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LATERAL PROJECTIONS OF NON-HOLONOMIC JETS

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Dealing with the holonomic or semi-holonomic jets, one has a unique natural projection of r-jets into s-jets, $s < r$. In the present paper we first introduce some further canonical projections (called „lateral") of non-holonomic r-jets into s-jets, $s < r$. Then we show that some natural properties of non-holonomic jets can be simply characterized by means of these lateral projections. For the sake of simplicity we restrict ourselves to C^∞-manifolds of finite dimension, though it seems to be easy to extend all investigations to arbitrary Banach manifolds in the sense of [4]. I would like to express my thanks to doc. I. Kolář for suggesting the topic of this paper and for several useful discussions.

I. Lateral Projections

Our considerations are in the category C^∞. The standard notation of the theory of jets is used throughout the paper, see [2]; in particular, $J^r(M, N)$ or $\bar{J}^r(M, N)$ or $\tilde{J}^r(M, N)$ means the space of all holonomic or semi-holonomic or nonholonomic r-jets of a manifold M, dim $M = m$, into a manifold N, dim $N = n$, respectively.

On $\tilde{J}^r(R^m, R^n)$, we introduce the coordinates

$$(1) \qquad u^i, x^a_{k_1 \dots k_r}, \; a = 1, \dots, n, \; i = 1, \dots, m, \; k = 0, 1, \dots, m$$

by the following induction procedure, cf. [3]. On $J^0(R^m, R^n) = R^m \times R^n$, we have the natural coordinates u^i, x^a. Let $u^i, x^a_{k_1 \dots k_{r-1}}$ be the coordinates on $\tilde{J}^{r-1}(R^m, R^n)$ and let $X \in \tilde{J}^r(R^m, R^n)$, $X = j^1_u \sigma(v)$, where $\sigma(v)$ is a cross-section of $\tilde{J}^{r-1}(R^m, R^n)$ determined by some functions $y^a_{k_1 \dots k_{r-1}}(v)$. Then we put

$$(2) \qquad \begin{aligned} u^i(X) &= u^i(\alpha X)\,, \\ x^a_{k_1 \dots k_{r-1} 0}(X) &= y^a_{k_1 \dots k_{r-1}}(u)\,, \\ x^a_{k_1 \dots k_{r-1} i_r}(X) &= \partial_{i_r} y^a_{k_1 \dots k_{r-1}}(u)\,. \end{aligned}$$

It was shown by Virsík [5], that X is semi-holonomic if and only if

$$(3) \qquad x^a_{k_1 \dots k_r}(X) = x^a_{k'_1 \dots k'_r}(X)\,,$$

whenever the r-tuples $(k_1, \dots, k_r)$ and $(k'_1, \dots, k'_r)$ differ only by the displacement of zeros. Further, if u^i or x^a are some local coordinates on M or N, respectively, then u^i, x^a are naturally extended to some local coordinates $u^i, x^a_{k_1 \dots k_r}$ on $\tilde{J}^r(M, N)$.

Let $j^{r-1}_r : \tilde{J}^r(M, N) \to \tilde{J}^{r-1}(M, N)$ be the target projection. We define

$$j^s_r = j^s_{s+1} \dots j^{r-1}_r : \tilde{J}^r(M, N) \to \tilde{J}^s(M, N), \; s < r \, .$$

In the above-mentioned coordinates we have

$$x^a_{k_1, \dots k_s}(j^s_r X) = x^a_{k_1 \dots k_s \underbrace{0 \dots 0}_{(r-s)-times}}(X) \, . \tag{4}$$

Definition 1. *Let $X \in \tilde{J}^r(M, N)$, $X = j^1_u \sigma(v)$, where $\sigma(v)$ is a local cross-section of $\tilde{J}^{r-1}(M, N)$. Then $j^{s-1}_{r-1}\sigma(v)$ is a local cross-section of $\tilde{J}^{s-1}(M, N)$ and we define*

$$l^s_r X = j^1_u[j^{s-1}_{r-1}\sigma(v)] \in \tilde{J}^s(M, N) \, , \quad s \geq 1 \, . \tag{5}$$

The mapping $l^s_r = {}^1l^s_r : \tilde{J}^r(M, N) \to \tilde{J}^s(M, N)$ will be called the first lateral projection of $\tilde{J}^r(M, N)$ into $\tilde{J}^s(M, N)$.

