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THE FUBINI THEOREM AND CONVOLUTION OF VECTOR-VALUED MEASURES

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Let $X$ be a Banach algebra. Let $G$ be a compact Hausdorff topological semigroup. Denote $\mathcal{B}(G)$ the $\sigma$-algebra of Borel sets in $G$. If $m : \mathcal{B}(G) \to X$ and $n : \mathcal{B}(G) \to X$ are regular Borel measures both with finite variation, then their convolution is a regular Borel measure on $\mathcal{B}(G)$, with finite variation, with values in $X$ which can be defined in two equivalent ways.

In the first definition, for each Borel subset $D$ of $G$, $m \ast n(D)$ is defined to be $m \otimes n(E)$, where $E$ is the Borel subset $\{(s, t) : st \in D\}$ of $G \times G$ and $m \otimes n$ is the unique regular Borel measure on $\mathcal{B}(G \times G)$, with finite variation, with values in $X$ such that

$$
\int_{G \times G} g \, d(m \otimes n) = \int_{G} \{ \int_{G} g(s, t) \, dm(s) \} \, dn(t)
$$

for all continuous functions $g$ on $G \times G$.

In the second definition, $m \ast n$ is taken to be the unique regular Borel measure on $\mathcal{B}(G)$, with finite variation, with values in $X$ satisfying

$$
\int_{G} f \, d(m \ast n) = \int_{G} \{ \int_{G} f(st) \, dm(s) \} \, dn(t)
$$

for all continuous functions $f$ on $G$ [cf. 5].

We wish to prove that both definitions are equivalent, similarly as in a complex case [cf. 3 and 9]. Also the first definition makes it possible, in case $G$ is a group, to give $m \ast n$ explicitly by the formula

$$
m \ast n(D) = \int_{G} m(Dt^{-1}) \, dn(t) = \int_{G} n(s^{-1}D) \, dm(s)
$$

for each $D$ in $\mathcal{B}(G)$. For this and other purposes the Fubini theorem for vector-valued measures is needed. Thus we establish a theorem of this kind convenient for our purposes.
1. Vector-valued measures in product spaces

Let $X$, $Y$ and $Z$ be Banach spaces. Let a bilinear continuous mapping of $X \times Y$ into $Z$ be given, denoted by juxtaposition, $z = xy$, $x \in X$, $y \in Y$, $z \in Z$ ($|xy| \leq |x| |y|$). Let $S$ and $T$ be compact Hausdorff topological spaces. Denote by $\mathcal{B}(S)$, $\mathcal{B}(T)$ the $\sigma$-algebra of Borel sets in $S$, $T$, respectively. For our purposes it is convenient to introduce a vector-valued measure in the product space $S \times T$ by means of dominated operators introduced by Dinculeanu [cf. 4, p. 379] and we use the terminology from his book. By $C(S)$ is meant, as usual, the Banach space of all continuous functions $f : S \to C$ ($C = \text{real line or complex plane}$) equipped with the standard supremum norm. Following Dinculeanu [4, p. 379] we say that a linear operator $U : C(S) \to X$ is dominated if there is a regular positive Borel measure $\mu$ such that

$$|U(f)| \leq \int_S |f| \, d\mu$$

for every $f$ in $C(S)$. According to [4, p. 380] there is an isomorphism $U \leftrightarrow \mu$ between the set of the dominated linear operators $U : C(S) \to X$ and the set of the regular Borel measures $\mu : \mathcal{B}(S) \to X$ with finite variation $\mu = |\mu|$, given by the equality

$$U(f) = \int_S f \, d\mu, \text{ for every } f \in C(S).$$

The measure $\mu = |\mu|$ is a least positive regular measure $\mu$ dominating $U$.

