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ON SUBSERIES OF DIVERGENT SERIES

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The paper is a contribution to the study of various properties of subseries. It is closely related to the basic result concerning the subseries contained in the papers [1] and [2].

The first part of the paper gives further results on subseries including a refining of the fundamental result of the papers [1] and [2] and deals with the study of the dependence of the absolute convergence of a series on the convergence of its subseries.

The second part is devoted to the study of properties of a certain class of functions which are defined by means of subseries of a divergent series.

In the third part we shall prove certain metric results concerning so-called factorial transformations of infinite series.

In the fourth part of the paper we shall present some applications of the first part of the paper to the theory of atomic measures.

DEFINITIONS AND NOTATIONS

1. Let

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \tag{1}$$

be a (formal) series with real or complex terms, let

$$k_1 < k_2 < \ldots < k_n < \ldots$$

be an increasing sequence of natural numbers. The series

$$\sum_{n=1}^{\infty} a_{k_n} = a_{k_1} + a_{k_2} + \ldots + a_{k_n} + \ldots \tag{2}$$

is called a subseries of the series (1).

If a number $x \in (0,1\rangle$ is expressed by means of its (infinite) dyadic expansion

(3) $$x = \sum_{k=1}^{\infty} \varepsilon_k(x)\, 2^{-k}$$

($\varepsilon_k(x) = 0$ or 1 and $\varepsilon_k(x) = 1$ for an infinite number of k' s) and if the series

(4) $$(x) = \sum_{k=1}^{\infty} \varepsilon_k(x) a_k$$

corresponds to x, then (4) is a subseries of (1). Conversely, every subseries (2) of the series (1) can be written in the form (4), if we put into (3) $\varepsilon_{k_n}(x) = 1$ ($n = 1, 2, \ldots$) and $\varepsilon_k(x) = 0$ for $k \neq k_n$ ($n = 1, 2, 3, \ldots$). Thus we get a one to one correspondence between the set of all subseries of the series (1) and the interval $(0,1\rangle$.

2. $C(\sum_1^{\infty} a_n)$ $(D(\sum_1^{\infty} a_n))$ denotes in what follows the set of all those $x \in (0,1\rangle$ for which the series (x) is convergent (divergent). Further $D^*(\sum_1^{\infty} a_n)$ denotes the set of all those $x \in (0,1\rangle$ for which the series (x) is oscillating. Hence $D^*(\sum_1^{\infty} a_n) \subset D(\sum_1^{\infty} a_n)$. We put further $C^*(\sum_1^{\infty} a_n) = (0,1\rangle - D^*(\sum_1^{\infty} a_n)$.

3. Let

(5) $$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$$

be a series with real terms. Put

$$N_+ = \{n;\ a_n > 0\},\ N_- = \{n;\ a_n < 0\},\ N_0 = \{n;\ a_n = 0\}.$$

The series (5) will be called a series of the type (α), (β) and (γ) respectively, if

$$\sum_{n \in N_+} a_n = +\infty \quad \text{and} \quad \sum_{n \in N_-} |a_n| < +\infty,$$

$$\sum_{n \in N_-} a_n = -\infty \quad \text{and} \quad \sum_{n \in N_+} a_n < +\infty,$$

$$\sum_{n \in N_+} a_n = +\infty \quad \text{and} \quad \sum_{n \in N_-} a_n = -\infty$$

respectively, putting $\sum_{n \in M} a_n = 0$ if $M = \emptyset$.

If $\sum_{n=1}^{\infty} |a_n| = +\infty$, then evidently (5) is either of the type (α) or of the type (β) or of the type (γ) and just one of these possibilities may occur.

4. When n is fixed, the interval $(0,1\rangle$ is decomposed into 2^n intervals of „the rank n"

$$i_n^{(l)} = \left(\frac{l}{2}, \frac{l+1}{2^n}\right\rangle \quad (l = 0, 1, \dots 2^n - 1).$$

If $\frac{l}{2^n} = \sum_{k=1}^{n} \varepsilon_k 2^{-k}$ is a finite dyadic expansion of the number $l/2^n$, then every $x \in i_n^{(l)}$ has the infinite dyadic expansion $x = \sum_{k=1}^{\infty} \varepsilon_k(x)\, 2^{-k}$, where $\varepsilon_k(x) = \varepsilon_k$ $(k = 1, 2, \dots n)$. Then $i_n^{(l)}$ is said to be associated with the sequence $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$.

5. $|A|$ denotes the Lebesgue measure of the set A, $|A|_e$ denotes the outer Lebesgue measure of the set A.

6. A set $M \subset (0,1\rangle$ is said to be homogeneous in $(0,1\rangle$, if for every two intervals $I, I' \subset (0, 1\rangle$

$$\frac{|I \cap M|_e}{|I|} = \frac{|I' \cap M|_e}{|I'|}$$

holds. It is known that if $M \subset (0, 1\rangle$ is homogeneous in $(0, 1\rangle$ and measurable, then either $|M| = 0$ or $|M| = 1$ (see [4]).

7. Let g be a real function defined on $(0, 1\rangle$. g is said to have the Darboux property on $(0, 1\rangle$ if for every two points $x_1, x_2 \in (0, 1\rangle$ the following holds: if z lies between $g(x_1)$ and $g(x_2)$ then there exists x_0 between x_1 and x_2 such that $g(x_0) = z$.

8. Afunction g defined on $(0,1\rangle$ is called locally recurrent at point $x_0 \in (0, 1\rangle$ if each neighbourhood of x_0 contains a point $x \in (0, 1\rangle$, $x \neq x_0$ such that $g(x) = g(x_0)$ (see [5], [6]). A function g will be called strongly recurrent at point $x_0 \in (0, 1\rangle$ if each neighbourhood of the point x_0 contains an uncountable set of points $x \in (0, 1\rangle$ for which $g(x) = g(x_0)$.

9. Let $\{q_k\}_{k=1}^{\infty}$ be a sequence of natural numbers, $q_k > 1$ $(k = 1, 2, 3, \dots)$. Express every $x \in \langle 0,1)$ by means of its Cantor series

$$x = \sum_{k=1}^{\infty} \frac{\varepsilon_k(x)}{q_1 \cdot q_2 \cdots q_k},$$

where $\varepsilon_k(x)$ $(k = 1, 2, \dots)$ are integers, $0 \leqq \varepsilon_k(x) < q_k$ $(k = 1, 2, 3, \dots)$ and $\varepsilon_k(x) < q_k - 1$ for an infinite number of k' s (see [7] p. 113). Form the series

$$\sum_{k=1}^{\infty} \varepsilon_k(x) a_k. \tag{6}$$

The series (6) appear as a certain generalization of the subseries of the series (1).

In fact if we put $q_k = 2$ ($k = 1, 2, \ldots$), then with the exception of all dyadically rational numbers of the interval $\langle 0,1 \rangle$ the series (6) coincide with the subseries of the series (1).

If we choose in particular $q_k = k + 1$ ($k = 1, 2, 3, \ldots$), then the series (6) are said to be factorial transformations of the series (1). Let the symbols $C_1(\sum_1^\infty a_n)$, $(D_1(\sum_1^\infty a_n))$ denote the set of all those

$$x = \sum_{k=1}^{\infty} \frac{\varepsilon_k(x)}{(k+1)!} \in \langle 0,1),$$

for which (6) is convergent (divergent).

10. dim M denotes in this paper the Hausdorff dimension of the set M with respect to the system of measure functions $\mu^\alpha(t) = t^\alpha$, $0 \leqq t < +\infty$, $\alpha \in (0, 1)$ (see [8], [9]).

11. In the following T denotes the set of all dyadically rational numbers of the interval $(0,1\rangle$, i. e. the set of all numbers of the form $k/2^n$, $1 \leqq k \leqq 2^n$. We put $X = (0,1\rangle - T$.

12. If $\{s_n\}_{n=1}^\infty$ is any sequence of real numbers, then $\{s_n\}_n'$ denotes the set of all limit points of the sequence $\{s_n\}_{n=1}^\infty$. Further $\langle -\infty, +\infty \rangle$ denotes the set $(-\infty, +\infty) \cup \{+\infty\} \cup \{-\infty\}$.

13. Troughout the paper, if nothing else is said, the symbol $\sum_{n=1}^\infty a_n$ denotes a series with real terms.

14. A function g defined on $(0,1\rangle$ is said to be of the type α if the set of all its discontinuity points is dense in $(0, 1\rangle$ and simultaneously the complement of this set is dense in $(0, 1\rangle$.

15. a) If $\mathscr{S}$ is any σ-additive field of subsets of some set and μ is a measure on $\mathscr{S}$, then $M \in \mathscr{S}$ is said to be an atom if $\mu(M) > 0$ and for every $Z \subset M$, $Z \in \mathscr{S}$ either $\mu(Z) = 0$ or $\mu(Z) = \mu(M)$ holds.

b) A set $A \in \mathscr{S}$ is said to have the Darboux property if for each δ with $0 < \delta < \mu(A)$ there exists $B \subset A$, $B \in \mathscr{S}$ such that $\mu(B) = \delta$ (see [3]).

c) A set $E \in \mathscr{S}$ is said to be purely atomic if for every set $H \subset E$, $H \in \mathscr{S}$ which contains no atom, $\mu(H) = 0$ holds.

