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ON SOME GENERALIZATIONS OF LIE ALGEBRAS

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§ 1. Weak Lie pseudo-algebra.
Let $K$ be a commutative field and $L$ a vector space over $K$.

**Definition.** $L$ is a weak Lie pseudo-algebra (w. L. p.-a) if, to every pair of vectors $x, y$ in $L$, there corresponds a vector $z \in L$, called the product of $x$ and $y$, $z = [x, y]$, satisfying the following axioms:

1°. $[x, y + z] = [x, y] + [x, z] - [x, o]$
2°. $[x + y, z] = [x, z] + [y, z] - [o, z]$
3°. $[x, y] = [x, y] - (x - 1)[x, o] + \langle x, o \rangle y, \langle x, o \rangle \in K$
4°. $[x, o] = o$
5°. There exist $\mu, \nu \in K$ such that for every $x, y, z \in L$,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] - \mu[[x, y], o] +$$
$$+ [[y, z], o] + [[z, x], o] - \nu([o, x] + [o, y] + [o, z]) = o.$$

CONSEQUENCES FROM THE DEFINITION

1. From the axioms 1, 2, 4 it is possible to infer the anticommutativity of the multiplication:

$$(1) \quad [x, y] = -[y, x],$$

if the characteristic of the field $K$ is different from two.

2. The consequence 1 and the third axiom involve:

$$(2) \quad [x, y] = \alpha[x, y] - (x - 1)[o, y] - (y, \alpha)x.$$

3. If we take in the 5-th axiom $x = y = o$, we obtain

$$(1 - \nu)[o, z] = 0.$$  

Since in general $[o, z] \neq o$, it follows that $\nu = 1$. In the 5-th axiom there rests only the constant $\mu$ called "the constant of type".

4. $\sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j = \sum_{i=1}^{n} \sum_{j=1}^{m} [x_i, y_j] - (n - 1) \sum_{j=1}^{m} [o, y_j]$. 

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§ 2. Special classes of w. L. p.-a and examples

If \([x, o] = [o, x] = o, \forall x \in L\), we obtain Lie’s pseudo-algebras introduced by Herz [2] and when \([x, x] = o, \forall x \in L, \forall x \in K\), the weak Lie’s algebra (w. L. a).

Finally, if \([x, o] = [o, x] = o\) and \([x, x] = o, \forall x \in L, \forall x \in K\), we obtain the usual Lie algebras.

We shall give some examples of w. L. a and w. L. p.-a.

1. Let \(\mathcal{A}\) be a weak algebra [1], that means, a linear algebra in which the distributivity laws are replaced by:

\[
x(y + z) = xy + xz - x \cdot o, \quad (x + y)z = xz + yz - o \cdot z.
\]

If \(\mathcal{A}\) is endowed with the product:

\[
[x, y] = xy - yx,
\]
we obtain a w. L. a with the constant of the type \(\mu = o\). This is the weak Lie algebra associated to a weak algebra.

2. Let \(\Phi\) be the set of functions:

\[
f : [GF(2)]^n \to [GF(2)].
\]

By using the inner operation:

\[
[f, g] = f(x_1, \ldots, x_{n-1}, g(x_1, \ldots, x_n)) \oplus g(x_1, \ldots, x_{n-1}, f(x_1, \ldots, x_n))
\]

where the sign \(\oplus\) means the addition modulo 2, we obtain a w. L. a.

Indeed, using the common notations from the logical algebra, every \(f\) belonging to \(\Phi\) may be written as:

\[
f = x_n f_1 \oplus (1 \oplus x_n) f_0 = x_n (f_1 \oplus f_0) \oplus f_0,
\]

where \(f_1\) and \(f_0\) are respectively \(f(x_1, \ldots, x_{n-1}, o)\) and \(f(x_1, \ldots, x_{n-1}, 1)\).

Then:

\[
[f, g] = (x_n (g_1 \oplus g_0) \oplus g_0)(f_1 \oplus f_0) \oplus f_0 \oplus
\]

\[
\oplus (x_n (f_1 \oplus f_0) \oplus f_0)(g_1 \oplus g_0) \oplus g_0 = g_0 f_1 \oplus f_0 g_1 \oplus f_0 \oplus g_0.
\]
We obtain immediately the relation:
\[
[f, g \oplus h] = [f, g] \oplus [f, h] \oplus [f, o],
\]
\[
[f \oplus g, h] = [f, h] \oplus [g, h] \oplus [f, o],
\]
\[
[f, f] = o,
\]
and
\[
[[f, g], h] \oplus [[g, h], f] \oplus [[h, f], g] \oplus [f, o] \oplus [g, o] \oplus [h, o] = o,
\]
(Here \( a = -a \pmod{2} \) and \([f, o] = f_0\)).