Lemma 1. *Let $X \in \tilde{J}^r(R^m, R^n)$. Then the following holds*

$$x^a_{k_1 \dots k_s}(l^s_r X) = x^a_{k_1 \dots k_{s-1} \underbrace{0 \dots 0}_{(r-s)-times} k_s}(X) \, . \tag{6}$$

Proof. If $X = j^1_u \sigma(v)$, where $\sigma(v)$ is determined by some functions $y^a_{k_1 \dots k_{r-1}}(v)$, then, according to (4), $j^{s-1}_{r-1}\sigma(v)$ is determined by the functions $y^a_{k_1 \dots k_{s-1} \underbrace{0 \dots 0}_{r-s}}(v)$. Applying (2), we obtain our lemma.

Proposition 1. *Let $X \in \tilde{J}^r(M, N)$ and let $t < s < r$. Then we have*

(a) $j^t_s(l^s_r X) = j^t_r X$,

(b) $l^t_s(l^s_r X) = l^t_r X$.

Proof. By (4) and (6) we obtain

$$x^a_{k_1 \dots k_t}(j^t_s(l^s_r(X))) = x^a_{k_1 \dots k_t \underbrace{0 \dots 0}_{r-t}}(X) \, ,$$

$$x^a_{k_1 \dots k_t}(l^t_s(l^s_r(X))) = x^a_{k_1 \dots k_{t-1} \underbrace{0 \dots 0}_{r-t} k_t}(X) \, .$$

This implies directly Proposition 1.

Remark 1. The projections "j" and "l" do not commute, i. e. the jets $l^t_s(j^s_r X)$ and $j^t_s(l^s_r X)$ are different in generall since their coordinates are

$$x^a_{k_1 \dots k_t}(l^t_s(j^s_r X)) = x^a_{k_1 \dots k_{t-1} \underbrace{0 \dots 0}_{s-t} k_t \underbrace{0 \dots 0}_{r-s}}(X) \, ,$$

$$x^a_{k_1\ldots k_t}(j^t_s(l^s_r X)) = x^a_{k_1\ldots k_t \underbrace{0\ldots 0}_{r-t}}(X) \,.$$

Remark 2. In particular, every $X \in \tilde{J}^r(M, N)$ determines the following r jets of the first order, which are different in general (after colons, we write their coordinates in a local coordinate system)

(7)
$$j^1_r X : x^a_{k_1 \underbrace{0\ldots 0}_{r-1}},\ u^i \,,$$
$$l^1_2(j^2_r X) : x^a_{0 k_2 \underbrace{0\ldots 0}_{r-2}},\ u^i \,,$$
$$\vdots$$
$$l^1_t(j^t_r X) : x^a_{\underbrace{0\ldots 0}_{t-1} k_t \underbrace{0\ldots 0}_{r-t}},\ u^i \,,$$
$$\vdots$$
$$l^1_r X : x^a_{\underbrace{0\ldots 0}_{r-1} k_r},\ u^i \,.$$

Definition 2. *Let $X \in \tilde{J}^r(M, N)$, $X = j^1_u \sigma(v)$, where $\sigma(v)$ is a local cross-section of $\tilde{J}^{r-1}(M, N)$. We define the p-th lateral projection ${}^p l^s_r : \tilde{J}^r(M, N) \to \tilde{J}^s(M, N)$, $p \leq s$, by the following induction*

(a)
$${}^0 l^s_r X = j^s_r X \,,$$

(b)
$${}^p l^s_r X = j^1_u[{}^{p-1} l^{s-1}_{r-1} \sigma(v)] \,.$$

Remark 3. Recently, dealing with the prolongations of fibered manifolds, Virsík [6], introduced some similar projections.