16. The symbol $A \dot{-} B$ denotes the symmetric difference of the sets A, B, so that $A \dot{-} B = B \dot{-} A = (A - B) \cup (B - A)$.

1.

SOME RESULTS ON SUBSERIES OF DIVERGENT SERIES

In the paper [1] (see also [2]) the following theorem is proved.

Theorem A. *Let*

$$\sum_{n=1}^{\infty} a_n = +\infty, \quad a_n > 0, \quad a_n \to 0. \tag{7}$$

Then there exists for each $K > 0$ *such a subseries* $\sum_{n=1}^{\infty} a_{k_n}$ *of the series* (7) *that* $\sum_{n=1}^{\infty} a_{k_n} = K$.

We prove a certain refining of this theorem.

Theorem 1,1. *Let*

$$\sum_{n=1}^{\infty} a_n = +\infty, \quad a_n > 0, \quad a\text{-} \to 0. \tag{8}$$

Then there exists for every $K > 0$ *an uncountable set* (*of the power of the continuum*) *of numbers* $x \in (0,1\rangle$ *such that* $(x) = \sum_{k=1}^{\infty} \varepsilon_k(x) a_k = K$.

Proof. From the conditions $K > 0$ and $a_n \to 0$,we infer the existence of a sequence $i_1 < i_2 < \ldots < i_n < \ldots$ of natural numbers such that

$$\sum_{n=1}^{\infty} a_{i_n} < K. \tag{9}$$

Let N denote the set of all natural numbers. Let us put

$$N - \{i_1, i_2, \ldots i_n, \ldots\} = \{l_1, l_2, \ldots l_n, \ldots\}.$$

In view of the assumption (8) we have $\sum_{n=1}^{\infty} a_{l_n} = +\infty$, $a_{l_n} > 0$, $a_{l_n} \to 0$.

Let $\{\varepsilon'_{i_n}\}_{n=1}^{\infty}$ be a sequence of numbers 0, 1. Let us put $s = \sum_{n=1}^{\infty} \varepsilon'_{i_n} a_{i_n}$. Evidently $s < K$ (see [9]). On account of theorem A there exists such a sequence $\{\varepsilon'_{l_n}\}_{n=1}^{\infty}$ with the terms 0, 1, where 1 appears infinitely many times, that $\sum_{n=1}^{\infty} \varepsilon'_{l_n} a_{l_n} = K - s > 0$. Put

$$x' = \sum_{n=1}^{\infty} \varepsilon_n(x')\, 2^{-n}, \tag{10}$$

where $\varepsilon_n(x') = \varepsilon'_n$ $(n = 1, 2, \ldots)$. Then $x' \in (0, 1\rangle$ and (10) is the infinite dyadic expansion of x'. Evidently $(x') = \sum_{n=1}^{\infty} \varepsilon_n(x')a_n = K$.

Now if $\{\varepsilon''_{i_n}\}_{n=1}^{\infty}$ is a sequence of numbers 0, 1, different from the sequence $\{\varepsilon'_{i_n}\}_{n=1}^{\infty}$, we shall construct by means of the previous procedure $x'' \in (0, 1\rangle$, $x'' = \sum_{n=1}^{\infty} \varepsilon_n(x'')\, 2^{-n}$ ($\varepsilon_n(x'') = 0$ or 1 and $\varepsilon_n(x'') = 1$ for an infinite number of n such that

$$\varepsilon_{i_n}(x'') = \varepsilon''_{i_n} \quad (n = 1, 2, 3, \ldots) \tag{11}$$

and $(x'') = K$. As a consequence of the uniqueness of dyadic expansions, $x'' \neq x'$ holds. Since the set of all sequences $\{\varepsilon_{i_n}\}_{n=1}^{\infty}$ of numbers 0, 1 is uncountable and has the power of the continuum, we get in this way an uncountable (of the power of the continuum) set of numbers $x \in (0, 1\rangle$ for which $(x) = K$. The proof is completed.

Theorem 1,2. *Let $a_n \to 0$ and let $\sum_{n=1}^{\infty} a_n$ be a series of the type* (α) (*alerntatively of the type* (β) *or* (γ)). *Then for every $K \in (s, +\infty)$, $s = \sum_{n \in N_-} a_n$ (alternatively $K \in (-\infty, s')$, $s' = \sum_{n \in N_+} a_n$ or $K \in (-\infty, +\infty)$) there exists an uncountable set (of the power of the continuum) of numbers $x \in (0, 1\rangle$ such that $(x) = K$.*

Proof. Let $\sum_{n=1}^{\infty} a_n$ be of the type (α), $a_n \to 0$. Let us put $\varepsilon_n = 1$ for $n \in N_- \cup \cup N_0$. In view of Theorem 1,1 there exists a family (of the power of the continuum) of such sequences $\{\varepsilon_n\}_{n \in N_+}$ of numbers 0, 1 (where 1 appears infinitely many times) that $\sum_{n \in N_+} \varepsilon_n a_n = K - s > 0$. Let us put

$$x = \sum_{n=1}^{\infty} \varepsilon_n(x)\, 2^{-n}, \tag{12}$$

where $\varepsilon_n(x) = \varepsilon_n$ ($n = 1, 2, 3, \ldots$). Evidently $x \in (0, 1\rangle$, (12) is the infinite dyadic expansion of the number x, $(x) = K$ and those x form an uncountable set (of the power of the continuum).

If $\sum_{n=1}^{\infty} a_n$ is of the type (β), the proof runs in an analogous way.

If $\sum_{n=1}^{\infty} a_n$ is of the type (γ), $a_n \to 0$ and $K > 0$, then there exists by Theorem A sequence $\{\varepsilon_n\}_{n \in N_0 \cup N_-}$ of numbers 0, 1 such that $\sum_{n \in N_0 \cup N_-} \varepsilon_n a_n = -K$. It follows further from Theorem A that there exists an uncountable system of sequences $\{\varepsilon_n\}_{n \in N_+}$ of numbers 0, 1 (where 1 appears infinitely many times) so that

$\sum_{n\in N_+} \varepsilon_n a_n = 2K$. Now, the assertion follows immediately. The proof for $K \leqq 0$ is analogous.

The correctness of the following theorem may also be easily seen.

Theorem 1,3. a) *Let $\sum_{n=1}^{\infty} a_n$ be of the type* (α) *or* (γ). *Then there exists an uncountable set (of the power of the continuum) of numbers $x \in (0, 1\rangle$ such that* $(x) = +\infty$.

b) *Let $\sum_{n=1}^{\infty} a_n$ be of the type* (β) *or* (γ). *Then there exists an uncountable set (of the power of the continuum) of numbers $x \in (0, 1\rangle$ such that* $(x) = -\infty$.

Various properties of the sets $C(\sum_{1}^{\infty} a_n)$, $D(\sum_{1}^{\infty} a_n)$, where $\sum_{n=1}^{\infty} a_n$ is a divergent series, are studied in [10]. According to certain results of the paper [11], if $\sum_{n=1}^{\infty} a_n = +\infty$, $a_n > 0$, then the set $C(\sum_{1}^{\infty} a_n)$ is of the first category in $(0, 1\rangle$ It is proved in the paper [12], that if $\sum_{n=1}^{\infty} a_n$ is a non — absolutely convergent series with real terms, then for all $x \in (0, 1\rangle$ with the exception of points of a set of the first category we have

$$\liminf_{n\to\infty} \sum_{k=1}^{n} \varepsilon_k(x) a_k = -\infty, \quad \limsup_{n\to\infty} \sum_{k=1}^{n} \varepsilon_k(x) a_k = +\infty$$

($(0, 1\rangle$ is assumed to be a metric space with the Euclidean metric). To these results is related the following theorem which refines also the quoted results of the paper [12].

Theorem 1,4. a) *If $\sum_{n=1}^{\infty} a_n$ is of the type* (α) *or* (β), *then $C(\sum_{1}^{\infty} a_n)$ is of the first category in* $(0, 1\rangle$.

b) *If $\sum_{n=1}^{\infty} a_n$ is of the type* (γ), *then for all $x \in (0,1\rangle$ with the exception of the points of a set of the first category we have*

$$\liminf_{n\to\infty} s_n(x) = -\infty, \quad \limsup_{n\to\infty} s_n(x) = +\infty,$$

where $s_n(x) = \sum_{k=1}^{n} \varepsilon_k(x) a_k$.

c) *If $\sum_{n=1}^{\infty} a_n$ is of the type* (γ) *and $a_n \to 0$, then for all $x \in (0,1\rangle$ with the exception of points of a set of the first category* $\{s_n(x)\}_n' = \langle -\infty, +\infty\rangle$.

d) *If* $\liminf_{n\to\infty} |a_n| = 0$, a_n $(n = 1, 2, 3,\ldots)$ *are complex numbers, then* $C(\sum_1^\infty a_n)$ *is dense in* $(0, 1\rangle$.

