Roy O. Davies

On the Measurability of Functions of Two Variables

_Matematický časopis_, Vol. 23 (1973), No. 3, 285--289

Persistent URL: http://dml.cz/dmlcz/126884

Terms of use:

© Mathematical Institute of the Slovak Academy of Sciences, 1973

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to
digitized documents strictly for personal use. Each copy of any part of this document must contain
these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped
with digital signature within the project DML-CZ: The Czech Digital Mathematics
Library http://project.dml.cz
ON THE MEASURABILITY OF FUNCTIONS OF TWO VARIABLES

Roy O. DAVIES, Leicester and J. DRAVECKÝ Bratislava

Introduction

It is well known that when a real-valued function \( f \) of two real variables \( x, y \) is Lebesgue measurable in each variable separately it need not be measurable in \( (x, y) \), and that when \( f \) is continuous in each variable separately it need not be continuous in \( (x, y) \). However in the latter case \( f \) must be measurable: indeed Ursell proved [9] that if \( f \) is continuous in \( x \) for each \( y \) and measurable in \( y \) for each \( x \), then it must be measurable in \( (x, y) \). (Marczewski and Ryll-Nardzewski [5] and Neubrunn [7] gave generalizations with \( x \) running over a separable metric space.) This was extended by Michael and Ronnie [6] to the following: if \( f \) is measurable in \( y \) for almost all \( x \), is equal to zero outside a certain measurable set \( E \), and on \( E \) is continuous in \( x \) with respect to \( E \) for almost all \( y \), then \( f \) must be plane measurable. One of us recently showed [2] that this theorem, with a similar proof, applies in products of more general topological measure spaces. Here we go further, replacing \( \mathbb{R}^2 \) (\( \mathbb{R} \) — the real line) by a product \( X \times Y \) of general \( \sigma \)-finite measure spaces of which only \( X \) is (second-countable) topological. The method of proof is necessarily different from that in [6], which made use of the topology of \( \mathbb{R}^2 \); in fact it turns out to be somewhat simpler. After stating and proving our theorem we show that the second-countability of \( X \) cannot be dropped from the hypotheses.

Main theorem

Theorem 1. Let \( (X, \mu) \) be a \( \sigma \)-finite second-countable topological measure space(1) and let \( (Y, v) \) be any \( \sigma \)-finite measure space. If \( f: X \times Y \to \mathbb{R} \) is \( v \)-measurable in \( y \) for \( \mu \)-almost all \( x \), is \( \mu \times v \)-measurable on the complement of a certain \( \mu \times v \)-

(1) That is, the \( \sigma \)-algebra of subsets of \( X \) on which \( \mu \) is defined includes the Borel sets.
-measurable set $E$, and on $E$ is continuous in $x$ with respect to $E$ for $\tilde{v}$-almost all $y$, then $f$ must be $\mu \times \nu$-measurable.

Proof. Without loss of generality we may suppose that $\mu(X) < \infty$ and $\nu(Y) < \infty$. Since the completion of $\mu \times \nu$ is the same as that of $\tilde{\mu} \times \tilde{\nu}$, by $\mu$ and $\nu$ we may denote the already completed measures $\tilde{\mu}$ and $\tilde{\nu}$, respectively. On the other hand, sets of measure zero do not affect the conclusion of the theorem, and hence we may assume that the section $E^y$ is $\mu$-measurable and the section $\hat{f}^y : E^y \to R$ continuous for all $y$, and that $E \times x$ is $\nu$-measurable and $f_x : E_x \to R$ $\nu$-measurable for all $x$. Further we may suppose that $0 \leq f(x, y) \leq 1$ on $E$, since every real-valued function can be written (preserving continuity and measurability) as a difference of two non-negative ones and each non-negative function $g$ is equal to $\lim_n n \cdot g_n$, where for $g_n$ defined by $g_n(x, y) = \frac{1}{n} \inf \{ n, g(x, y) \}$ we have in fact $0 \leq g_n \leq 1$. We must show that $f$ is $\mu \times \nu$-measurable on $E$.