Let $m : \mathcal{B}(S) \to X$ and $n : \mathcal{B}(T) \to Y$ be regular Borel measures with finite variation, $\mu = |m|$, $\nu = |n|$, respectively. Then the mappings

$$U(f) = \int_S f \, dm, \quad f \in C(S),$$

$$V(g) = \int_T g \, dn, \quad g \in C(T)$$

are the dominated operators from $C(S)$ into $X$, $C(T)$ into $Y$, respectively. Take now $h$ in $C(S \times T)$. Then for every $s \in S$, the mapping $t \to h(s, t)$ is a continuous function on $T$. Further the mapping from $S$ into $Z$, given by the relation

$$s \to \int_T h(s, t) \, dn(t)$$

is continuous. We have

$$|\int_S \left\{ \int_T h(s, t) \, dn(t) \right\} \, dm(s)| \leq \int_S \left\{ \int_T |h(s, t)| \, dn(t) \right\} \, d|m|(s).$$
It is easy to see that the mapping given by

\[ h \mapsto \int_S \left\{ \int_T h(s, t) \, d|m|(t) \right\} \, d|m|(s), \quad h \in C(S \times T), \]

is a positive linear functional on \( C(S \times T) \) and thus the mapping \( W \), given by the formula

\[ W(h) = \int_S \left\{ \int_T h(s, t) \, dn(t) \right\} \, dm(s), \quad h \in C(S \times T), \]

is a dominated linear operator on \( C(S \times T) \) into \( Z \) [4, p. 392]. Therefore there exists a regular Borel measure \( \nu : \mathcal{B}(S \times T) \rightarrow Z \) with finite variation \( \varphi = |\nu| \) such that

\[ W(h) = \int_{S \times T} h \, d\nu, \quad \text{for every} \ h \in C(S \times T). \]

We denote the measure \( \nu \) by \( \nu = m \otimes n. \) Similarly \( |m| \otimes |n| \) is a unique positive regular Borel measure on \( \mathcal{B}(S \times T) \) such that

\[ \int_S \left\{ \int_T h(s, t) \, d|m|(t) \right\} \, d|m|(s) = \int_{S \times T} h \, d|m| \otimes |n| \]

for every \( h \in C(S \times T). \)

Since we have

\[ |W(h)| \leq \int_{S \times T} |h(s, t)| \, d|m| \otimes |n|(s, t) \]

and \( |m \otimes n| \) is a least positive regular Borel measure \( \beta \) such that

\[ |W(h)| \leq \int_{S \times T} |h(s, t)| \, db(s, t), \]

we obtain \( \varphi = |m \otimes n| \leq |m| \otimes |n|. \) Clearly

\[ \int_{S \times T} h \, dm \otimes n = \int_S \left\{ \int_T h(s, t) \, dn(t) \right\} \, dm(s) \]

for every function \( h \in C(S \times T). \)

We remark that \( |m| \otimes |n|, |m \otimes n| \) and \( m \otimes n \) are defined on the \( \sigma \)-algebra \( \mathcal{B}(S \times T) \) which contains the product \( \sigma \)-algebra \( \mathcal{B}(S) \times \mathcal{B}(T). \) The inclusion \( \mathcal{B}(S) \times \mathcal{B}(T) \subset \mathcal{B}(S \times T) \) may be proper if neither \( S \) nor \( T \) is metrisable [cf. 2]. Therefore \( |m| \times |n| \) as defined in [1] or \( m \times n \) as defined in [6] need not be a Borel measure [cf. 7]. Thus \( |m| \otimes |n| \) is the unique regular Borel extension of \( |m| \times |n| \) and \( m \otimes n \) is the unique regular Borel extension of \( m \times n. \)

Since every function in \( C(S \times T) \) can be uniformly approximated by function which are finite sums of type

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with \( f_i \in C(S) \) and \( g_i \in C(T) \), all functions in \( C(S \times T) \) are \( m \times n \)-integrable [4, p. 138] and we may write

\[
\int_{S \times T} h \, dm \otimes n = \int_{S \times T} h \, dm \times n = \int_S \{ \int_T h(s, t) \, dn(t) \} \, dm(s)
\]

for every \( h \in C(S \times T) \).

2. The Fubini theorem

We take the measures \( m \) and \( n \) as in Section 1. The proof of the Fubini theorem is based on some lemmas.

**Lemma 1.** Let \( \mu = |m| \). For every function \( f \in L^1(S, \mu) \) there exists a sequence \((f_n)\) of the functions in \( C(S) \) converging to \( f \) in mean and \( \mu \)-almost everywhere.