Proof. We shall prove b). The proof of a), which is similar, will be omitted.

Put $X = (0, 1\rangle - T$, X is assumed in what follows to be a metric space with the Euclidean metric. Let p be a natural number and let $A(p)$ denote the set of all those $x \in X$, for which $s_n(x) \leqq p$ $(n = 1, 2, 3,\ldots)$. Since the functions s_n $(n = 1, 2,\ldots)$ are continuous on X, $A(p)$ is closed in X. We shall show that $A(p)$ is nowhere dense in X. In view of its closeness it is sufficient to prove, that

$$[(a, b) \cap X] \cap (X - A(p)) \neq \emptyset \tag{13}$$

for each interval $(a, b) \subset (0, 1\rangle$. This being so, let $(a, b) \subset (0, 1\rangle$. Choose m, l, $0 \leqq l \leqq 2^m - 1$ such that $i_m^{(l)} \subset (a, b)$. Let $i_m^{(l)}$ be associated with the sequence $\varepsilon_1, \varepsilon_2,\ldots, \varepsilon_m$. Let us define $x_0 = \sum_{k=1}^{\infty} \varepsilon_k(x_0)\, 2^{-k}$ in the following way: $\varepsilon_k(x_0) = \varepsilon_k$ $(k = 1, 2,\ldots m)$, $\varepsilon_k(x_0) = 0$ for $k \in (N_- \cup N_0) \cap \{m + 1, m + 2, m + 3,\ldots\}$, $\varepsilon_k(x_0) = 1$ for $k \in N_+ \cap \{m + 1, m + 2,\ldots\}$. Evidently $(x_0) = \sum_{k\ 1}^{\infty} \varepsilon_k(x_0) a_k = + \infty$, so that $x_0 \in X - A(p)$ and $x_0 \in i_m^{(l)} \cap X \subset (a, b) \cap X$.

Let us put $A = \bigcup_{p-1}^{\infty} A(p)$. Then A is a set of the first category in X and thus also of the first category in $(0, 1\rangle$. Obviously A is the set of all those $x \in X$ for which $\limsup_{n\to\infty} s_n(x) < + \infty$.

In a similar way it is possible to show that also the set B of all those $x \in X$ for which $\liminf_{n\to\infty} s_n(x) > - \infty$ is of the first category in $(0, 1\rangle$ and thus in view of the countability of the set T the affirmation of b) follows immediately.

The part c) follows easily from b). In fact, let $\sum_{n=1}^{\infty} a_n$ be of the type (γ) and $a_n \to 0$. Then according to b) there exists a residual set $M \subset (0, 1\rangle$ such that for $x \in M$ we have $\liminf_{n\to\infty} s_n(x) = - \infty$, $\limsup_{n\to\infty} s_n(x) = + \infty$.

In view of these relations it is possible to assign to each $\zeta \in (- \infty, + \infty)$ a sequence of natural numbers

$$n_1 < n_2 < n_3 < \ldots < n_k < \ldots$$

such that $s_{n_1}(x) < \zeta$, n_2 is the least of those $n > n_1$, for which $s_n(x) \geqq \zeta$, n_3 is the least of those $n > n_2$, for which $s_n(x) < \zeta$ etc. Then obviously

(14) $$|s_{n_k}(x) - \zeta| \leqq |\varepsilon_{n_k}(x) a_{n_k}| \leqq |a_{n_k}| \quad (k = 2, 3, \ldots),$$

in view of the assumption $a_n \to 0$ we get $\zeta \in \{s_n(x)\}'_n$.

We shall prove part d). From the assumption the existence of such sequence of natural numbers

$$k_1 < k_2 < \ldots < k_n < \ldots$$

follows that

(15) $$\sum_{n=1}^{\infty} |a_{k_n}| < +\infty.$$

Evidently it suffices to prove: if m, l are two integers, m natural and $0 \leqq l \leqq \leqq 2^m - 1$, then $i_m^{(l)} \cap C(\sum_1^\infty a_n) \neq \emptyset$. Let $i_m^{(l)}$ be associated with the sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$. Let us define $x_0 = \sum_{k=1}^{\infty} \varepsilon_k(x_0)\, 2^{-k}$ in the following way: $\varepsilon_k(x_0) = \varepsilon_k$ ($k = 1, 2, \ldots m$), further $\varepsilon_k(x_0) = 1$, if $k > m$ and $k = k_j$ (for suitable j), and $\varepsilon_k(x_0) = 0$, if $k > m$, $k \neq k_i$ ($i = 1, 2, 3, \ldots$). Evidently $x_0 \in i_m^{(l)}$ and according to (15) $x_0 \in C(\sum_1^\infty a_n)$ holds.

Theorem 1,5. *Let $\sum_{n\ 1}^{\infty} a_n$ be a series with complex terms, let $\sum_{n-1}^{\infty} |a_n| = +\infty$. Then $C(\sum_1^\infty a_n)$ is a set of the first category in $(0, 1\rangle$.*

Proof. Let us put $a_n = a'_n + i a''_n$, a'_n, a''_n are real numbers. From the assumption of the Theorem it follows that at least one of the series $\sum_{n-1}^{\infty} |a'_n|$, $\sum_{n\ 1}^{\infty} |a''_n|$ is divergent. The conclusion of the Theorem is then easy consequence of Theorem 1,4 and of the evident inclusion

$$C(\sum_1^\infty a_n) \subset C(\sum_1^\infty a'_n) \cap C(\sum_1^\infty a''_n).$$

In the well-known collection of problems of mathematical analysis [14] (see [14] p. 44), the problems of the following type can be found: Let $\mathscr{U}$ be a set of subseries of the series

(16) $$\sum_{n\ 1}^{\infty} a_n - a_1 + a_2 + \ldots + a_n + \ldots .$$

The question arises if the convergence of all series belonging to $\mathscr{U}$ implies the absolute convergence of the series (16). If e. g. the system of all subseries of the series (16) is taken for $\mathscr{U}$, then the convergence of all subseries of the system $\mathscr{U}$ implies the absolute convergence of the series (16). If the system of all subseries of the form $\sum_{s=1}^{\infty} a_{k+sl}$ (k, l are natural numbers) of the series (16) is taken for $\mathscr{U}$, then the convergence of all the series of the system $\mathscr{U}$ does not imply the absolute convergence of the series (16). The following theoren which may be taken as a topological criterion of the absolute convergence of series, shows the origin of the different answers to the given question, when the choices of the sets $\mathscr{U}$ differ. Observe that in the second case the set $\mathscr{U}$ is countable, since the set of all the arithmetic sequences $\{k + sl\}_{s=1}^{\infty}$, where l, k are natural numbers, is countable. Hence in this case the set of all those $x \in (0, 1\rangle$, for which $(x) \in \mathscr{U}$, is of the first category in $(0, 1\rangle$.

Theorem 1,6. *The series* $\sum_{n=1}^{\infty} a_n$ *with complex terms is absolutely convergent if and only if there exists such a set* $H \subset (0, 1\rangle$ *of the second category in* $(0, 1\rangle$ *that for every* $x \in H$ *the series* $(x) = \sum_{k=1}^{\infty} \varepsilon_k(x)\, a_k$ *is convergent.*

Proof. If $\sum_{n=1}^{\infty} |a_n| < +\infty$, then we can put $H = (0,1\rangle$. Suppose, conversely, that there exists $H \subset (0,1\rangle$ of the second category in $(0, 1\rangle$ such that for $x \in H$ the series (x) is convergent. If $\sum_{n=1}^{\infty} |a_n| = +\infty$, then according to Theorem 1,5 $C(\sum_1^{\infty} a_n)$ is of the first category in $(0. 1\rangle$ and from the evident inclusion $H \subset C(\sum_1^{\infty} a_n)$ we see that H is of the first category in $(0, 1\rangle$. But this is a contradiction with the assumption of the Theorem. Hence $\sum_{n=1}^{\infty} |a_n| < +\infty$ and the proof is completed.

Note 1,1. In connection with Theorem 1,6 a question arises if it is possible to construct in the interval $(0, 1\rangle$ a set $H \subset (0, 1)$ such that for every interval $(a, b) \subset (0, 1\rangle$ the sets $H \cap (a, b)$ and $(a, b) - H$ would be of the second category in $(0, 1\rangle$. From the results of the paper [15] (see also [13] p. 54) it follows that such a decomposition is possible. The proofs of the results contained in [15] are not constructive and thus they do not give a direction how such a decomposition is to be made.