Let $G_1, G_2, \ldots$ be a countable basis for the non-empty open sets in $X$. Given any $n$, define points $x_{n1}, x_{n2}, \ldots \in G_n$ by induction as follows: let

$$k_{ns} = \sup\{v(E_x \setminus \bigcup_{\tau < s} E_{x_{\tau s}}); x \in G_n\},$$

and select $x_{ns} \in G_n$ with

$$v(E_{x_{ns}} \setminus \bigcup_{\tau < s} E_{x_{\tau s}}) \geq \frac{1}{2} k_{ns}.$$ 

Denote by $F_n$ the set $\bigcup_{s=1}^\infty E_{x_{ns}}$, and by $H_n$ the set $G_n \times F_n$.

**Assertion I.** $$(\mu \times \nu) [E \cap (G_n \times Y) \setminus H_n] = 0.$$

**Proof of Assertion I.** Observe first that $G_n \times Y$ and $H_n$ are $\mu \times \nu$-measurable, and therefore so is the set $K_n = E \cap (G_n \times Y) \setminus H_n$. Hence in view of Fubini's theorem it will be sufficient to show that $v[(K_n)_{x}] = 0$ for all $x \in G_n$. Now for $x \in G_n$ we have $(K_n)_x = E_x \setminus \bigcup_{s=1}^\infty E_{x_{ns}}$. Consequently, if $v[(K_n)_x] = d > 0$, then $k_{ns} \geq d$ for all $s = 1, 2, \ldots$, and

$$v(Y) \geq v(\bigcup_{s=1}^\infty E_{x_{ns}}) = \sum_{s=1}^\infty v(E_{x_{ns}} \setminus \bigcup_{\tau < s} E_{x_{\tau s}}) = \infty,$$

a contradiction. Our assertion is proved.

From Assertion I it follows that the set

$$Z = \bigcup_{n=1}^\infty [E \cap (G_n \times Y) \setminus H_n]$$

286
has $\mu \times v$-measure zero, and it will be sufficient to prove that $f|(E \setminus Z)$ is $\mu \times v$-measurable. Let

$$D = \{x_{ns}; n = 1, 2, \ldots, s = 1, 2, \ldots\}.$$  

For each $n$ define a function $f_n: E \setminus Z \to R$ as follows:

if $(x, y) \in (E \setminus Z) \setminus (G_n \times Y)$ then $f_n(x, y) = 1$;

if $(x, y) \in (E \setminus Z) \cap (G_n \times Y)$ then $f_n(x, y) = \sup \{f(w, y); w \in D \cap G_n \text{ and } (w, y) \in E\}.$

Observe that if $(x, y) \in (E \setminus Z) \setminus (G_n \times Y)$ then $(x, y) \in G_n \times F_n$, so $x \in G_n$ and $y \in E_{x_n}$ for some $s$; hence $y \in E_w$ for some $w \in D \cap G_n$, that is, $w \in D \cap G_n$ and $(w, y) \in E$ for some $w$, so the supremum is over a non-empty set. Since $f_n$ is obviously $\mu \times v$-measurable, it will be enough to prove the following.

**Assertion II.** On $E \setminus Z$ we have $f = \inf_n f_n$.

Proof of Assertion II. (a) To show that $f(x, y) \leq \inf_n f_n(x, y)$ on $E \setminus Z$, we must show that $f(x, y) \leq f_n(x, y)$ for all $n$. This is obvious if $(x, y) \in (E \setminus Z) \setminus (G_n \times Y)$, because then $f(x, y) \leq 1 = f_n(x, y)$. Hence we may suppose that $(x, y) \in (E \setminus Z) \cap (G_n \times Y)$; in particular $x \in G_n$. It will be enough to show that $f(x, y) - \varepsilon \leq f_n(x, y)$ for every $\varepsilon > 0$.

In view of the continuity of $f$, there is an open set $G$ containing $x$ such that $f(z, y) \geq f(x, y) - \varepsilon$ for all $z \in G \cap E_y$. For some $m$ we have $x \in G_m \subset G \cap G_n$. Then $(x, y) \in (E \setminus Z) \cap (G_m \times Y)$, and as observed earlier there exists $w \in D \cap G_m$ with $(w, y) \in E$. Then $f(w, y) \geq f(x, y) - \varepsilon$ and, since $w \in D \cap G_n$, $f_n(x, y) \geq f(w, y) \geq f(x, y) - \varepsilon$ as required.

