), for sufficiently large \( n \). Thus \( f_{n} \) converges to \( f \).

On the other hand let \( f \notin \mathcal{D} \). By Lemma 8, there is either \( \lim_{n \to \infty} f_{n}^{*} = f^{*} \), or \( \lim_{n \to \infty} f_{n}^{-} = f_{-}^{*} \), hence by Lemma 7 either \( \lim_{n \to \infty} f_{n}(z) = f^{*}(z) \), or \( \lim_{n \to \infty} f_{n}^{-}(z) < \lim_{n \to \infty} f_{n}(z) \). So there is some open interval \( J \subset \mathbb{R} \) such that \( f(z) = (f_{n}(z), f_{*}(z)) \subset J \) and there is an \( n \) as large as we want such that \( (f_{n}(z), f_{n}(z)) \notin J \). Now the set \( G \) of closed subintervals of \( J \) is a neighbourhood of \( f(z) \) such that there is some \( n \) arbitrary large with \( f_{n}(z) \notin G \). Thus \( f_{n} \) fails to converge to \( f \), \ q. \ e. \ d.
5. Some Sufficient Conditions for a Limit of Darboux Functions to be a Darboux Function

Since \( \mathcal{D} \) is not closed under the uniform limits (see Bruckner, Ceder and Weiss [2]) from Theorem 2 it follows that there is a sequence \( \{f_n\}_{n=1}^\infty \) of Darboux functions such that \( \lim_{n \to \infty} f_n = f \), but \( \tilde{f}_n \) fails to converge to \( \tilde{f} \). In the present section we shall consider the sequences \( \{f_n\}_{n=1}^\infty \) of Darboux functions \( f_n : \mathbb{R}_0 \to \mathbb{R}_0 \) with the following property: There is a function \( f \) such that \( \{f_n\}_{n=1}^\infty \) converges pointwise to \( f \) and \( \tilde{f}_n \) to \( \tilde{f} \). For such sequences some sufficient and necessary conditions for \( f \) to be in \( \mathcal{D} \) are shown below. At first we note that in general \( \tilde{f}_n \to \tilde{f} \) does not imply \( f \in \mathcal{D} \) as it is shown in the following example.

**Example.** Define \( f_n : \mathbb{R}_0 \to \mathbb{R}_0 \) by

\[
f_n(x) = \begin{cases} 
1 + \frac{1}{n} \sin \left( \frac{1}{x} \right) & \text{if } 0 < x \leq \frac{1}{n}, \\
\pi(1 - nx) \quad & \text{if } \frac{1}{n} < x \leq \frac{1}{n}, \\
\pi - 1 & \text{if } \frac{1}{n} < x, \\
0 & \text{if } \frac{1}{n} < x \leq 0, \\
1 & \text{if } x \leq 0,
\end{cases}
\]

and let \( f(x) = 0 \) for \( x > 0 \), and \( f(x) = 1 \) for \( x \leq 0 \). Clearly every \( f_n \) is in \( \mathcal{D} \) and \( \lim_{n \to \infty} f_n = f \notin \mathcal{D} \). On the other hand, \( \tilde{f}_n(0, +) = \left( 1 - \frac{1}{n}, 1 + \frac{1}{n} \right) \), and for \( z \neq (0, +) \), \( \tilde{f}_n(z) = f_n(z') \), where \( z' \) is the real part of \( z \). Similarly for every \( z \), \( \tilde{f}(z') = f(z') \), where \( z' \) is the real part of \( z \). Thus \( \tilde{f}_n \) converge to \( \tilde{f} \).

The next theorem gives a sufficient condition for the limit of a sequence of Darboux functions to be also Darboux.

For the sake of simplicity, if \( I \) is an interval in \( \mathbb{R} \), and \( \varepsilon > 0 \), let \( O\varepsilon(I) \) denote the open \( \varepsilon \)-neighbourhood of \( I \) (in \( \mathbb{R} \)).

**Theorem 3.** Let \( \{f_n\}_{n=1}^\infty \) be a sequence of Darboux functions \( f_n : \mathbb{R}_0 \to \mathbb{R}_0 \) converging pointwise to a function \( f \), and let \( \tilde{f}_n \) converge pointwise to \( \tilde{f} \). If for every \( \varepsilon > 0 \), and every \( m \), there is some \( n > m \) such that
for every $z \in S$, then $f$ is a Darboux function.