Lemma 2. *Let $X \in \tilde{J}^r(R^m, R^n)$. Then*

(8)
$$x^a_{k_1\ldots k_s}({}^p l^s_r X) = x^a_{k_1\ldots k_{s-p} \underbrace{0\ldots 0}_{r-s} k_{s-p+1}\ldots k_s}(X) \,.$$

Proof. For $p = 1$, Lemma 2 coincides with Lemma 1. By the induction hypothesis, the coordinates of ${}^{p-1} l^{s-1}_{r-1} \sigma(v)$ are $y^a_{k_1\ldots k_{s-p} \underbrace{0\ldots 0}_{r-s} k_{s-p+1}\ldots k_{s-1}}(v)$, provided we have used the notation of Lemma 1. Then we deduce (8) directly by (2).

· **Proposition 2.** *Let $X \in \tilde{J}^r(M, N)$. The composition of some lateral projections obeys the following rules*

(a)
$${}^p l^t_s({}^q l^s_r X) = {}^p l^t_r X \quad for \quad p \leq q \leq p + s - t \,,$$

(b)
$${}^p l^t_s({}^q l^s_r X) = {}^{q+t-s} l^t_{r-s+t}({}^p l^{r-s+t}_r X) \quad for \quad p + s - t < q \,.$$

The proof is quite analogous to the proof of Proposition 1.

Definition 3. *We shall say that an s-jet* $Y \in \tilde{J}^s(M, N)$ *is subordinated to an r-jet* $X \in \tilde{J}^r(M, N)$, $\alpha X = \alpha Y$, $\beta X = \beta Y$, $s < r$, *if there exist some integers* $i_1, \dots, i_j, i_1 + \dots + i_j = s$ *and* $a_1, \dots, a_{j+1}, a_1 + \dots + a_{j+1} = r - s$, *such that the following holds*

$$(9) \qquad x^a_{k_1 \dots k_s}(Y) = x^a_{\underbrace{0\dots0}_{a_1} k_1 \dots k_{i_1} \underbrace{0\dots0}_{a_2} k_{i_1+1} \dots k_{i_1+i_2} \underbrace{0\dots0}_{a_3} k_{i_1+i_2+1} \dots \underbrace{0\dots0}_{a_j} k_{i_1+i_2+\dots+i_{(j-1)}+1} \dots k_{i_1+\dots+i_j} \underbrace{0\dots0}_{a_{j+1}}}(X)$$

in some local coordinates.

Proposition 3. *An s-jet Y is subordinated to an r-jet X if and only if Y can be derived from X by a sequence of lateral projections.*

Proof. Assume that X and Y satisfy (9). Then a straightforward evaluation based on (8) gives

$$Y = {}^{i_1 \dots i_j}l^s_{s+a_1}({}^{i_2+\dots+i_j}l^{s+a_1}_{s+a_1+a_2}(\dots {}^{i_j}l^{s+a_1+\dots+a_{j-1}}_{s+a_1+\dots+a_j}(j^{s+a_1+\dots+a_j}_r(X)) \dots)) .$$

Remark 4. In particular, Proposition 3 shows that Definition 3, in which some local coordinates are used, has an invariant meaning.

II. Some Applications

We first deduce that one can characterize some special non-holonomic jets by means of some properties of their lateral projections. In accordance with Ehresmann [2], we introduce the following concepts. Let Φ be a submanifold of $\tilde{J}^k(M, N), k \geq 1$. Then $\bar{J}^1(\Phi) \subset \tilde{J}^{k+1}(M, N)$ will denote the set of all elements of the form $j^1_u\sigma$, where σ is a local cross-section of Φ satisfying the additional condition

$$(10) \qquad j^1_u[j^{k-1}_{k}\sigma] = \sigma(u) .$$

By induction, we introduce

$$(11) \qquad \bar{J}^s(\Phi) = \bar{J}^1(\bar{J}^{s-1}(\Phi)) .$$

In particular, if $k = 1$, $s = r - 1$ and $\Phi = J^1(M, N)$, we obtain the space $\bar{J}^r(M, N)$ of all semi-holonomic r-jets of M into N, i.e. $\bar{J}^{r-1}(J^1(M, N)) = \bar{J}^r(M, N)$.