If $X = \sum_{k=1}^{\infty} \varepsilon_k(x)\, 2^{-k}$ is the infinite dyadic expansion of the number $x \in (0, 1\rangle$, then $p(n, x)$ will denote the sum $\sum_{k=1}^{n} \varepsilon_k(x)$. It is proved in the papers [16] and

[17] that if $\sum_{n=1}^{\infty} a_n$ is a series with real or complex terms and $\sum_{n=1}^{\infty} |a_n| = +\infty$, then there exists an $x \in \langle 0,1)$ such that $\lim_{n\to\infty} p(n, x)/n = 0$ and $(x) = \sum_{k=1}^{\infty} \varepsilon_k(x)a_k$ is divergent. In connection with this assertion we shall prove the following result:

Theorem 1,7. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series with coplex terms and let* $\sum_{n=1}^{\infty} |a_n| = +\infty$. *Then for all* $x \in (0, 1\rangle$ *with the exception of the points of a set of the first category in* $(0, 1\rangle$ *the following relations hold simultaneously*:

(i) $$(x) = \sum_{k=1}^{\infty} \varepsilon_k(x)a_k$$

is divergent,

(ii) $$\left\{\frac{p(n, x)}{n}\right\}_n' = \langle 0, 1\rangle.$$

Proof. Let M be the set of all those $x \in (0,1\rangle$ for which (i) and (ii) hold. Let M_1 be the set of all those $x \in (0,1\rangle$ for which (ii) holds. According to [18] the set M_1 is residual in $(0, 1\rangle$. Evidently $M = M_1 \cap D(\sum_1^{\infty} a_n)$ and the conclusion follows from Theorem 1,5.

Note 1,2. a) In the preceding theorem (ii) cannot be replaced by the condition $\lim_{n\to\infty} p(n, x)/n = 0$ (Cf. with the mentioned results of the papers [16], [17]), since $\{x \in (0, 1\rangle;\ \lim_{n\to\infty} p(n, x)/n = 0\}$ is of the first category in $(0, 1\rangle$ (see [18]).

b) It is showed in the paper [10] that if $\sum_{n=1}^{\infty} a_n$ is a divergent series with positive terms and

$$a_1 \geqq a_2 \geqq \ldots \geqq a_n \geqq \ldots,$$

then

(*) $$\liminf_{n\to\infty} \frac{p(n, x)}{n} = 0$$

is a necessary condition for the convergence of the series (x). Theorem 1,7 shows that (*) is not a sufficient condition for the convergence of the series (x).

c) If $\mathscr{U}$ is any system of subseries of the series

$$(17) \qquad \sum_{n\ 1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$$

we shall denote by $E(\mathscr{U})$ the set of all those $x \in (0, 1\rangle$ for which $(x) \in \mathscr{U}$. If $E(\mathscr{U})$ is a set of the second category then according to Theorem 1,6 the convergence of all the series of the system $\mathscr{U}$ implies the absolute convergence of the series (17). Thus the ,,topological magnitude" of the set $E(\mathscr{U})$ guarantees that the convergence of all the series of the system $\mathscr{U}$ implies the absolute convergence of the series (17). The following example shows that the ,,metrical magnitude" of the set $E(\mathscr{U})$ has here no influence similar to the mentioned influence of the ,,topological magnitude" of the set $E(\mathscr{U})$. Let e. g. $E(\mathscr{U})$ consist of all those $x \in (0,1\rangle$ for which (ii) holds. Then $E(\mathscr{U})$ is residual (see [18]) and thus from the convergence of the series (x), $x \in E(\mathscr{U})$, the absolute convergence of the series (17) follows. But from the metrical point of view $E(\mathscr{U})$ is a ,,poor" set, since not only its Lebesgue measure but even its Hausdorff dimension is zero (see [8]).

The following Theorems concern the topological structure and the metric properties of the sets $C(\sum_1^{\infty} a_n)$, $D(\sum_1^{\infty} a_n)$ and some related sets.

Theorem 1,8. *Let $\sum_{n=1}^{\infty} a_n$ be an arbitrary series with real terms. Then each of the sets $C(\sum_1^{\infty} a_n)$, $C^*(\sum_1^{\infty} a_n)$ is of the type $F_{\sigma\delta}$ in $(0, 1\rangle$.*

Corollary. *Each of the sets $D(\sum_1^{\infty} a_n)$, $D^*(\sum_1^{\infty} a_n)$ is of the type $G_{\delta\sigma}$ in $(0, 1\rangle$.*

Proof of the Theorem. According to the Cauchy-Bolzano criterion for the convergence of series we get

$$(18) \qquad C(\sum_1^{\infty} a) \cap X = \bigcap_{k=1}^{\infty} \bigcup_{p\ 1}^{\infty} \bigcap_{n=p}^{\infty} \bigcap_{m=p}^{\infty} A(k, m, n),$$

where

$$A(k, m, n) = \left\{ x \in X; \quad |\sum_{l=n}^{m} \varepsilon_l(x) a_l| < \frac{1}{k} \right\}.$$

Obviously $A(k, m, n)$ is a union of a finite number of sets of the form $i_r^{(j)} \cap X$, $r \quad \max(m, n)$ and therefore $A(k, m, n)$ is closed in X. Then it follows from (18) that $C(\sum_1^{\infty} a_n) \cap X$ is of the type $F_{\sigma\delta}$ in X and so may be expressed in the form $X \cap P$, where P is of the type $F_{\sigma\delta}$ in $(0,1\rangle$. But X is evidently of the type G_δ in $(0,1\rangle$ and so $C(\sum_1^\infty a_n) \cap X$ is of the type $F_{\sigma\delta}$ in $(0,1\rangle$. Since $C(\sum_1^\infty a_n) \cap T \subset T$, the set $C(\sum_1^\infty a_n) \cap T$ is of the type F_σ in $(0,1\rangle$. Hence we get immediately that $C(\sum_1^\infty a_n)$ is of the type $F_{\sigma\delta}$ in $(0,1\rangle$.

If we denote by $G_1 = G_1(\sum_1^\infty a_n)$ and alternatively by $G_2 = G_2(\sum_1^\infty a_n)$ the set of all those $x = \sum_{k=1}^\infty \varepsilon_k(x)\, 2^{-k} \in (0,1\rangle$, for which $\sum_{k-1}^\infty \varepsilon_k(x) a_k = +\infty$ alternatively $\sum_{k=1}^\infty \varepsilon_k(x) a_k = -\infty$, then evidently

$$C^*(\sum_1^\infty a_n) = C(\sum_1^\infty a_n) \cup G_1(\sum_1^\infty a_n) \cup G_2(\sum_1^\infty a_n). \tag{19}$$

Thus it suffices to prove that each of the sets $G_1(\sum_1^\infty a_n)$, $G_2(\sum_1^\infty a_n)$ is of the type $F_{\sigma\delta}$ in $(0,1\rangle$. We shall prove it for $G_1(\sum_1^\infty a_n)$, the proof for $G_2(\sum_1^\infty a_n)$ being analogous.

From the definition of $G_1 = G_1(\sum_1^\infty a_n)$ we have

$$G_1 \cap X = \bigcap_{k=1}^\infty \bigcup_{p=1}^\infty \bigcap_{n=p}^\infty B(k, n), \tag{20}$$

where $B(k, n) = \{x \in X; \sum_{l=1}^n \varepsilon_l(x) a_l > k\}$. $B(k, n)$ is evidently closed in X and the same reasoning as in the case of examining the structure of $C(\sum_1^\infty a_n)$ leads, using (20), to the fact that $G_1(\sum_1^\infty a_n)$ is of the type $F_{\sigma\delta}$ in $(0, 1\rangle$.

Theorem 1,9. *Let* $\sum_{n=1}^\infty |a_n| = +\infty$. *Then for almost all* $x \in (0, 1\rangle$ *we have* $\sum_{k\ 1}^\infty |\varepsilon_k(x) a_k| = +\infty$.

Proof. Denote by U_1 (U_2) the set of all those $x \in (0, 1\rangle$ for which $\sum_{k=1}^\infty |\varepsilon_k(x) a_k| < +\infty$ ($\sum_{k=1}^\infty |\varepsilon_k(x) a_k| = +\infty$). Evidently U_1, U_2 are disjoint and $U_1 \cup U_2 = (0, 1\rangle$. Further U_1 and U_2 are measurable. The measurability of U_1 follows from the fact, that U_1 is the domain of convergence of the sequence $\{s_n(x)\}_{n=1}^{\infty}$ $s_n(x) = \sum_{k=1}^{n} \varepsilon_k(x)a_k$ and s_n $(n = 1, 2, 3, \dots)$ are measurable since they are simple (on $(0, 1\rangle$) functions.

