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*Matematický časopis*, Vol. 23 (1973), No. 3, 267--269

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A NOTE ON THE COMPLETENESS OF $L_q$

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There is a connection between the completeness of $L_q$ and the completeness of the metric space of all sets of finite measure (see [1]). It has been shown in [2] that the completeness of the measure space can be formulated and proved by means of some properties of the families of sets of "small measure". We use a similar method in the present paper to prove a generalization of an $L_q$-completeness theorem.

First we introduce a sequence $\{G_n\}^{\infty}_{n=0}$ of sets of extended real valued measurable functions defined on a set $S$ and satisfying some axioms. An example of such a sequence is the following. Let $(S, \sum, \mu)$ be a finite measurable algebra, $G_0 = \{f\text{-measurable, } \int_S |f|^q d\mu < \infty, G_n = \{f, f \in G_0, \int_S |f|^q d\mu < 2^{-n}\}$.

The operations $f + g$, $xf$ etc. are defined as usually, only we put $\infty + (-\infty) = (-\infty) + (\infty) = 0$, $0 \cdot \infty = 0$. Hence we list the axioms:

I. If $f \in G_n$, then $|f| \in G_n$, $n = 0, 1, 2, \ldots$

II. If $f \in G_m$, $g$ is a measurable function such that $|g| \leq f$ on $S$, then also $g \in G_n$.

III. If $f, g \in G_n$, then $f + g \in G_n$, $f \cdot g \in G_0$ for $f, g \in G_0$.

IV. If $f_n \in G_0$, $n = 1, 2, 3, \ldots$, $f_n \nrightarrow f$, $f_{n+1} \not\subset f_n \in G_n$, then also $f \in G_0$ ($f_n \not\subset f$ if $f_n(x) \leq f_{n+1}(x)$ and $\lim_{n \to \infty} f_n(x) = f(x)$ for every $x \in S$).

V. If $\{\lambda_n\}^{\infty}_{n=1}$ is a sequence of real valued constant functions and $\lim_{n \to \infty} \lambda_n = 0$, then to any $n$ there is $m$ such that the constant function $f(x) = \lambda_m$, $x \in S$, belongs to $G_n$.

VI. For every real nonzero constant $\lambda$ and positive integer $n$ there exists an index $m$ such that $f \in G_m$, implies $\lambda f \in G_n ((\lambda f)(x) = \lambda f(x)$ for every $x \in S$).

VII. If $f_n \to f$ (i.e. for every $x \in S$ $\lim_{n \to \infty} f_n(x) = f(x)$), $f_n \in G_{k+1}$ for $n = 0, 1, 2, \ldots$, then $f \in G_k$. 

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VIII. If \( f \in G_0, \; M = \{ x : |f(x)| < \infty \} \) and \( g \) measurable, \( g \cdot \chi_M \in G_i, \) then \( g \in G_i. \)

**Theorem.** Let \( q \geq 1, \; A = \{ f \in G_0, \; |f|^q \in G_0 \}, \; U_n = \{ (f, g) : |f - g|^q \in G_n \} \) \((n = 0, 1, 2, \ldots)\) and \( \mathcal{B} = \{ U_{n+1} \}. \) Then \((A, \mathcal{B})\) is a complete uniform pseudometricizable space. Furthermore, there is a translation invariant pseudometric \( d \) on \( A \) such that \( d \) and \( \mathcal{B} \) generate the same uniformity on \( A, \) and \( \lambda \in \mathcal{B}, \{f_n\}_{n=1}^\infty \) in \( A, \) \( d(f_n, 0) \to 0 \) imply \( d(\lambda f_n, 0) \to 0. \)

**Proof.** Let \( q > 1. \)

We prove the completeness of \((A, \mathcal{B}).\) The base \( \mathcal{B} \) of \( A \) is countable. Hence \( A \) is complete if every Cauchy sequence is convergent (see [3]). Let \( f_n \to f \) denote the convergence in \((A, \mathcal{B}).\) It means: \( f \in A \) and to every \( k \) there exists \( N_0 \) such that \( (f_n, f) \in U_k \) for \( n \geq N_0. \) A sequence \( \{f_n\}_{n=1}^\infty \) is Cauchy in \((A, \mathcal{B})\) if for each \( k \) there exists \( N \) such that \( (f_n, f_m) \in U_k \) for \( n, m \geq N. \)