Lemma 3. *Let* $X \in \tilde{J}^r(R^m, R^n)$. *Then* $X \in \bar{J}^q(\tilde{J}^{r-q}(R^m, R^n))$ *if and only if*

$$(12) \qquad x^a_{k_1 \dots k_{r-q-1} k_{r-q} \dots k_r}(X) = x^a_{k_1 \dots k_{r-q-1} k'_{r-q} \dots k'_r}(X)$$

whenever the $(q + 1)$*-tuples* $(k_{r-q}, \dots, k_r)$ *and* $(k'_{r-q}, \dots, k'_r)$ *differ only by the displacement of zeros.*

Proof. We shall proceed by induction with respect to q. Let $q = 1$. If $X = j^1_u\sigma(v)$, where σ satisfies (10), then (2) and (10) imply.

$$x^a_{k_1\dots k_{r-2}0k}(X) = x^a_{k_1\dots k_{r-2}k0}(X)\,, \tag{13}$$

which is (12) for $q = 1$. Conversely, let X satisfy (13). Consider the section $\sigma(v)$ determined by the functions

$$y^a_{k_1\dots k_{r-1}}(v) = x^a_{k_1\dots k_{r-1}i}(v^i - u^i) + x^a_{k_1\dots k_{r-1}0}\,. \tag{14}$$

Then $\sigma(v)$ satisfies (10) and $X = j^1_u\sigma(v)$. Further, assume that Lemma 3 holds for $q - 1$ and we have to deduce it for q. Let $X = j^1_u\sigma(v)$, where $\sigma(v)$ is a cross--section of $\bar{J}^{q-1}(\bar{J}^{r-q}(R^m, R^n))$, so that its coordinate functions satisfy

$$y^a_{k_1\dots k_{r-q-1}k_{r-q}\dots k_{r-1}}(v) = y^a_{k_1\dots k_{r-q-1}k'_{r-q}\dots k'_{r-1}}(v)$$

whenever the q-tuples $(k_{r-q}, \dots, k_{r-1})$ and $(k'_{r-q}, \dots, k'_{r-1})$ differ only by displacement of zeros. Then (2) and (10) imply (12). Conversely, let the coordinates of X have the above-mentioned property. Consider the section $\sigma(v)$ determined by (14). Then, by the induction hypothesis, $\sigma(v)$ is a cross--section of $\bar{J}^{q-1}(\bar{J}^{r-q}(R^m, R^n))$ and one sees easily that it satisfies (10). Hence $X \in \bar{J}^q(\bar{J}^{r-q}(R^m, R^n))$, QED.

Proposition 4. *Let $X \in \tilde{J}^r(M, N)$. Then $X \in \bar{J}^q(\bar{J}^{r-q}(M, N))$ if and only if*

$$\begin{aligned} j^{r-1}_r X &= {}^1l^{r-1}_r X = {}^2l^{r-1}_r X = \dots = {}^ql^{r-1}_r X \\ j^{r-2}_r X &= {}^1l^{r-2}_r X = {}^2l^{r-2}_r X = \dots = {}^{q-1}l^{r-2}_r X \\ &\vdots \\ j^{r-q}_r X &= {}^1l^{r-q}_r X\,. \end{aligned} \tag{15}$$

Proof. This is a direct consequence of Lemma 3 and of the coordinate formulae for lateral projections.

Corollary 1. *A non-holonomic r-jet X of M into N is semi-holonomic if and only if*

$$\begin{aligned} j^{r-1}_r X &= {}^1l^{r-1}_r X = {}^2l^{r-1}_r X = \dots = {}^{r-2}l^{r-1}_r X = {}^{r-1}l^{r-1}_r X \\ j^{r-2}_r X &= {}^1l^{r-2}_r X = {}^2l^{r-2}_r X = \dots = {}^{r-2}l^{r-2}_r X \\ &\vdots \\ j^2_r X &= {}^1l^2_r X = {}^2l^2_r X \\ j^1_r X &= {}^1l^1_r X\,. \end{aligned} \tag{16}$$

Proof. In Proposition 4, we set $q = r - 1$.

Remark 5. This Corollary was also established by Virsík [6].

As an example of iterated applications of Proposition 4, we state the following obvious.

