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NOTE ON A THEOREM OF KURATOWSKI—SIERPIŃSKI

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It is known that many important functional properties of a function $f: X \to Y$ are valid also for its associated graph function $f_y: X \to X \times Y$, where $f_y(x) = (x, f(x))$ (for example continuity). It is not so easy to see what happens when we investigate in this connection the Darboux property, i.e., "the mean-value property". There are examples of real functions of the real variable, which on each interval attain each real value and have no fixed points [4]. Such functions clearly have the Darboux property (images of connected sets are connected), but their graphs are not connected. Such functions cannot belong to the 1st Baire class. Indeed, it was shown by Kuratowski and Sierpiński in [3] that this property is substantial. In that paper the following theorem was proved: If f is a real function of the real variable from the 1st Baire class having the Darboux property, then its graph is connected. As the simple example below demonstrates, for a more general domain this theorem may be false. The main problem treated in this paper is the question of how we can extend the domain and the range of considered functions that a certain generalization of the Kuratowski-Sierpiński theorem is still valid, respectively how we must at same time restrict the class of considered functions.

For real valued functions on a more general domain like E_1 the Darboux property is generalized in several ways. For example if f maps connected subsets of X onto connected sets, then this function is called connected (in this meaning we shall understand the Darboux property of the function f in Example 1 and in the following). When f maps the closures of elements of a topological base $\mathscr{B}$ onto connected sets [5], then we say that f is $\mathscr{B}$-connected. In another generalization f maps closed connected subsets of X onto connected sets.

Example 1. Let $X = \{(x, \sin 1/x) \text{ for } x > 0\} \cup \{0, 0\}$ $(X \subset E_2)$, let $f(0, 0) = 1$ and $f(x, y) = y$ otherwise. Then f belongs to the 1st Baire class and has the Darboux property, but its graph is not connected on any connected subset containing the point (0, 0).

More generally we immediately have

Proposition 1. *Let X and Y be arbitrary Hausdorff topological spaces, let A be a connected subset of X and let $f: X \to Y$. Suppose that there exists $x_0 \in A$ such that for no net $\{x_\alpha\}$, $x_\alpha \in A - \{x_0\}$ converging to x_0 the assertion* $\lim_\alpha f(x_\alpha) = f(x_0)$ *holds. Then $f_\gamma(A)$ is not connected.*

Proof. $f_\gamma(A) = B_1 \cup B_2$ where $B_1 = f_\gamma(A - \{x_0\})$ and $B_2 = f_\gamma(x_0)$. Evidently $B_1 \neq \emptyset \neq B_2$, and $\bar{B}_1 \cap B_2 = \emptyset = B_1 \cap \bar{B}_2$ since $\lim_\alpha (x_\alpha, f(x_\alpha)) = (x_0, f(x_0))$ if and only if $\lim_\alpha x_\alpha = x_0$ and $\lim_\alpha f(x_\alpha) = f(x_0)$.

If f is a real function of the real variable whose graph is on some interval I connected, then it is easy to see that its graph function f_γ has the Darboux property on I (is connected). But already in the case $X = E_2$, $Y = E_1$ this is not true. Take for example the function $f: E_2 \to E_1$ defined by $f(x, y) = x$ for $(x, y) \in [0,1] \times [0,1]$ and $f(x, y) = 0$ otherwise. f has a connected graph on E_2, but the function f_γ is not connected. (It is even not $\mathscr{B}$-connected, for $\mathscr{B}$ being the base of open intervals in E_2.)

In the proof of the Kuratowski—Sierpiński theorem essentially the following property of Darboux functions is used:

(*) For every x_0 from the interior of the domain there exist two sequences $\{s_n\}$, $\{t_n\}$ so that $s_n < x_0 < t_n$, $\lim_{n\to\infty} s_n = x_0 = \lim_{n\to\infty} t_n$ and $\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} f(t_n) = f(x_0)$. This property can be reformulated in the following way: If $x_0 \in A$, where A is a connected (but not a one-point) subset of E_1, then there exists a sequence $\{x_n\}$, $x_n \in A - \{x_0\}$, such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Example 2 at the end shows that this property of real valued Darboux functions is already not valid for $X = E_2$.