Using the properties of this algebra we obtained some applications in the switching theory [5, 6].

3. Let \( L \) be a distributive algebra in the usual sense.

If we introduce the product
\[
[x, y] = xy - yx + x - y,
\]
we obtain a w. L. a.

We have:
\[
[x, y + z] = [x, y] + [x, z] - [x, o],
\]
\[
[x + y, z] = [x, z] + [y, z] - [o, z],
\]
\[
[x, x] = o; \quad [x, o] = x,
\]
and
\[
[[x, y], z] + [[y, z], x] + [[z, x], y] + \{[[x, y], o] + [[x, z], o] +
\]
\[
+ [[z, x], o]\} - ([x, o] + [y, o] + [z, o]) = o.
\]

For this w. L. a the constant of type is \( \mu = 1 \).

4. Let \( \mathcal{F}(\tau) \) be the field of meromorphic functions of the variable \( \tau \) in a simply connected region and \( R_n \) an \( n \)-dimensional vector space of \( n \)-tuples, defined over \( \mathcal{F}(\tau) \).

Let \( \{u_i\}, i = 1, n \) be a basis in \( R_n \). We suppose that \( R_n \) is differentiably closed, namely the vectors \( u_i \) verify the equations
\begin{equation}
\frac{du_i}{d\tau} = \alpha_i u_j(\tau).
\end{equation}

If \( u \in R_n \), then \( u = u_i \alpha_i, \alpha_i \in \mathcal{F}(\tau) \).
We introduce the mapping $T : R^n \rightarrow R^n$, by

$$Tu = \frac{du_i}{d\tau} \alpha_i + u_i \frac{d^2\lambda}{d\tau}. \quad (6)$$

It is clear that:

$$T(u + v) = Tu + Tv.$$

Then

$$T(\lambda u) = \lambda Tu + \frac{d\lambda}{d\tau} u.$$  

Because formula (6) contains the term $\frac{du_i}{d\tau} \alpha_i$ the transformation $T$ depends on the basis chosen in $R^n$.

It is clear that there are many transformations $T$. Let us consider the matrix $U$ of the basis, $U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$; because $u_i$ are vectors the $U$-type is $n \times n$.

If we take a new basis, related to the basis $U$ by the relation $U = AV$, where $A = (A_{ij})$ is an invertible $n \times n$ matrix, from (5) it follows that:

$$\frac{dA_{ij}}{d\tau} v_j + A_{ij} \frac{dv_j}{d\tau} = \alpha_i \alpha h A_hk v_k,$$

or:

$$A_{ij} \frac{dv_j}{d\tau} = \left( - \frac{dA_{ik}}{d\tau} + \alpha_i \alpha h A_hk \right) v_k,$$

and if we denote by $B = (B_{ij})$ the inverse of the matrix $A$, we obtain the new coefficients of the equation (5):

$$\tilde{\alpha}_{ij} = B_{ip} \left( - \frac{dA_{pj}}{d\tau} + \alpha_p \alpha h A_hf \right).$$

Denote by $\mathcal{F}$ the set of all transformations $T$.

Let us consider the operation

$$[\ldots] : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F},$$
defined by
\[ [T, U] = T \circ U - U \circ T + T - U, \]
where
\[ (T \circ U)u = T(Uu). \]

To make \( F \) a w. L. p.-a. we interpret addition and scalar multiplication as the ordinary addition of two transformations and multiplication of a transformation by a scalar and use the product defined by (7).