**Proof.** The space \( C(S) \) is dense in \( L^1(S, \mu) \) [4, p. 325]. So for every natural number \( n \) there exists a sequence \((h_n)\) in \( C(S) \) such that

\[
\int_S |h_n - f| \, d\mu < \frac{1}{n}.
\]

Thus the sequence \((h_n)\) converges to \( f \) in mean. According to [4, p. 130] the sequence \((h_n)\) contains a subsequence \((f_n)\) converging \( \mu \)-almost everywhere and in mean to \( f \).

**Lemma 2.** Let \( Z \) be a set of \( \mu \otimes v \)-measure 0 in \( S \times T \). Then for \( \mu \)-almost \( s \in S \) we have \( v(Z_s) = 0 \), i.e. there exists a set \( P \) of \( \mu \)-measure 0 such that \( v(Z_s) = 0 \) for \( s \notin P \).

**Proof.** We have, using the Fubini theorem for positive Borel measures [8, p. 153]

\[
0 = \mu \otimes v(Z) = \int_S c_Z \, d\mu \otimes v = \int_S \{ \int_T c_Z(s, t) \, dv(t) \} \, d\mu(s) = \int_S \{ \int_T c_Z(s, t) \, dv(t) \} \, d\mu(s) = \int_S v(Z_s) \, d\mu(s),
\]

where \( c_Z \) denotes the characteristic function of the set \( Z \).

**Theorem 1** (Fubini). Let \( f \) be a scalar function on \( S \times T \). Let \( f \in L^1(S \times T, \mu \otimes v) \), \( \mu = |m| \), \( v = |n| \). Then

- \( f \) is \( m \otimes n \)-integrable;
- for \( \mu = |m| \)-almost all \( s \), the map \( f_s: t \to f(s, t) \), is in \( S^1(T, v) \);
the map given by

\[ s \to \int_{T} f_s \, dn \]

for \( \mu \)-almost all \( s \) (and defined arbitrarily for other \( s \)) is in \( L^1_y(S, \mu) \) and we have

\[ \int_{S \times T} f \, d(m \otimes n) = \int_{S} \{ \int_{T} f(s, t) \, dn(t) \} \, dm(s) . \]

Proof. The fact that \( f \) is \( m \otimes n \)-integrable follows [4, p. 132] from the inequality

\[ |m \otimes n| \leq |m| \otimes |n| = \mu \otimes v. \]

By Lemma 1 there exists a sequence \( (f_n) \) in \( C(S \times T) \) converging to \( f \mu \otimes v \)-almost everywhere and in mean, i.e.

\[ \lim_{n \to \infty} \int_{S \times T} |f(s, t) - f_n(s, t)| \, d\mu \otimes v(s, t) = 0. \]

From there we have

\[ \lim_{n \to \infty} \int_{S \times T} |f(s, t) - f_n(s, t)| \, d|m \otimes n|(s, t) = 0, \]

therefore

\[ \lim_{n \to \infty} \int_{S \times T} \left( f(s, t) - f_n(s, t) \right) \, d|m \otimes n(s, t) = 0, \]

that is

\[ \lim_{n \to \infty} \int_{S \times T} f_n(s, t) \, dm \otimes n(s, t) = \int_{S \times T} f(s, t) \, dm \otimes n(s, t). \]

Let \( Z \) be a set of \( \mu \otimes v \)-measure 0 in \( S \times T \) such that \( (f_n) \) converges to \( f \) outside \( Z \) and \( P \) denote a set of \( \mu \)-measure 0 in \( S \) (Lemma 2) such that for \( s \notin P \) we have

\[ v(Z_s) = 0. \]

If \( s \notin P \), it follows that \( (f_{n,s}) \) converges pointwise to \( f_s \) on the complement of \( Z_s \).

For each \( n \) the map \( g_n : s \to f_{n,s} \) is a map of \( S \) into \( C(T) \subset L^1(T, v) \). The sequence \( (g_n) \) is Cauchy in \( L^1_{x(v)}(S, \mu) \). In fact, we have

\[ N_1(g_n - g_m) = \int_{S} |g_n - g_m|_{L^1_v} \, d\mu = \int_{S} |g_n(s) - g_m(s)|_{L^1_v} \, d\mu(s) = \]

\[ = \int_{S} \int_{T} |f_n(s, t) - f_m(s, t)| \, dv(t) \, d\mu(s) = \int_{S \times T} |f_n - f_m| \, d\mu \otimes v \to 0, \]

as \( m, n \to \infty \). Since the space \( L^1_{x(v)}(S, \mu) \) is complete there is a function
$g : S \to L^1(T, \nu)$ such that $(g_n)$ (taking subsequences if necessary) converges to $g$ $\mu$-almost everywhere and in mean, i.e.