Proof: Let $\delta > 0$, and $z_0 \in S$. Since $(f_n)_* \text{converges to } f_*$ there is some $m_0$ such that

\begin{equation}
m' > m_0 \implies (f_{m'})_*(z_0) > f_*(z_0) - \frac{\delta}{3}.
\end{equation}

Put in (7) $m = m_0$, and $\epsilon = \frac{\delta}{3}$. Since $(f_i)_*, m < i \leq n$, are lower semi-continuous there is a neighbourhood $O(z_0)$ of $z_0$ such that $z \in O(z_0)$ implies

\begin{equation}
(f_i)_*(z) > (f_i)_*(z_0) - \frac{\delta}{3},
\end{equation}

where $m < i \leq n$ (see (6)). Now from (7) it follows that for every $z \in O(z_0)$ there is some $n_z$ with $m + 1 \leq n_z \leq n$ such that

\begin{equation}
f_*(z) > (f_{n_z})_*(z) - \frac{\delta}{3} > (f_{n_z})_*(z_0) - \frac{2\delta}{3} > f_*(z) - \delta
\end{equation}

(here the second inequality follows from (9), and the third from (8)). Hence $f_*(z) \geq f_*(z_0)$ for every $z \in O(z_0)$ and consequently $f_*$ is lower semi-continuous. A similar argument shows that $f^*$ is upper semi-continuous and hence by Lemma 5, $f \in \mathcal{D}$, q. e. d.

The next theorem is more general than Theorem 3. It gives a sufficient condition for the limit $f$ of a sequence $\{f_n\}_{n=1}^\infty$ of functions to be in $\mathcal{D}$, where $f_n$ are arbitrary functions $f_n : R_0 \to R_0$ such that $\tilde{f}_n \to \tilde{f}$. First we prove the following lemma:

**Lemma 9.** Let $T$ be a first countable topological space. Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions $f_n : T \to R$ which converges pointwise to a function $f$. Then $f$ is lower (upper) semi-continuous if and only if for every $a \in T$ and every $\epsilon > 0$ there is a neighbourhood $O(a)$ of a such that for every $z \in O(a)$ and every $k$ there is some $m$ with

$$f_{k+m}(z) > f(a) - \epsilon \quad \text{(resp. } f_{k+m}(z) < f(a) + \epsilon);$$

in symbols

\begin{equation}
\forall \forall \exists \forall \exists f_{k+m}(z) > f(a) - \epsilon \quad \text{(resp. } f_{k+m}(z) < f(a) + \epsilon)\quad \forall a \in O(a) \forall \exists k \exists m \quad \epsilon > 0.
\end{equation}
Proof: Because of the symmetry it suffices to prove the Lemma for lower semi-continuous functions. Thus assume the condition (10) to be satisfied. Let \( \{z_n\}_{n=1}^{\infty} \) be a sequence converging in \( T \) to \( a \). We can assume \( z_n \in O(a) \), for every \( n \). Since \( f_n \) converge to \( f \) there is a \( k_1 \) such that \( (fz_1) > f_k(z_1) - \varepsilon \), for \( k > k_1 \). In general, let \( k_n \) be a positive integer such that for every \( k > k_n \), \( f(z_n) > f_k(z_n) - \varepsilon \). From (10) it follows that there is a sequence \( \{m_i\}_{i=1}^{\infty} \) of positive integers such that \( f_{k_n+m_n}(z_n) > f(a) - \varepsilon \), for every \( n \). Hence
\[
f(z_n) > f_{k_n+m_n}(z_n) - \varepsilon > f(a) - 2\varepsilon
\]
and hence
\[
\liminf_{n \to \infty} f(z_n) \geq f(a) - 2\varepsilon;
\]
thus \( \liminf_{n \to \infty} f(z_n) \geq f(a) \) and consequently (see (6)) \( f \) is lower semi-continuous.

Now assume that a sequence \( \{f_n\}_{n=1}^{\infty} \) converges to a lower semi-continuous function \( f \) and that contrary to what we wish to show the condition (10) is not satisfied. Then
\[
\exists \exists \forall \exists \exists \forall \exists f_{k+m}(z) \leq f(a) - \varepsilon.
\]
Hence in every neighbourhood of \( a \) there is a point \( z \) such that, for every \( m \), \( f_{k+m}(z) \leq f(a) - \varepsilon \), so that \( \lim_{m \to \infty} f_{k+m}(z) = f(z) \leq f(a) - \varepsilon \). But in this case \( f \) cannot be lower semi-continuous (see (6)) in \( a \). The contradiction finishes the proof of the Lemma.

Now we are able to prove the following.

Theorem 4. Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of functions \( f_n : R_0 \to R_0 \) converging pointwise to a function \( f \) such that \( f_n \) converges to \( f \). Then \( f \) is in \( \mathcal{D} \) if and only if for every \( a \in S \), and \( \varepsilon < 0 \), there is a neighbourhood \( O(a) \) of \( a \) such that for every \( z \in O(a) \),
\[
\tilde{f}(z) \subset O(\tilde{f}(a)).
\]

Proof: \( S \) is a first countable topological space (see the section 2 above) hence Lemma 9 can be applied. Replace the functions \( f_{k+m}, f \), in (10) by \( (f_{k+m})^*, f^* \), resp. \( (f_{k+m})^*, f^* \), to obtain the condition
\[
\forall \forall \exists \forall \exists \exists f_{k+m}(z) = \langle (f_{k+m})^*(z), (f_{k+m})^*(z) \rangle \subset O_{\varepsilon/2}(\tilde{f}(a));
\]
From this the Theorem follows.
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