Thus
\[
[T + U, V]u = ((T + U) \circ V)u - (V \circ (T + U))u + (T + U)u - Vu = (T \circ V - V \circ T + T - V)u + (U \circ V - V \circ U + U - V)u + Vu = ([T, V] + [U, T])u + Vu.
\]

Since
\[
[0, V] = -V,
\]
we obtain
\[
[T + U, V] = [T, V] + [U, V] - [0, V].
\]

Now, by using (7) and (6), we have:
\[
[\alpha T, U]u = (\alpha T \circ U)u - (U \circ \alpha T)u + \alpha Tu - Uu = ((\alpha T \circ U)u - (\alpha U \circ T)u + d\alpha/d\tau Tu) + \alpha Tu - Uu = (\alpha T \circ U)u - (\alpha U \circ T)u - d\alpha/d\tau Tu + \alpha Tu - Uu = \alpha [T, U]u - (\alpha - 1)[0, U]u - \langle U, \alpha \rangle Tu.
\]

For \( \forall T \in F \) by comparing with the relation (2), we have
\[
\langle T, \alpha \rangle = d\alpha/d\tau.
\]

Finally, for the axiom 5, let us calculate:
\[
[[T, U], V] + [[U, V], T] + [[V, T], U] = 1)
\]
\[
= [TU - UT + T - U, V] + [UV - VU + U - V, T] + [VT - TV + V - T, U] = (TU - UT + T - U)V - V(TU - UT + T - U) + TU - UT + T - U - V + (UV - VU + U - V)T - T(UV - VU + U - V) + UV - VU + U - V - T + (VT - TV + V - T)U - U(VT - TV + V - T) + \]

1) Here for simplicity we write \( TU \) for \( T \circ U \).
Thus we obtain

\[[T, U], V] + [[U, V], T] + [[V, T], U] + [T, 0] + [U, 0] + [V, 0] +
+ [[T, U], 0] + [[U, V], 0] + [[V, T], 0] = 0.

§ 3. The transformation \langle \ldots \rangle in a w. L. p.-a

For the transformation \langle \ldots \rangle : L \times K \rightarrow K, the w. L. p.-a acts like a domain of operators over the field \( K \).

Now, we shall deal with some properties of these transformations.

**Proposition 1. If the characteristic of the field \( K \) is not two, then \langle \ldots \rangle : L \times \times K \rightarrow K is a bilinear application.**

We note next that according to the axiom 2,

\[ [x + y, az] = [x, az] + [y, az] - [o, az], \]

on the other hand according to the axioms 3 and 2,

\[ [x + y, az] = \varepsilon [x + y, z] - (\varepsilon - 1)[x + y, o] + \langle x + y, \varepsilon z =
\]

\[ = \varepsilon [x, z] + \varepsilon [y, z] - \varepsilon [o, z] - (\varepsilon - 1)[x, o] -
\]

\[ - (\varepsilon - 1)[y, o] + \langle x + y, \varepsilon z =
\]

\[ = [x, z] - (\varepsilon - 1)[x, o] + \langle x, \varepsilon z + \varepsilon [y, z] -
\]

\[ - (\varepsilon - 1)[y, o] + \langle y, \varepsilon z - \varepsilon [o, z] - \langle o, \varepsilon z. \]

Hence,

\[ \langle x + y, \varepsilon z = \langle x, \varepsilon z + \langle y, \varepsilon z - \langle o, \varepsilon z. \]

This relation holds for \( \forall z \in L \), therefore:

\[ \langle x + y, \varepsilon = \langle x, \varepsilon + \langle y, \varepsilon - \langle o, \varepsilon. \]

If we set \( y = -x \), taking into account proposition 2, when the characteristic of the field is different from two, we obtain \( \langle 0, \varepsilon = 0 \) and so

\[ \langle x + y, \varepsilon = \langle x, \varepsilon + \langle y, \varepsilon. \]

Hence

\[ [x, (\varepsilon + \beta)y] = [x, \varepsilon y + \beta y] = [x, \varepsilon y] + [x, \beta y] - [x, o] =
\]

\[ = \varepsilon [x, y] - (\varepsilon - 1)[x, o] + \langle x, \varepsilon y + \beta [x, y] - (\beta - 1)[x, o] + \]

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\[ + \langle x, \beta \rangle y - [x, o] = \]
\[ = (\alpha + \beta)[x, y] - (\alpha + \beta - 1)[x, o] + \langle x, \alpha \rangle y + (x, \beta \rangle y. \]

On the other hand,
\[ [x, (\alpha + \beta)y] = (\alpha + \beta)[x, y] - (\alpha + \beta - 1)[x, o] + \langle x, \alpha + \beta \rangle y, \]
therefore
\[ \langle x, \alpha + \beta \rangle = \langle x, \alpha \rangle + \langle x, \beta \rangle. \]