$$\lim_{n \to \infty} \int_S |g_n - g| \, d\mu = \lim_{n \to \infty} \int_S |g_n(s) - g(s)| \, d\mu(s) = 0.$$ 

This means that there is a set $Q$ of $\mu$-measure 0 in $S$ such that for $s \notin Q$, the sequence $(g_n(s)) = (f_{n,s})$ is Cauchy in $L^1(T, \nu)$, i.e.

$$\int_T |g_n(s) - g_m(s)| \, d\nu = \int_T |f_{n,s} - f_{m,s}| \, d\nu \to 0,$$

as $m, n \to \infty$ for $s \notin Q$.

If $s \notin P \cup Q$, we know that $(f_{n,s}(t))$ converges to $f_s(t)$ for $\nu$-almost all $t \in T$. Hence by [4, p. 133] we conclude that $f_s \in L^1(T, \nu) \subset L^1(T, \mu)$ and that $(f_{n,s})$ is $L^1(T, \nu)$-convergent to $f_s$, so that

$$\int_T |f_{n,s} - f_s| \, d\nu \to 0,$$

as $m, n \to \infty$, for all $s \notin P \cup Q$, i.e. $\int f_{n,s} \, d\nu$ converges to $\int f_s \, d\nu$ for $s \notin P \cup Q$.

Finally, we note that the map $h_n$,

$$h_n(s) = \int_T f_{n,s} \, d\nu,$$

is a continuous function from $S$ into $Y$, $h_n \in C_Y(S) \subset L^1_Y(S, \mu)$. Furthermore, $(h_n)$ is Cauchy in $L^1_Y(S, \mu)$,

$$\int_S |h_n - h_m| \, d\mu = \int_S |h_n(s) - h_m(s)| \, d\mu(s) =
= \int_S \int_T |f_{n,s} - f_{m,s}| \, d\nu \, d\mu(s) \leq \int_S \int_T |f_{n,s} - f_{m,s}| \, d\nu \, d\mu(s) \to 0,$$

as $m, n \to \infty$, and since for $s \notin P \cup Q$ $h_n(s)$ converges to

$$h(s) = \int_T f_s \, d\nu,$$

$(h_n)$ is $L^1_Y(S, \mu)$-convergent to $h$, and $h$ is in $L^1_Y(S, \mu)$.

For $n \to \infty$ we have

$$\int_S \int_T |f_{n,s} \, d\nu \, dm(s) - \int_S \int_T |f_s \, d\nu \, dm(s)| \leq \int_S \int_T |f_{n,s} - f_s| \, d\nu \, d\mu(s) \to 0,$$

i.e.

$$\lim_{n \to \infty} \int_S \int_T f_{n,s}(t) \, d\nu(t) \, dm(s) = \int_S \int_T f(s, t) \, d\nu(t) \, dm(s),$$

but

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\[
\lim_{n \to \infty} \int_S \int_T f_n(s, t) \, dn(t) \, dm(s) = \lim_{n \to \infty} \int_S f_n(s, t) \, dm \otimes n(s, t) = \int_{S \times T} f(s, t) \, dm \otimes n(s, t),
\]

i.e.
\[
\int_{S \times T} f(s, t) \, d(m \otimes n) = \int_S \{ \int_T f(s, t) \, dn(t) \} \, dm(s).
\]

**Corollary.** Let \( Q \) be a Borel set in \( S \times T \). Then we have
\[
\int_{S \times T} c_Q \, d(m \otimes n) = \int_S \left( \int_T c_Q(t) \, dn(t) \right) \, dm(s).
\]

### 3. Images of measures and the convolution formula

Let \( T \) and \( S \) be compact Hausdorff spaces and suppose that \( p : T \to S \) is a continuous function. Let \( X \) be a Banach space and \( m : \mathcal{B}(T) \to X \) a regular Borel measure with finite variation \( \mu \) on \( T \). For every \( A \in \mathcal{B}(S) \) we put
\[
n(A) = m(p^{-1}(A))
\]
and
\[
v(A) = \mu(p^{-1}(A)).
\]