From Lemma 1 of the paper [19] (see also [4]) it easily follows that U_1 is homogeneous in $(0, 1\rangle$ and therefore either $|U_1| = 0$ or $|U_1| = 1$. If $|U_1| = 1$, then the measure of the set $g(U_1 \cap (0, \frac{1}{2}\rangle) \subset (\frac{1}{2}, 1\rangle$, $g(t) = 1 - t$, is equal to $1/2$ and there exists an $x_0 \in T$ such that x_0 and also $1 - x_0$ belongs to U_1. Hence $\sum_{k=1}^{\infty} |\varepsilon_k(x_0)a_k| < +\infty$ and simultaneously $\sum_{k=1}^{\infty} |\varepsilon_k(1 - x_0)a_k| = \sum_{k=1}^{\infty} |(1 - \varepsilon_k(x_0))a_k| < +\infty$, consequently $\sum_{k=1}^{\infty} |a_k| < +\infty$ in contradiction to the assumption of the Theorem. Hence $|U_1| = 0$ and the Theorem is proved.

Theorem 1,10. a) *If* $\sum_{n=1}^{\infty} a_n + = \infty$, *then* $|C(\sum_1^{\infty} a_n)| = 0$.

b) *If* $\sum_{n-1}^{\infty} a_n = -\infty$, *then* $|C(\sum_1^{\infty} a_n)| = 0$.

c) *If* $\sum_{n=1}^{\infty} a_n$ *oscillates, then* $|C^*(\sum_1^{\infty} a_n)| = 0$.

Proof. a), b) may be proved in a way analogous to the one used in the proof of Theorem 1,9.

In the case c) the measurability of each of the sets

$$C(\sum_1^{\infty} a_n), \quad G_1(\sum_1^{\infty} a_n), \quad G_2(\sum_1^{\infty} a_n)$$

may be seen by a similar method as in the proof of the preceding Theorem when the measurability of U_1 was proved. The homogeneity of each of the sets may be easily seen by means of Lemma 1 of the paper [19]. Using a method similar to that used in proving $|U_1| = 0$ in Theorem 1,9 we find that

$$|C(\sum_1^{\infty} a_n)| = |G_1(\sum_1^{\infty} a_n)| = |G_2(\sum_1^{\infty} a_n)| = 0.$$

2.

PROPERTIES OF THE FUNCTIONS $f(\sum_1^{\infty} a_n)$.

Throughout the following part of the paper we shall assume a_n $(n = 1, 2, 3, \dots)$ to be real numbers, $\sum_{n=1}^{\infty} |a_n| = +\infty$.

The function $f = f(\sum_1^\infty a_n)$ will be defined in the following way: If $x \in C\,(\sum_1^\infty a_n)$, we put $f(x) = \dfrac{(x)}{1 + |(x)|}$ (see (3), (4)). If $(x) = +\infty$, then $f(x) = 1$. If $(x) = -\infty$, then $f(x) = -1$. If (x) oscillates, we put $f(x) = 0$.

Theorem 2,1. a) *Let $a_n \to 0$ and let $\sum_{n=1}^\infty a_n$ be of the type (α) or (β). Then the set of all discontinuity points of the function $f = f(\sum_1^\infty a_n)$ coincides with the set $C(\sum_1^\infty a_n) \cup (T - \{1\})$.*

b) *Let $a_n \to 0$ and let $\sum_{n=1}^\infty a_n$ be of the type (γ). Then $f(\sum_1^\infty a_n)$ is discontinuous at each point of the interval $(0, 1\rangle$.*

Proof. a) We shall restrict ourselves to the case that $\sum_{n=1}^\infty a_n$ is of the type (α). Put $T' = T - \{1\}$ and let $x_0 \neq 1$, $x_0 \notin C(\sum_1^\infty a_n) \cup T'$. Then

$$x_0 \in D(\sum_1^\infty a_n) \tag{21}$$

and $f(x_0) = 1$. Let $\varepsilon > 0$. Choose $K > 0$ such that $1/(1 + K) < \varepsilon$. Since $x_0 \neq 1$ fulfils (21) and $\sum_{n=1}^\infty a_n$ is of the type (α), there exists such an interval $i_m^{(l)}$ of the m-th rank, that x_0 lies in the interior of this interval and for each point x which lies in the interior of that interval, $(x) = \sum_{k=1}^\infty \varepsilon_k(x)\, a_k > K$ holds. Hence if $(x) < +\infty$, then

$$|f(x) - f(x_0)| = \left|1 - \frac{(x)}{1 + |(x)|}\right| = \frac{1}{1 + (x)} < \frac{1}{1 + K} < \varepsilon.$$

Thus f is continuous at the point x_0. Similarly it may be seen that f is left-continuous at the point 1.

Now, let $x_0 \in C(\sum_1^\infty a_n)$, so that $f(x_0) < 1$. It follows from Theorem 1,4 that $D(\sum_1^\infty a_n)$ is dense in $(0, 1\rangle$. Consequently each nieghbourhood of the point x_0 contains a point x ($x \in D(\sum_1^\infty a_n)$), for which $f(x) = 1$. This proves the discontinuity of f at the point x_0.

Finally let $x_0 \in T' = T - \{1\}$. Then $f(x_0) = 1$.
Let

$$x_0 = \frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2^2} + \ldots + \frac{\varepsilon_r}{2^r}, \quad r \geqq 1,\ \varepsilon_r = 1,$$

be the finite dyadic expansion of the number x_0. Then its infinite dyadic expansion is

$$x_0 = \frac{\varepsilon_1(x_0)}{2} + \ldots + \frac{\varepsilon_{r-1}(x_0)}{2^{r-1}} + \frac{0}{2^r} + \frac{1}{2^{r+1}} + \frac{1}{2^{r+2}} + \cdots,$$

where $\varepsilon_i(x_0) = \varepsilon_i$ for $i = 1, 2, \ldots r - 1$. Let $k_1 < k_2 < \ldots < k_n < \ldots$ be a sequence of natural numbers such that $k_1 > r$ and $\sum_{n-1}^{\infty} |a_{k_n}| < +\infty$. Such a sequence exists according to the assumption $a_n \to 0$. Now, let us define the numbers x_v as follows:

$$x_v = \sum_{k=1}^{\infty} \varepsilon_k(x_v) 2^{-k}, \quad \varepsilon_k(x_v) = \varepsilon_k(x_0) \quad (k = 1, 2, \ldots r - 1),$$

$\varepsilon_r(x_v)$ 1, $\varepsilon_n(x_v) = 0$ for $n > r$, $n \neq k_i$ $(i = 1, 2, \ldots)$, $\varepsilon_{k_i}(x_v) = 0$ for $i \leqq v$ and $\varepsilon_{k_i}(x_v) = 1$ for $i > v$.

Evidently $x_v \to x_0$ $(v \to \infty)$ and

$$|f(x_v)| \leqq \frac{\sum_{i=1}^{r} |a_i| + \sum_{i=1}^{\infty} |a_{k_i}|}{1 + \sum_{i=1}^{r} |a_i| + \sum_{i=1}^{\infty} |a_{k_i}|} = \delta_r < 1$$

for $v - 1, 2, 3, \ldots$, where δ_r depends only on r and does not depend on v. The discontinuity of f at the point x_0 follows immediately.

b) Let $\sum_{n-1}^{\infty} a_n$ be of the type (γ). Let $x_0 \in (0, 1\rangle$. The following two cases may occur:

1. $|f(x_0)| = 1$. 2. $|f(x_0)| < 1$.

In the case 1 we use the fact that on a residual and consequently dense in $(0, 1\rangle$ set (see Theorem 1,4 c)) $f(x) = 0$ holds. This implies the discontinuity of f at x_0 immediately.

In the case 2 let δ be an arbitrary positive number. Let m be chosen so large that an interval $i_m^{(l)}$ $(0 \leqq l \leqq 2^m - 1)$ with the property $x_0 \in i_m^{(l)} \subset$

$\subset (x_0 - \delta, x_0 + \delta)$ exists. Let $i_m^{(l)}$ be associated with the sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$. Let $x_1 = \sum_{k=1}^{\infty} \varepsilon_k(x_1) 2^{-k}$ be defined as follows: $\varepsilon_k(x_1) = \varepsilon_k (k = 1, 2, \ldots m)$, futher $\varepsilon_k(x_1) = 0$ for $k \in (N_- \cup N_0) \cap \{m + 1, m + 2, \ldots\}$ and $\varepsilon_k(x_1) = 1$ for $k \in N_+ \cap \{m + 1, m + 2, \ldots\}$. Evidently $x_1 \in i_m^{(l)}$ and $f(x_1) = 1$. This implies the discontinuity of f at the point x_0.

According to Theorem 1,10 and the known Lebesgue criterion on Riemann integrability, we get the following result.