Nevertheless under a certain special assumption a property similar to (*) is valid:

Theorem 1. *Let (X, ϱ) and (Y, ϱ_1) be arbitrary metric spaces and let $f: X \to Y$ be a connected function. Let $A \subset X$ be a non-trivial (not a one point set) connected set and let x_0 be a point of local connectedness of A. Then there exists a sequence $\{x_n\}$, $x_n \in A - \{x_0\}$ such that* $\lim_{n\to\infty} x_n = x_0$ *and* $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Proof. Let U_1 be a connected neighbourhood of x_0 in the relative topology of A, let $x_1 \in U_1$, $x_1 \neq x_0$ (x_0 cannot be an isolated point of A) and let U_2 be a connected neighbourhood of x_0 such that $U_2 \subset O(x_0, \varrho(x_0, x_1)/2) \cap A$. Since $f(U_2)$ is a connected set containing $f(x_0)$, there is a point $x_2 \in U_2$ such that $\varrho_1(f(x_2), f(x_0)) \leq \max(\varrho_1(f(x_1), f(x_0))/2, 1/2)$. Similarly we choose $U_3, U_4, \ldots$.

Obviously U_n is a connected neighbourhood of x_0 such that $U_n \subset O(x_0, \varrho(x_0, x_{n-1})/2) \cap A$, $x_n \in U_n$ and $\varrho_1(f(x_n), f(x_0)) \leq \max(\varrho_1(f(x_{n-1}), f(x_0))/2, 1/n)$. It is easy to see that the chosen sequence $\{x_n\}$ has the required properties.

Corollary. Let (X, ϱ) and (Y, ϱ_1) be metric spaces. If X is a locally connected space, then for every connected function $f: X \to Y$, for every non-trivial connected subset $A \subset X$ and for every point $x_0 \in \operatorname{int} A$ there exists a sequence $\{x_n\}$, $x_n \in A - \{x_0\}$ for $n = 1, 2, 3, \ldots$ which satisfies $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) = f(x_0)$.

For the proof of our Theorem 2 we need the following definition.

Definition. *Let X and Y be metric spaces. We shall say that a function $f: X \to Y$ has the property $\mathscr{A}$ if for any open non-trivial connected set $C \subset X$ and a point $x_0 \in \bar{C}$ there is a sequence $\{x_n\}$, $x_n \in C - \{x_0\}$ such that* $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Proposition 2. *Let X and Y be metric spaces, let $f: X \to Y$ be a connected function and let $x_0 \in \bar{C}$, where $C \subset X$. Suppose that there is a sequence of connected sets $C_n \subset X$, $n = 1, 2, 3, \ldots$ such that for every n $x_0 \in C_n$, $\emptyset \neq C_n - \{x_0\} \subset C$ and $\lim_{n\to\infty} \operatorname{diam} C_n = 0$. Then there is a sequence $\{x_n\} \subset C - \{x_0\}$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) = f(x_0)$.*

Proof. Since $f(C_n)$ are connected subsets of Y containing $f(x_0)$, for each n there is an $x_n \in C_n - \{x_0\}$ such that $\varrho(f(x_n), f(x_0)) < 1/n$.

We immediately obtain a corollary that in the case of $X = E_1$ connected functions have the property $\mathscr{A}$. For $X = E_2$ this is not true (see Example 2).

Obviously if X and Y are metric spaces, $f: X \to Y$ has the property $\mathscr{A}$ and $D \subset X$ is an open set, then f/D has also the property $\mathscr{A}$. Moreover, if D is connected, then $f/\bar{D}$ has also the property $\mathscr{A}$.

Theorem 2. *Let X be a locally connected complete metric space, let Y be a separable metric space and let $f: X \to Y$ be a function from the 1st Baire class having the property $\mathscr{A}$. Then $f_\gamma(A)$ is connected for each component $A \subset X$.*

Proof. Let A be a component of X, let $B = f_\gamma(A)$ and suppose that B is not connected, i.e., that $B = B_1 \cup B_2$, where $B_1 \neq \emptyset \neq B_2$ and $\bar{B}_1 \cap B_2 = \emptyset = B_1 \cap \bar{B}_2$. Put $A_1 = \{x \in A : f_\gamma(x) \in B_1\}$ and $A_2 = \{x \in A : f_\gamma(x) \in B_2\}$. Then clearly $A_1 \neq \emptyset \neq A_2$, $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$ and since A is connected either $\bar{A}_1 \cap A_2 \neq \emptyset$ or $A_1 \cap \bar{A}_2 \neq \emptyset$. Since A is closed $\bar{A}_1 \cap \bar{A}_2 \subset \subset \bar{A}_1 \cup \bar{A}_2 = A$.