**Proposition 2.** If the characteristic of the field \( K \) is not two, then
\[ \langle \beta y, \alpha \rangle = \beta \langle y, \alpha \rangle. \]

Let us calculate the expression
\[ [\alpha x, \beta y] = \beta[\alpha x, y] - (\beta - 1)[\alpha x, o] + \langle \alpha x, \beta y \rangle = \]
\[ = \beta(\alpha x, y) - (\alpha - 1)[o, y] - \langle y, \alpha \rangle x - \]
\[ - (\beta - 1)\alpha[x, o] + \langle \alpha x, \beta y \rangle, \]
therefore
\[ [\alpha x, \beta y] = \alpha \beta(x, y) - \beta(\alpha - 1)[o, y] - \langle y, \alpha \rangle x + \langle \alpha x, \beta y \rangle = \]
\[ = \alpha \beta([x, y] - [x, o] - [o, y]) + \alpha[x, o] + \beta[o, y] - \]
\[ - \langle y, \alpha \rangle x + \langle \alpha x, \beta y \rangle. \]

Then
\[ [\beta y, \alpha x] = - [\alpha x, \beta y] = \alpha \beta([y, x] - [y, o] - [o, x]) + \]
\[ + \beta[y, o] + \alpha[o, x] - \langle x, \beta \rangle y + \langle \beta y, \alpha \rangle x, \]
whence
\[ \langle y, \alpha \rangle \beta x - \langle \alpha x, \beta \rangle y = \beta \langle y, \alpha \rangle x - \langle x, \beta \rangle y, \]
or
\[ (\beta \langle y, \alpha \rangle - \langle \beta y, \alpha \rangle)x + (\alpha \langle x, \beta \rangle - \langle \alpha x, \beta \rangle)y = 0. \]

As this relation holds \( \forall x, y \in L \), we get the necessary equality.

**Proposition 3.** We have
\[ \langle x, \alpha \beta \rangle = \langle x, \alpha \rangle \beta + \langle x, \beta \rangle \alpha, \]

**hence the elements of a w. L. p.-a may be interpreted like differentiations over the field \( K \).**
Indeed

\[ [x, \alpha \beta y] = [x, \alpha(\beta y)] = \alpha[x, \beta y] - (\alpha - 1)[x, o] + \langle x, \alpha \beta y \rangle = \alpha \beta[x, y] - (\alpha - 1)[x, o] + \langle x, \beta \alpha y \rangle - (\alpha - 1)[x, o] + \langle x, \beta \alpha y \rangle - \langle x, \alpha \beta y \rangle. \]

On the other hand

\[ [x, (\alpha \beta)y] = \alpha \beta[x, y] - (\alpha \beta - 1)[x, o] + \langle x, \alpha \beta y \rangle. \]

Comparing these two expressions we obtain the relation (8).

**Proposition 4.** \( \langle x, \pm n \rangle = 0, x \in L, \) where \( n = 1 + \ldots + 1, (n \text{ times}), \) 1 being the unity in \( K. \)

If in axiom 3 we put \( \alpha = 1, \) it is easy to see that \( \langle x, 1 \rangle = 0, \forall y \in L, \)

hence \( \langle x, 1 \rangle = 0. \)

Thus \( \langle x, o \rangle = [x, o] - [x, o] = o. \)

Thus \( \langle x, -1 \rangle = \langle x, -1 \rangle + \langle x, 1 \rangle = \langle x, 1 - 1 \rangle = \langle x, o \rangle = o. \)

Taking into account the proposition 1, we obtain P. 4.

**Corollary**

\[ [x, - y] = [y, x] + 2[x, o] \]
\[ [- x, y] = [y, x] + 2[o, y]. \]

**Proposition 5.** \( \langle x, - \alpha \rangle = - \langle x, \alpha \rangle. \)

Indeed:

\[ \langle x, - \alpha \rangle = \langle x, (-1)\alpha \rangle = \langle x, -1 \rangle \alpha + \langle x, \alpha \rangle(-1) = - \langle x, \alpha \rangle, \]

using the propositions 3 and 4.