Theorem 2,2. *If $a_n \to 0$ and $\sum_{n=1}^{\infty} a_n$ is of the type (α) or (β), then $f(\sum_1^{\infty} a_n)$ is integrable (in the Riemann sense) on $\langle 0, 1\rangle$. If $a_n \to 0$ and $\sum_{n-1}^{\infty} a_n$ is of the type (γ), then $f(\sum_1^{\infty} a_n)$ is not integrable (in the Riemann sense) on $\langle 0, 1\rangle$.*

Note 2,1. If $\sum_{n=1}^{\infty} a_n$ is of the type (α) or (β), then Theorem 1,4 implies the function $f(\sum_1^{\infty} a_n)$ to be of the type α (see [20]). It can be easily seen that the set of all discontinuity points of a function of the type α is of the first category (see [20]). The last fact implies again that if $\sum_{n=1}^{\infty} a_n$ is of the type (α) or (β) and $a_n \to 0$, then $C(\sum_1^{\infty} a_n)$ is of the first category in $(0, 1\rangle$.

From Theorem 1,2 the following result may be easily obtained.

Theorem 2,3. a) *Let $a_n \to 0$ and let $\sum_{n=1}^{\infty} a_n$ be of the type (α). Then*

$$f((0, 1\rangle) = \left(\frac{s}{1 + |s|}, 1\right\rangle$$

if N_- and N_0 are finite and

$$f((0, 1\rangle) = \langle \frac{s}{1 + |s|}, 1\rangle$$

if at least one of the sets N_-, N_0 is infinite ($s = \sum_{n \in N_-} a_n$).

b) *Let $a_n \to 0$ and let $\sum_{n-1}^{\infty} a_n$ be of the type (γ). Then $f((0, 1\rangle) = \langle -1, 1\rangle$.*

Note 2,2. An analogous affirmation to that appearing in part a) may be formulated for the case of $\sum_{n\ 1}^{\infty} a_n$ being of the type (β).

Several examples of functions with the Darboux property and discontinuous at each point of the interval $(0, 1\rangle$ are known (see [21], [22]). From the following Theorem the existence of one class of such functions follows.

Theorem 2,4. a) *Let* $a_n \to 0$, *let* $\sum_{n=1}^{\infty} a_n$ *be either of the type* (α) *or* (β). *Let* $x_1 < x_2$, $x_1, x_2 \in (0, 1\rangle$, $f(x_1) \neq f(x_2)$ $(f = f(\sum_1^{\infty} a_n))$. *Let* z *lie between the numbers* $f(x_1), f(x_2)$. *The there exists a set (of the power of the continuum) of numbers* x *lying between* x_1 *and* x_2 *such that* $f(x) = z$.

b) *Let* $a_n \to 0$ *and let* $\sum_{n=1}^{\infty} a_n$ *be of the type* (γ). *Then* $f = f(\sum_1^{\infty} a_n)$ *assumes each of its values on every interval at the points of a set of the power of the continuum.*

Corollary. *If* a_n $(n = 1, 2, 3, \ldots)$ *are real numbers,* $a_n \to 0$, $\sum_{n-1}^{\infty} |a_n| = +\infty$, *then* $f(\sum_1^{\infty} a_n)$ *has on the interval* $(0, 1\rangle$ *the Darboux property.*

Proof of the Theorem. Let $\sum_{n=1}^{\infty} a_n$ be of the type (α). Let $x_1 < x_2$ and let e. g. $f(x_1) < z < f(x_2)$ (in the case $f(x_1) > z > f(x_2)$ the proof would run in an analogous way). Choose m so large that the numbers x_1, x_2 belong to two different intervals of the rank m, $x_1 \in i_m^{(k_1)}$, $x_2 \in i_m^{(k_2)}$, $k_1 \neq k_2$. Since $f(x_1) < < f(x_2) \leqq 1$, we have $f(x_1) < 1$, so that $x_1 \notin T$ (all the points of T have in their dyadic expansions from a certain index all the digits equal to 1 and therefore $f(x) = 1$ for $x \in T$). In view of the fact that the function

$$\varphi(t) = \frac{t}{1 + |t|}, \quad t \in (-\infty, +\infty)$$

is increasing and continuous, a $\delta > 0$ exists such that

$$z = \frac{\sum_{k=1}^{\infty} \varepsilon_k(x_1) a_k + \delta}{1 + \left|\sum_{k-1}^{\infty} \varepsilon_k(x_1) a_k + \delta\right|}. \tag{22}$$

Define $x = \sum_{k-1}^{\infty} \varepsilon_k(x) 2^{-k}$ in this way: $\varepsilon_k(x) = \varepsilon_k(x_1)$ for $k = 1, 2, \ldots m$. Further, for $k > m$ we put $\varepsilon_k(x) = 1$ if $\varepsilon_k(x_1) = 1$. Denote by Z the set of all those $k > m$ for which $\varepsilon_k(x_1) = 0$. Then evidently Z is infinite and $\sum_{k \in Z} a_k = +\infty$. According to Theorem 1,2 we shall construct a sequence $k_i \in Z$ $(i = 1, 2, 3, \ldots)$, $k_1 < k_2 < \cdots < k_n < \ldots$

such that

(23) $$\sum_{i=1}^{\infty} a_{k_i} = \delta.$$

We put $\varepsilon_{k_i}(x) = 1$ $(i = 1, 2, 3, \ldots)$ and $\varepsilon_k(x) = 0$ for $k \in Z$, $k \neq k_i$ $(i = 1, 2, 3, \ldots)$. Thus $\{\varepsilon_k(x)\}_{k=1}^{\infty}$ is defined for all k, $x \in i_m^{(k_1)}$ and therefore $x < x_2$. Evidently $x_1 < x$ and in view of (22), (23) $f(x) = z$ holds.

On account of Theorem 1,2 (applied to the series $\sum_{k \in Z} a_k$) there exists for the number $\delta > 0$ an uncountable set (of the power of the continuum) of sequences $k_1 < k_2 < \ldots < k_n < \ldots$, for which (23) holds and so according to the preceding construction there exists also an uncountable set (of the power of the continuum) of such x for which $x_1 < x < x_2$ and $f(x) = z$.

Now let $\sum_{n-1}^{\infty} a_k$ be of the type (γ). Let $I \subset (0, 1\rangle$. If $u \in (-1, 1)$ we shall find K such that $\frac{K}{1 + |K|} = u$. We put $K = +\infty$ if $u = 1$ and $K = -\infty$ if $u = -1$. For suitable m, l we shall have $i_m^{(l)} \subset I$. Let $i_m^{(l)}$ be associated with the sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$. Since $\sum_{i=m+1}^{\infty} a_i$ is again of the type (γ) and $a_i \to 0$, Theorems 1,2 and 1,3 imply the existence of such an uncountable set (of the power of the continuum) of sequences $\{\eta_i\}_{i}^{\infty}{}_{m+1}$ of numbers 0, 1 (in which 1 appears infinitely many times) that $\sum_{i-m+1}^{\infty} \eta_i a_i = K - \sum_{i=1}^{m} \varepsilon_i a_i$. Now if we put $x = \sum_{i\ 1}^{\infty} \varepsilon_i(x) 2^{-i}$, $\varepsilon_i(x) = \varepsilon_i$ $(i = 1, 2, \ldots m)$ and $\varepsilon_i(x) = \eta_i$ $(i = m + 1, m + 2, \ldots)$, we see that $x \in i_m^{(l)}$, $f(x) = u$ and the set of all such x has the power of the continuum. The proof is completed.

Theorem 2,5. *The function* $f = f(\sum_1^{\infty} a_n)$ *is strongly locally recurrent at each point of the interval* $(0, 1\rangle$ *if* $\sum_{n=1}^{\infty} a_n$ *fulfils one of these conditons:*

(j) $\sum_{n-1}^{\infty} a_n$ *is of the type* (α) *and both the sets* N_-, N_0 *are finite,*

(k) $\sum_{n\ 1}^{\infty} a_n$ *is of the type* (α), N_- *is finite and* N_0 *infinite,*

(l) $\sum_{n\ 1}^{\infty} a_n$ *is of the type* (α) *and both the sets* N_-, N_0 *are infinite,*

(m) $\sum_{n-1}^{\infty} a_n$ *is of the type* (γ).

Note 2,3. It is possible to prove an analogical result for the type (β). An easy consideration enables us to see that if $\sum_{n=1}^{\infty} a_n$ is of the type (α), N_- is infinite and N_0 finite, then $f(\sum_1^\infty a_n)$ is not locally recurrent at an arbitrary point, the infinite dyadic expansion of which differs from the dyadic expansion of the number $x_0 = \sum_{k=1}^\infty \varepsilon_k(x_0)2^{-k}$ ($\varepsilon_k(x_0) = 1$ for $k \in N_-$ and $\varepsilon_k(x_0) = 0$ for other k) at most in those k which belong to the set N_0.