Since f belongs to the 1st Baire class, f_γ belongs there as well and therefore by assumptions of the theorem there exists a point of continuity $x_0 \in \bar{A}_1 \cap \bar{A}_2$ of $f_\gamma / \bar{A}_1 \cap \bar{A}_2$ ([2], p. 430). Obviously $\bar{A}_1 \cap \bar{A}_2 = \bar{A}_1 \cap A_2 \cup A_1 \cap \bar{A}_2$. Let $x_0 \in \bar{A}_1 \cap A_2$ (the second possibility $x_0 \in A_1 \cap \bar{A}_2$ is only the interchange of notations). Then there are two possibilities:

1) there is a sequence $\{x_n\}$, $x_n \in A_1 \cap \bar{A}_2$ converging to x_0, or
2) there is an $\varepsilon > 0$ such that $\{x \in \bar{A}_1 \cap \bar{A}_2 : \varrho(x, x_0) < \varepsilon\} \subset \bar{A}_1 \cap A_2$.

In the case 1) $\lim_{n\to\infty} f_\gamma(x_n) = f_\gamma(x_0) \in B_2$ and since $f_\gamma(x_n) \in B_1$ for each n, we obtain that $\bar{B}_1 \cap B_2 \neq \emptyset$, a contradiction. In the following we show that the case 2) leads also to the same contradiction $\bar{B}_1 \cap B_2 \neq \emptyset$, which will prove the theorem.

Let us have the case 2). Since X is locally connected, there is a connected neighbourhood U of x_0 contained in $\{x \in X : \varrho(x, x_0) < \varepsilon\}$. Let $x_1 \in A_1 \cap U$ ($\neq \emptyset$ since $x_0 \in \bar{A}_1$). Then $\varrho(x_1, x_0) < \varepsilon$ and since $A_1 \cap A_2 = \emptyset$. $x_1 \notin \bar{A}_1 \cap \bar{A}_2$ and therefore $x_1 \in A_1 - \bar{A}_2$. Put $A_1 - \bar{A}_2 = D$ and let C be the component of D containing x_1. Since U is connected and contains the point $x_1 \in C$ as well as the point $x_0 \in U - C$, there must be $\mathrm{Fr}(C) \cap U \neq \emptyset$. Let $x_2 \in \mathrm{Fr}(C) \cap U$. Since C is a component of the open set $D = A \cap (X - \bar{A}_2)$ and X is locally connected, C is open, hence $\mathrm{Fr}(C) = \bar{C} - C = \bar{C} - D \subset \bar{A}_1 - (A_1 - \bar{A}_2) = \bar{A}_1 \cap \bar{A}_2$. Thus $x_2 \in \bar{A}_1 \cap \bar{A}_2$ and $\varrho(x_2, x_0) < \varepsilon$ and therefore $x_2 \in A_2$. Since $x_2 \in \bar{C}$ where C is open and connected by the property $\mathscr{A}$ of f there is a sequence $\{x_n\}$, $x_n \in C \subset A_1 - \bar{A}_2$ such that $\lim_{n\to\infty} x_n = x_2$ and $\lim_{n\to\infty} f(x_n) = f(x_2)$. Thus $f_\gamma(x_2) \in \bar{B}_1 \cap B_2$ and so we obtain the required contradiction.

As the following example demonstrates, the property $\mathscr{A}$ is not necessary for the connectedness of $f_\gamma(A)$, A being a component of X, even if $X = E_2$, $Y = E_1$ and f is a connected function from the 1st Baire class.

Example 2. Define $f: E_2 \to E_1$ as follows: $f(x, y) = \cos x$ for $x \leq 0$, $y \in E_1$ and $f(x, y) = \sin 1/x$ for $x > 0$ and $y \in E_1$. Obviously f is a connected function from the 1st Baire class and $f_\gamma(E_2)$ is a connected set in E_3. Let $C = \{(x, y) \in E_2,\ 0 < x < 1/10$ and $|y - \sin 1/x| < x^4\}$. Then C is open and connected in E_2 and $(0, 0) \in \bar{C}$.

Evidently for no sequence $\{x_n\}$, $x_n \in C$ converging to $(0, 0)$ the assertion $\lim_{n\to\infty} f(x_n) = f(0, 0)$ is valid and thus f does not have the property $\mathscr{A}$.

Example 2 also shows that, contrary to the case of real functions of a real variable, the connectedness of a function $f: E_2 \to E_1$ does not imply the connectedness of its associated graph function f_γ (for example, $f_\gamma(C \cup (0, 0))$ is not connected).

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