Let \( \{f_n\}_{n=1}^\infty \) be a Cauchy sequence in \((A, \mathcal{B})\) and let \( i \geq 1 \) be given. By V there is \( \lambda > 0 \) such that

\[ \frac{1}{\lambda^{p-1}} \in G_{i+1}, \text{ where } p = \frac{q}{q-1}. \]

By VI there is \( m_i \) such that

\[ (\lambda q)^{-1} G_{m_i} \subset G_{i+1}. \]

Since \( \{f_n\}_{n=1}^\infty \) is Cauchy, there exists \( k'_i \) such that

\[ (f_n, f_m) \in U_{m_i} \text{ for all } n, m \geq k'_i. \]

From (2) and (3) it follows that

\[ (\lambda q)^{-1} |f_n - f_m|^q \in G_{i+1} \text{ for all } n, m \geq k'_i. \]

The inequality

\[ a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q} \quad (a, b \geq 0) \]

implies \( a = \lambda, \; b = |f_n(x) - f_m(x)|, \; x \in S): \)

\[ |f_n - f_m| \leq (\lambda q)^{-1} |f_n - f_m|^q + \frac{1}{p} \lambda^{p-1} \quad (n, m \geq k'_i). \]

But (1), (4), (5), III and II imply

\[ f_n - f_m \in G_i \text{ for all } n, m \geq k'_i. \]

Let \( \{k_i\}_{i=1}^\infty \) be a strictly increasing sequence of integers such that \( k_i \geq k'_i \) (for example, \( k_i = \max \{k'_i, \ldots, k'_i\} + 1 \)). Then (6) implies \( f_{k_{i+1}} - f_{k_i} \in G_i \) \((i \geq 1), \) since \( k_{i+1} > k_i \geq k'_i. \)

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Put now \( h_0 = |f_k|, \ h_i = |f_{k+i} - f_k|, \ i = 1, 2, \ldots \). Then \( \sum_{i=0}^n h_i \leq \sum_{i=0}^\infty h_i \),

\[
h_n = \sum_{i=0}^n h_i - \sum_{i=0}^{n-1} h_i \in G_n, \text{ hence } \sum_{i=0}^\infty h_i \in G_0 \text{ according to IV.}
\]

Finally define

\[
f(x) = f_k(x) + \sum_{i=1}^\infty (f_{k+i}(x) - f_k(x)),
\]

if \( \sum_{i=0}^\infty h_i(x) \) converges and

\[
f(x) = 0
\]

in the opposite case. Then \( f \) is a measurable function, for which \( |f| \leq \sum_{i=0}^\infty h_i \in G_0 \), hence \( f \in G_0 \) according to II. Put \( M = \{x:\sum_{i=0}^\infty h_i(x) < \infty\} \). Evidently \( f_k \cdot \chi_M \rightarrow \rightarrow f \cdot \chi_M \). According to VII and to VIII \( f_k \Rightarrow f \). Now it is not difficult to prove that \( |f|^q \in G_0 \) and also \( f_n \Rightarrow f \).

The base \( \mathcal{B} \) gives on \( A \) a base of neighbourhoods of 0, which form a topology on \( A \): the discrete product of these neighbourhoods forms a topology on \( A \times \times A = \{(x, y) : x \in A, y \in A\} \). Since the function \( f(x, y) = x + y \) from \( A \times A \) into \( A \) is a continuous function (III) and the function \( g(x, x) = xx \) from \( \mathcal{B} \times A = \{(x, x), \ x-\text{real number}, x \in A\} \) into \( A \) is a continuous function too, \( A \) is a linear topological space. One can easily define on \( A \) a translation invariant pseudometric \( d \), such that \( d \) generates the same uniformity on \( A \) as \( \mathcal{B} \), and the following holds true: for every sequence \( \{f_n\}_{n=0}^\infty \) of elements of \( A \), if \( d(f_n, 0) \rightarrow 0 \), then \( d(\lambda f_n, 0) \rightarrow 0 \) for every real number \( \lambda \ ([4,5]) \).

In a case \( q = 1 \) the proof is simple.

Let us remark that the space \( A \) needs not be separated.

REFERENCES