Proof of the Theorem. The affirmation easily follows from Theorem 2,4 if $\sum_{n=1}^\infty a_n$ fulfils the condition (m). Let $\sum_{n=1}^\infty a_n$ fulfil (j). On account of [6] it suffices to verify that for every t from the range of f, the set $E_t(f) = \{x \in (0, 1\rangle; f(x) = t\}$ is dense in itself. Theorem 2,3 implies that the range of f is the interval $\left(\frac{s}{1+|s|}, 1\right\rangle$, $s = \sum_{n \in N_-} a_n$. This being so, let $t \in \left(\frac{s}{1+|s|}, 1\right\rangle$. If $t = 1$, then from Theorem 1,4 we infer that $E_1(f)$ is dense in itself. Suppose therefore that $\frac{1+|s|}{s} < t < 1$. Let $f(x_0) = t$, $x_0 = \sum_{k=1}^\infty \varepsilon_k(x_0)2^{-k} \in (0, 1\rangle$, let $\delta > 0$. There exists m such that if l is suitably chosen, $0 \leqq l \leqq 2^m - 1$ we have $x_0 \in i_m^{(l)} \subset (x_0 - \delta, x_0 + \delta)$. Since $f(x_0) = t < 1$, the series $(x_0) = \sum_{k=1}^\infty \varepsilon_k(x_0)a_k$ is convergent. Put $u = \sum_{k=m+1}^\infty \varepsilon_k(x_0)a_k$. The series $\sum_{k=m+1}^\infty a_k$ is evidently of the type (α). As a consequence of the assumption (j) we have $s^* < u < +\infty$, where $s^* = \sum_{k>m,\, a_k<0} a_k$. Hence for the series $\sum_{k=m+1}^\infty a_k$ the assumptions of Theorem 1,2 are fulfilled and therfore in view of Theorem 1,2 there exists such an infinite set (of the power of the continuum) of sequences $\{\eta_k\}_{k=m+1}^\infty$ of numbers 0, 1 (where 1 appears infinitely many times) that $\sum_{k=m+1}^\infty \eta_k a_k = u$. Put $x = \sum_{k=1}^\infty \varepsilon_k(x)2^{-k}$, $\varepsilon_k(x) = \varepsilon_k(x_0)$ for $k \leqq m$, $\varepsilon_k(x) = \eta_k$ for $k > m$. Evidently $x \in i_m^{(l)}$, so $x \in (x_0 - \delta, x_0 + \delta)$ and $f(x) = f(x_0) = t$. Moreover the set of such x is obviously uncountable of the power of the continuum.

In a similar way the assertion in the cases (k), (l) may be proved.

We shall show that every function $f = f(\sum_1^\infty a_n)$ is Borel measurable.

Theorem 2,6. a) *If $\sum_{n=1}^\infty a_n$ is of the type (α) or (β), then $f(\sum_1^\infty a_n)$ is a function of the second Baire class (on $(0, 1\rangle$).*

b) *If $\sum_{n=1}^\infty a_n$ is of the type (γ), then $f(\sum_1^\infty a_n)$ is a function of the third Baire class (on $(0, 1\rangle$).*

Proof. a) Let $\sum_{n=1}^{\infty} a_n$ be of the type (α) (for the type (β) the proof runs in an analogous way). It follows from the definition of $f = f(\sum_1^\infty a_n)$ that f is a limit of a sequence of simple functions f_n ($n = 1, 2, 3, \ldots$) defined on $(0, 1\rangle$ as follows: for $x = \sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k} \in (0, 1\rangle$

$$f_n(x) = \frac{\sum_{i=1}^{n} \varepsilon_i(x) a_i}{1 + \left| \sum_{i=1}^{n} \varepsilon_i(x) a_i \right|} .$$

Since f_n ($n = 1, 2, 3, \ldots$) are of the first Baire class, f is of the second Baire class on $(0, 1\rangle$ (see [13] p. 299—300).

Let $\sum_{n=1}^{\infty} a_n$ be of the type (γ). Then it suffices to prove (see [13] p. 282) that for every real a each of the sets $M^a = \{x \in (0, 1\rangle; f(x) < a\}$, $M_a = \{x \in (0, 1\rangle; f(x) > a\}$ is a set of the type $F_{\sigma\delta\sigma}$ in $(0, 1\rangle$.

Consider at first the set M^a. If $a > 1$ or $a \leqq -1$, M^a is evidently of the type $F_{\sigma\delta\sigma}$ in $(0, 1\rangle$. Hence let $-1 < a \leqq 1$. Two cases can occur. 1. $0 < a \leqq \leqq 1$ 2. $-1 < a \leqq 0$.

1. Evidently $M^a = D^*(\sum_1^\infty a_n) \cup \{x \in C^*(\sum_1^\infty a_n); f(x) < a\}$. Put $R^a = \{x \in \in C^*(\sum_1^\infty a_n); f(x) < a\}$. Then $R^a = (R^a \cap X) \cup (R^a \cap T)$. Since $R^a \cap T \subset T$, we see that $R^a \cap T$ is of the type F_σ and consequently also of the type $F_{\sigma\delta\sigma}$ in $(0, 1\rangle$. Further $D^*(\sum_1^\infty a_n)$ is of the type $G_{\delta\sigma}$ in $(0, 1\rangle$ (see the corollary following Theorem 1,8), thus it suffices to prove that $R^a \cap X$ is of the type $F_{\sigma\delta\sigma}$ in $(0, 1\rangle$.

The following equality is obvious:

$$R^a \cap X = \bigcup_{k=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcap_{n=p}^{\infty} B(k, n), \tag{24}$$

where $B(k, n) = \left\{ x \in C^*(\sum_1^\infty a_n) \cap X; f_n(x) < a - \frac{1}{k} \right\}$. But $B(k, n)$ is closed in $X \cap C^*$ ($C^* = C^*(\sum_1^\infty a_n)$) and so as a consequence of the equality (24) $R^a \cap X$ is of the type F_σ in $C^* \cap X$. Hence

$$R^a \cap X = (C^* \cap X) \cap M, \tag{25}$$

where M is of the type F_σ in $(0, 1\rangle$. Since X is of the type G_δ in $(0, 1\rangle$, (25) and Theorem 1,8 give that $R^a \cap X$ is of the type $F_{\sigma\delta\sigma}$ in $(0, 1\rangle$.

2. Evidently $M^a = \{x \in C^*(\sum_1^\infty a_n);\ f(x) < a\}$, since $a \leqq 0$ and $f(x) = 0$ for $x \in D^*(\sum_1^\infty a_n)$. Further we have $M^a = (M^a \cap X) \cup (M^a \cap T)$ and similarly as in the case 1 we see that it suffices to prove that $M^a \cap X$ is of the type $F_{\sigma\delta\sigma}$ in $(0, 1\rangle$. This can be done by a similar procedure to the one used in the case 1.

In the same way it may be proved that each of the sets M_a is of the type $F_{\sigma\delta\sigma}$ in $(0, 1\rangle$. The proof is completed.

It follows from Theorem 2,6 that each of the functions $f = f(\sum_1^\infty a_n)$ is Lebesgue measurable and owing to its boundedness also integrable (in the Lebesgue sense) on $\langle 0, 1\rangle$. The question arises about the value of the Lebesgue integral $\int_0^1 f(t)\mathrm{d}t$ of this function. We can expect that its value will essentialy depend on the properties of the series $\sum_{n=1}^\infty a_n$. The following Theorem is an easy consequence of Theorem 1,10.

Theorem 2,7. *If* $\sum_{n=1}^\infty a_n$ *is of type* (α), *then* $\int_0^1 f(t)\mathrm{d}t = 1$. *If* $\sum_{n-1}^\infty a_n$ *is of type* (β), *then* $\int_0^1 f(t)\mathrm{d}t = -1$. *If* $\sum_{n=1}^\infty a_n$ *oscillates, then* $\int_0^1 f(t)\mathrm{d}t = 0$.

3.

ON THE FACTORIAL TRANSFORMATIONS OF INFINITE SERIES

In this part of the paper a metrical result, which is an analogy of certain metrical results of the paper [10] on subseries, will be proved.

Theorem 3,1. *Let* $\sum_{n-1}^\infty a_n = +\infty$, *let there exist* j *such that*

$$a_j \geqq a_{j+1} \geqq \dots \geqq a_{j+k} \geqq \dots$$

Then $\dim C_1(\sum_1^\infty a_n) = 0$.

Before we prove the Theorem, we shall prove an auxiliary result. If

$$x = \sum_{k=1}^{\infty} \frac{\varepsilon_k(x)}{(k+1)!} \in \langle 0, 1),$$

we put $h_n(x) = \sum_{i=1}^{n} \varepsilon_i(x)$.

Lemma 3,1. *Let* $\sum_{n=1}^{\infty} a_n$ *fulfil the assumption of Theorem* 3,1. *Let* $x \in C_1\,(\sum_1^{\infty} a_n)$. *Then*

$$\liminf_{n\to\infty} \frac{h_n(x)}{n} = 0.$$

Proof of the Lemma. Let

$$\liminf_{n\to\infty} \frac{h_n(x)}{n} > 0.$$

Then there exist numbers $r \geqq \max(2, j)$ and $\delta > 0$ such that $h_n(x) \geqq \delta n$ for $n \geqq r$. By means of Abel's partial summation we get $\sum_{k=r}^{r+p} \varepsilon_k(x) a_k = -h_{r-1}(x) a_r + h_r(x)(a_r - a_{r+1}) + \ldots + h_{r+p}(x) a_{r+p} \geqq -h_{r-1}(x) a_r + \delta(a_r + a_{r+1} + \ldots + a_{r+p})$. Further $a_r + a_{r+1} + \ldots + a_{r+p} \to +\infty$ if $p \to \infty$, so that $x \in D_1(\sum_1^{\infty} a_n)$ and thus $x \notin C_1(\sum_1^{\infty} a_n)$.