Since \( p^{-1}(A) \in \mathcal{B}(T) \) for every \( A \in \mathcal{B}(S) \), \( n \) and \( v \) are well defined, \( n \) has finite variation, \( |n| \leq v \), and \( n \) is regular [4, p. 402—403]. The regular Borel measure \( n : \mathcal{B}(S) \to X \) is called the image of \( m \) by the function \( p \) and is denoted \( p(m) \) [4]. Then \( v \) is denoted \( p(\mu) \) and the inequality \( |n| \leq v \) is now written \( |p(\mu)| \leq v \leq p(|m|) \). Since \( \mu \) is bounded, \( p(\mu) \) is bounded.

Let now \( S = \mathbb{G} \) be a compact Hausdorff topological semigroup, and \( T = \mathbb{G} \times \mathbb{G} \). Let \( m : \mathcal{B}(\mathbb{G}) \to X \) and \( n : \mathcal{B}(\mathbb{G}) \to Y \) be two regular Borel measures with finite variation \( \mu \) and \( v \), respectively. Let \( \mu^1 \cdot v \) and \( m^1 \cdot n \) denote the measures, which are the images of \( \mu \otimes v \), \( m \otimes n \), respectively by the semigroup operation \( p(s, t) = st \),
\[
\mu^1 \cdot v = p(\mu \otimes v), \quad m^1 \cdot n = p(m \otimes n).
\]

Let \( f \in C(S) \). Then \( f \in \mathcal{L}^1(\mathbb{G}, \mu^1 \cdot v) \) and \( f \circ p \in \mathcal{L}^1(\mathbb{G} \times \mathbb{G}, \mu \otimes v) \) [4, p. 404] and we have
\[
\int_{\mathbb{G} \times \mathbb{G}} f \circ p \, dm \otimes n = \int_{\mathbb{G}} f \, dp(m \otimes n),
\]
in other words.
\[
\int_{\mathcal{G} \times \mathcal{G}} f(st) \, dm \otimes n(s, t) = \int_{\mathcal{G}} f \, dm_n \, n.
\]

Since the last equality holds for every function \( f \in C(\mathcal{G}) \), we have

\[
\int_{\mathcal{G}} f \, dm_n \, n = \int_{\mathcal{G}} f(st) \, dm \otimes n(s, t) = \int_{\mathcal{G}} f \, dm_n \, n
\]

for every \( f \in C(\mathcal{G}) \). However, this means that

\[ m \ast n = m \ast n \]

on \( \mathcal{B}(\mathcal{G}) \) [4, p. 326].

If \( \mathcal{G} \) is a group, then the convolution formula is an easy consequence of Corollary of Theorem 1.

**Theorem 2.** Let \( \mathcal{G} \) be a compact Hausdorff group, \( m \) and \( n \) regular Borel measures on \( \mathcal{B}(\mathcal{G}) \) with finite variation and with values in \( X \) and \( Y \), respectively. Then, for each Borel subset \( D \) of \( \mathcal{G} \)

(1) \( t \mapsto m(Dt^{-1}) \)

is an \( n \)-integrable function on \( \mathcal{G} \) and we have

(2) \( m \ast n(D) = \int_{\mathcal{G}} m(Dt^{-1}) \, dn(t) \).

**Proof.** We have, putting \( E = p^{-1}(D) \),

\[
\int_{\mathcal{G}} c_{E}(s, t) \, dm(s) = m(Dt^{-1}),
\]

and

\[ m \ast n(D) = m \otimes n(E) = \int_{\mathcal{G} \times \mathcal{G}} c_{E} \, dm \otimes n = \int_{\mathcal{G}} \left\{ \int_{\mathcal{G}} c_{E}(s, t) \, dm(s) \right\} \, dn(t), \]

using the fact that if \( g \in L^1(\mathcal{G}, \mu \ast \nu) \), then \( g \circ p \in L^1(\mathcal{G} \times \mathcal{G}, m \otimes n) \) and we have

\[
\int_{\mathcal{G} \times \mathcal{G}} g \circ p \, dm \otimes n = \int_{\mathcal{G}} g \, dm \ast n
\]

[cf. 4. p. 404], in particular for \( g = c_{D} \).

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