(b) Finally we show that $f(x, y) \geq \inf_n f_n(x, y)$ on $E \setminus Z$; that is, given $\varepsilon > 0$ we have $f(x, y) + \varepsilon \geq f_m(x, y)$ for some $m$. As above, there is an open set $G$ containing $x$ such that $f(z, y) \leq f(x, y) + \varepsilon$ for all $z \in G \cap E_y$. For some $m$ we have $x \in G_m \subset G$. Then $(x, y) \in (E \setminus Z) \cap (G_m \times Y)$, and for every $w \in D \cap G_m$ with $(w, y) \in E$ we certainly have $w \in G \cap E_y$ and therefore $f(w, y) \leq f(x, y) + \varepsilon$. Hence $f_m(x, y) \leq f(x, y) + \varepsilon$, as required.

A counter-example

Our proof that the second-countability hypothesis is essential in Theorem 1 will be based on two key notions: Sierpiński's paradoxical decomposition of $R^2$ [8] and the density topology on $R$ (see [3]).

**Theorem 2.** There exists a $\sigma$-finite topological measure space $(X, \mu)$, a $\sigma$-finite measure space $(Y, v)$, and a function $f: X \times Y \to R$ such that $f_x$ is $v$-measurable for all $x$ and $f^y$ is continuous for all $y$, but $f$ is not $\mu \times v$-measurable.
Proof. Let \( S_\lambda \) be the least possible cardinality for a subset of \((0, 1)\) having positive outer Lebesgue measure, and choose a set \( S \subset (0, 1) \) of cardinality \( S_\lambda \) with \( m^*(S) > 0 \). Let \((S, \mu)\) be the measure space in which the \(\sigma\)-algebra consists of the intersections with \( S \) of the Lebesgue measurable subsets of \((0, 1)\), and in which \( \mu \) is outer Lebesgue measure on this \(\sigma\)-algebra. We consider \((S \times S, \mu \times \mu)\), the first factor being endowed with the topology induced on \( S \) by the density topology on \( \mathbb{R} \).

Let \( \prec \) be a well ordering of \( S \) of type \( \omega_\lambda \). Define \( M = \{(x, y); x \prec y\} \) and observe that \((S \times S \setminus M)\) has measure zero for all \( x \in S \) and \( M^y \) has measure zero for all \( y \in S \). In particular \( M^y \) is a closed set with respect to the density topology on \( \mathbb{R} \). We can choose a set \( K = K(y) \) in \( \mathbb{R} \setminus M^y \) which is closed in the ordinary topology, such that \( K \cap S \) has positive \( \mu \)-measure. By the Remark after Theorem 3 of [3], there is a function \( f^y \) from \((0, 1)\) to \( \langle 0, 1 \rangle \) which is continuous with respect to the density topology, such that \( f^y(x) = 1 \) on \( M^y \) and \( f^y(x) = 0 \) on \( K(y) \).

Let \( f: S \times S \to \langle 0, 1 \rangle \) be defined by \( f(x, y) = f^y(x) \) for \((x, y) \in S \times S \). For each fixed \( x \), \( f_x \) differs from the characteristic function of \( M_x \) on a set of measure zero only, and so

\[
\int_S f(x, y)d\mu(y) = \mu(M_x) = \mu(S),
\]

while

\[
\int_S f(x, y)d\mu(x) \leq \mu(S) - \mu[K(y) \cap S] < \mu(S),
\]

an application of Fubini’s theorem yields the desired non-measurability of \( f \).

Remarks

In view of our results, it is natural to ask whether if \((X, \mu)\) is an arbitrary \(\sigma\)-finite topological measure space and \( f: X \times X \to \mathbb{R} \) is continuous in \( y \) for all \( x \) and continuous in \( x \) for all \( y \), the function \( f \) is necessarily \( \mu \times \nu \)-measurable. One of us has shown [1] that the answer is negative, assuming the existence of a non-measurable cardinal, but that the answer is positive in the special case when \( X \) is \( \mathbb{R} \) with the density topology and \( \mu \) is Lebesgue measure. The latter result resolves a problem of Mišik recently quoted by Lipiński [4].

REFERENCES


288

Received June 20, 1972

Department of Mathematics
The University
Leicester
England

Katedra matematickej analýzy
Prírodovedeckej fakulty UK
Bratislava
Czechoslovakia