Proof of the Theorem. Let the assumptions of the Theorem be fulfilled. Then for an arbitrary

$$x = \sum_{k=1}^{\infty} \frac{\varepsilon_k(x)}{(k+1)!} \in \langle 0, 1)$$

(the factorial expansion of the number x)

$$h_n(x) = N_n(1, x) + 2N_n(2, x) + \ldots + kN_n(k, x) + \ldots \tag{26}$$

holds, where $N_n(k, x)$ denotes the number of k' s in the sequence $\varepsilon_1(x), \varepsilon_2(x), \ldots, \ldots, \varepsilon_n(x)$ (in (26) on the right hand $N_n(k, x) = 0$ for $k > n$). By means of simple estimation we get from (26)

$$h_n(x) \geqq \sum_{k=1}^{\infty} N_n(k, x). \tag{27}$$

If $x \in C_1(\sum_1^{\infty} a_n)$, then as a consequence of (27) and Lemma 3,1 we have

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^{\infty} N(k, x)}{n} = 0.$$

Observe that

$$\frac{N_n(0, x)}{n} = 1 - \frac{\sum_{k=1}^{\infty} N(k, x)}{n}.$$

Thus for $x \in C_1(\sum_1^{\infty} a_n)$ we have $\limsup_{n\to\infty} \frac{N_n(0,x)}{n} = 1$, hence the inclusion

$$C_1(\sum_1^{\infty} a_n) \subset \left\{ x \in \langle 0,1); \limsup_{n\to\infty} \frac{N_n(0, x)}{n} = 1 \right\} = W_1 \tag{28}$$

holds. On account of a certain result of the paper [23] (see [23] Theorem 5) dim $W_1 = 0$ holds and then (28) implies dim $C_1(\sum_1^{\infty} a_n) = 0$. The proof is completed.

4.

APPLICATIONS OF SOME RESULTS ON SUBSERIES TO THE THEORY OF ATOMIC MEASURES

Let $\mathscr{S}$ be some σ-field of subsets of a given set, let μ be a measure on $\mathscr{S}$. Let A, B be two atoms of $\mathscr{S}$. Then $A \cap B \subset A$ and the following cases may occur:

$$(\text{i}_1)\ \mu(A \cap B) = 0 \qquad (\text{i}_2)\ \mu(A \cap B) = \mu(A).$$

In the case (i_1) the sets A, B are taken as two different atoms. In the case (i_2) $\mu(A \cap B) > 0$ and so in view of the inclusion $A \cap B \subset B$ we have $\mu(A \cap \cap B) > 0$ and so in view of the inclusion $A \cap B \subset B$ we have $\mu(A \cap B) = = \mu(B)$. Then evidently $\mu(A) = \mu(B)$. In this case the sets A, B are not taken as two different atoms. Thus we identify the atoms which differ only in a set of measure zero.

Now, let $E \in \mathscr{S}$ be a purely atomic set, let $\mu(E) = +\infty$. Let

$$E_1, E_2, \ldots E_n, \ldots \tag{29}$$

be all the (mutually different) atoms contained in E, let $\mu(E_n) < +\infty$ ($n = = 1, 2, 3, \ldots$). It is proved in the paper [3] by means of Theorem A that E has the Darboux property if and only if there exists a sequence of natural numbers

$$p_1 < p_2 < \ldots < p_n < \ldots \tag{30}$$

such that

$$(31) \qquad \sum_{n=1}^{\infty} \mu(E_{p_n}) = +\infty, \quad \mu(E_{p_n}) \rightarrow 0.$$

In connection with this fact, an application of our Theorem 1,1 leads to the following result.

Theorem 4,1. *Let $E \in \mathscr{S}$ be a purely atomic set with the Darboux property, let $\mu(E) = +\infty$. Let (29) be all (mutually different) atoms contained in E, let $\mu(E_n) < +\infty$ ($n = 1, 2, \ldots$). Then to each $a > 0$ there exists an infinite system (of the power of the continuum) of sets $D \subset E$, mutually differing in a set of a positive measure such that $\mu(D) = a$.*

Proof. Since E has the Darboux property, there exists (30) such that (31) holds. Put $a_n = \mu(E_{p_n})$ ($n = 1, 2, 3, \ldots$). Then the series $\sum_{n=1}^{\infty} a_n$ fulfils the assumptions of Theorem 1,1. As a consequence of that theorem there exists such a set V (of the power of the continuum) of numbers $x = \sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k} \in$ $\in (0, 1\rangle$ for which $(x) = \sum_{k=1}^{\infty} \varepsilon_k(x) a_k = a$. We put $D_x = \bigcup_{k; \varepsilon_k(x)=1} E_{p_k}$ if $x \in V$. Evidently $\mu(D_x) = a$.

Now, let $x', x'' \in V$, $x' \neq x''$ and let e. g. $\varepsilon_j(x') = 1$, $\varepsilon_j(x'') = 0$. Then $D_{x'} -$ $- D_{x''} \supset E_{p_j} - \bigcup_{i \neq j} E_{p_i}$,

$$\mu(D_{x'} - D_{x''}) \geqq \mu(E_{p_j} - \bigcup_{i \neq j} E_{p_i} \cap E_{p_j}) = \mu(E_{p_j}) > 0,$$

since $\mu(E_{p_i} \cap E_{p_j}) = 0$ for $i \neq j$. The proof is finished.

Let E fulfil the assumptions of Theorem 4,1. Denote by $\mathscr{U}_0$ the system of all such $H \in \mathscr{S}$, $H \subset E$, which contain an infinite number of atoms (in the sense of the agreement made at the beginning of this part of the paper a set $H \in \mathscr{S}$ contains an atom A exactly when $\mu(A \cap H) > 0$). Assign to each $H \in \mathscr{U}_0$ the number $\delta(H) = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k} \in (0, 1\rangle$, where $\varepsilon_k = 1$, if H contains the atom E_k, and $\varepsilon_k = 0$ in the other case. It can be easily seen that if $H', H'' \in \mathscr{U}_0$, then $\delta(H') \neq \delta(H'')$ if and only if $\mu(H' \dot{-} H'') > 0$.

The construction of the numbers $\delta(H)$ is similar to the construction of the numbers $\varrho(M)$, which are assigned to the sets of natural numbers in the additive number theory (see [24] p. 17).

We put $\delta\{\mathscr{U}\} = \{\delta(H); H \in \mathscr{U}\}$, if $\mathscr{U} \subset \mathscr{U}_0$. The „magnitude" of the system $\mathscr{U}$ may be measured by the „magnitude" of the set $\delta\{\mathscr{U}\}$.

If $H \in \mathscr{U}_0$ and $\delta(H) = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}$, then we put $p(n, H) = \sum_{k-1}^{n} \varepsilon_k$.

The applications of Theorems 1,4 and 1,7 give the following result completing Theorem 5 of the paper [3].

Theorem 4,2. *Let E be a purely atomic set, let $\mu(E) = +\infty$ and let* (29) *be all (mutually different) atoms contained in E, $\mu(E_n) < +\infty$ $(n = 1, 2, 3, \ldots)$.*

a) *Let $\mathcal{U}_1$ denote the system of all $H \in \mathcal{U}_0$, for which $\mu(H) < +\infty$. Then $\delta\{\mathcal{U}_1\}$ is of the first category in* $(0, 1\rangle$.

b) *Let $\mathcal{U}_2$ denote the system of all such $H \in \mathcal{U}_0$, for which $\mu(H) = +\infty$ and simultaneously* $\left\{\frac{p(n, H)}{n}\right\}'_n = \langle 0, 1\rangle$. Then $\delta\{\mathcal{U}_2\}$ is residual in $(0, 1\rangle$.

Note 4,1. As we have already noted assigning a set of numbers $\delta\{\mathcal{U}\}$ to the system $\mathcal{U} \subset \mathcal{U}_0$ makes it possible to measure the value of the „magnitude" of the system $\mathcal{U}$ by means of $\delta\{\mathcal{U}\}$. Thus the system $\mathcal{U}$ may be said to be of the measure 0 or to be of the first category if the corresponding set $\delta\{\mathcal{U}\}$ is of the measure 0 or of the first category (see [3]). In this sense it is possible to formulate the result a) (it would be possible to make it also for b)) of Theorem 4,2 in the following way: For all $H \in \mathcal{U}_0$ with the exception of sets forming a system of the first category we have $\mu(H) = +\infty$.

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