Corollary 2. *Let* $X \in \tilde{J}^r(M, N)$. *Then* $X \in \tilde{J}^{q-1}(J^1(\tilde{J}^{r-q}(M, N)))$ *if and only if*

$$\begin{aligned}
j_r^{r-1}X &= {}^1l_r^{r-1}X = {}^2l_r^{r-1}X = \ldots = {}^{q-1}l_r^{r-1}X \\
j_r^{r-2}X &= {}^1l_r^{r-2}X = {}^2l_r^{r-2}X = \ldots = {}^{q-2}l_r^{r-2}X \\
&\vdots \\
j_r^{r-q+1}X &= {}^1l_r^{r-q+1}X \\
j_r^{r-q-1}X &= {}^1l_{r-q}^{r-q-1}(j_r^{r-q}X) = {}^2l_{r-q}^{r-q-1}(j_r^{r-q}X) = \ldots = {}^{r-q-1}l_{r-q}^{r-q-1}(j_r^{r-q}X) \\
j_r^{r-q-2}X &= {}^1l_{r-q}^{r-q-2}(j_r^{r-q}X) = {}^2l_{r-q}^{r-q-2}(j_r^{r-q}X) = \ldots = {}^{r-q-2}l_{r-q}^{r-q-2}(j_r^{r-q}X) \\
&\vdots \\
j_r^1X &= {}^1l_{r-q}^1(j_q^{r-q}X)
\end{aligned}$$

Now we shall show that the lateral projections can be also used for a simple characterization of invertibility and regularity of non-holonomic jets.

Proposition 5. *Assume* $\dim M = \dim N$. *A non-holonomic r-jet* X *of* M *into* N *is invertible if and only if all the jets of the first order* (7) *are regular.*

Proof. First assume that the jets (7) are regular. We shall proceed by induction. For $r = 1$, we get a well-known result. Assume that our assertion is true for $r - 1$. Set $Y = j_r^{r-1}X$. Since

$$j_{r-1}^1 Y = j_r^1 X,\ l_2^1(j_{r-1}^2 Y) = l_2^1(j_r^2 X), \ldots, l_{r-1}^1 Y = l_{r-1}^1(j_r^{r-1}X),$$

Y is invertible by the induction hypothesis. Moreover, since the subset of all invertible elements is open, we may write $X = j_u^1\sigma(v), \sigma(u) = Y$, where $\sigma(v)$ is local cross-section of $\tilde{J}^{r-1}(M, N)$ all elements of which are invertible. Further, since l_r^1X is regular, we may assume that the local map $\varphi(v) = \beta\sigma(v)$ of M into N is a local diffeomorphism. Hence $\xi(z) = \sigma^{-1}(\varphi^{-1}(z))$ is a local cross-section of N into $\tilde{J}^{r-1}(M, N)$. Put $Z = j_w^1\xi(z)$, $w = \varphi(u)$. Using the definition of the composition of non-holonomic jets, [2], one finds easily $ZX = j_u^r \operatorname{id}_M$. $XZ = j_w^r \operatorname{id}_N$. Thus, X is invertible.

Conversely, assume that X is invertible. Let $x^i_{k_1 \ldots k_r}$, u^i or $z^j_{k_1 \ldots k_r}$, w^i be the coordinates of X or X^{-1} respectively in a local coordinate system. According to a paper by Dekrét [1], we have $x^i_{0 \ldots 0 i_t 0 \ldots 0} z^{i_t}_{0 \ldots 0 j_t 0 \ldots 0} = \delta^i_j$ for every $t = 1, \ldots, r$. Hence $\det |x^i_{0 \ldots 0 i_t 0 \ldots 0}| \neq 0$ for every t, i.e. all the jets $j_r^1X, \ldots, l_t^1 j_r^t X, \ldots, l_r^1X$ are regular, QED.

Definition 4. *Let* $X \in \tilde{J}^r(M, N)$, $\dim M \leq \dim N$. *We shall say that* X *is* *regular, if there exists a jet* $Z \in \tilde{J}^r(M, N)$, $\alpha Z = \beta X$, $\beta Z = \alpha X = u$, *such that* $ZX = j^r_u \, \mathrm{id}_M$.

Proposition 6. *A non-holonomic r-jet of M into N*, $\dim M \leq \dim N$, *is regular if and only if all the jets of the first order* (7) *are regular.*

The proof is quite similar to the proof of Proposition 5.

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