**Proposition 6.** For every \( \alpha \in K \) and \( x, y \in L, \) the following holds:

\( \mu(\langle x, \alpha \rangle - \langle y, \alpha \rangle)[z, o] + (\langle [x, y], \alpha \rangle - \langle x, \langle y, \alpha \rangle \rangle + \langle y, \langle x, \alpha \rangle \rangle)z = o. \)

Let us consider now the axiom 5:

\[ [[x, y], az] + [[y, az], x] + [[az, x], y] - \mu([[x, y], o] + [[y, az], o] + [[az, x], o]) - ([o, x] + [o, y] + [o, az]) = o. \]

Hence, since:

\[ [[x, y], az] = az[[x, y], z] - (a - 1)[[x, y], o] + \langle x, \alpha \rangle z, \]
\[ [[y, az], x] = ax[[y, z], x] - \langle y, \alpha \rangle [o, x] + \langle x, \alpha \rangle [y, o] - \langle x, \alpha \rangle [y, z] + \langle y, \alpha \rangle [z, x] + (1 - \alpha)[[y, o], x] - \langle x, \langle y, \alpha \rangle \rangle z, \]
\[ [[az, x], y] = ax[[z, x], y] + \langle x, \alpha \rangle [o, y] + \langle y, \alpha \rangle [o, x] + \]

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\[ + (1 - a)[[o, x], y] - \langle y, \alpha \rangle [z, x] - \langle x, \alpha \rangle [z, y] + \langle y, \langle x, \alpha \rangle \rangle z, \]

and
\[
[[y, \alpha z], o] = \alpha[[y, z], o] - (\alpha - 1)[[y, o], o] + \langle y, \alpha \rangle [z, o],
[[\alpha z, x], o] = \alpha[[z, x], o] - (\alpha - 1)[[o, x], o] - \langle x, \alpha \rangle [z, o],
\]

it follows that:
\[
\alpha([[x, y], z] + [[y, z], x] + [[z, x], y]) + \langle [x, y], \alpha \rangle - \langle x, \langle y, \alpha \rangle \rangle + \\
+ \langle y, \langle x, \alpha \rangle \rangle z + (1 - \alpha)([[o, x], y] + [[x, y], o] + [[y, o], x]) - \\
- \mu([[x, y], o] + \alpha[[y, z], o] + \alpha[[z, x], o] - (\alpha - 1)[[y, o], o] - \\
- (\alpha - 1)[[o, x], o] + \langle y, \alpha \rangle [z, o] - \langle x, \alpha \rangle [z, o]) - [o, x] - \\
- [o, y] - \alpha [o, z] = o,
\]

and using axiom 5, we obtain relation (9).

**Corollary.** If \( \mu = 0 \) or \( [z, o] = 0 \) we obtain Herz’s relation
\[
\langle [x, y], \alpha \rangle = \langle x, \langle y, \alpha \rangle \rangle - \langle y, \langle x, \alpha \rangle \rangle.
\]

§ 4. The adjoint transformation in a w. L. p.-a

We introduce in a w. L. p.-a the adjoint transformation by:
\[
ad(x)y = [x, y] - [x, o] - [o, y].
\]

**Proposition 7.** In a w. L. p. a. ad is a pseudo-linear transformation. **Indeed,** we have:
\[
ad (x)xy = [(x, ay) - [x, o] - [o, ay] = \\
= \alpha[x, y] - (\alpha - 1)[x, o] - [x, o] + (x, \alpha) y - \\
- \alpha[o, y] = \alpha[x, y] - \alpha[x, o] - \alpha[o, y] + (x, \alpha) y = \\
= \alpha ad (x)y + (x, \alpha) y,
\]

and
\[
ad (x)(y + z) = [x, y + z] - [x, o] - [o, y + z] = \\
= [x, y] + [x, z] - [x, o] - [x, o] - [o, y] - [o, z] - \\
= ad (x)y + ad(x)z,
\]

hence
\[
ad (x)(xy + \beta z) = \alpha ad (x)y + \beta ad (x)z + \langle x, \alpha \rangle y + \langle x, \beta \rangle z.
\]

If \( \langle x, \alpha \rangle = 0, \forall x \in L \), then ad is a linear transformation (the case of weak Lie algebras).
Proposition 8. There hold the following equalities:

\[ \text{ad } (x + y) = \text{ad } x + \text{ad } y \]

The proof of these relations is an immediate one.

If in the set of the adjoint transformations of a weak Lie algebra we introduce the ordinary addition and multiplication by a scalar, and use the product:

\[ ([\text{ad } x, \text{ad } y])z = (\text{ad } [x, y])z, \]

then this set becomes a weak Lie algebra.

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