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ON SOME PROPERTIES OF HAAR MEASURE

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In this paper we shall generalize two theorems concerning the Lebesgue measure for the case of the Haar measure in any locally compact topological group. The first of them is the Lebesgue density theorem. The second contains the statement that the Lebesgue measure in the n-space is equal to the n-dimensional Hausdorff measure. These assertions are based on the Vitali covering theorem proved for a general case in paper [2].

1. Definitions and notations. Throughout the article we shall suppose that X is a locally compact topological group. By m we denote a regular Haar measure defined on the system of all Borel subsets of X (we shall use the terminology according to Book [3]), while m^* denotes the outer measure induced by m.

For the convenience of readers we wish to summarise some results of paper [2] which will be used.

1,1. Definition. *A sequence $\{S_n\}$ of subsets of X is called the sequence of quasispheres, if for some $\delta > 0$ and all positive integers n we have*

$$S_{n+1}^{-1} S_{n+1} \subset S_n, \tag{1}$$

$$m^*(S_{n+1}) > \delta\, m^*(S_n). \tag{2}$$

1,2. Definition. *Let Y be a topological space, $A \subset Y$, K be a system of subsets of Y. We say that K is a Vitali covering of the set A, if to any $x \in A$ and any neighbourhood V of x there is $U \in K$ such that $x \in U \subset V$.*

1.3. Definition. *Let $A \subset X$, K be a system of closed Borel subsets of X of positive measure. K is a regular Vitali covering of the set A if there exist a sequence of quasispheres $\{S_n\}$ and a real function M on A such that to any $x \in A$ and any neighbourhood V of x there are $U \in K$, $y \in X$ and a positive integer n such that $x \in U \subset V$, $U \subset S_n$, $m^*(S_n) \leqq M(x)\ m(U)$.*

1,4. Theorem. *Let K be a regular Vitali covering of a set A. Let $A \subset \bigcup_{k=1}^{\infty} V_k$, where V_k are open sets, $m(V_k) < \infty$. Then there exists a sequence $\{U_k\}$ of pairwise disjoint sets from K for which $m^*\left(A - \bigcup_{k=1}^{\infty} U_k\right) = 0$.*

Proof. Theorem 3.5, [2], 87.

2. The Lebesgue density theorem. The main result of this paragraph is Theorem 2,5. Its proof is based on an assertion from paper [4]. First we shall verify the assumptions of this assertion.

2,1. Definition. *Let K be a system of closed subsets of X. By $\mathscr{D}(K)$ we denote the system of all outer measures h with the fololwing property: If $L \subset K$, $M \subset X$, L is a Vitali covering of the set M, then there is a sequence $\{L_i\}$ of pairwise disjoint sets from L such that $h(M - \bigcup_{i=1}^{\infty} L_i) = 0$.*

2,2. Lemma. *If K is a regular Vitali covering of the set $N = \bigcup \{E : E \in K\}$ and the set N is σ-bounded, then $m^* \in \mathscr{D}(K)$.*

Proof. *Let $M \subset X$, $L \subset K$, L* be a Vitali covering of M. Clearly L is a regular Vitali covering of the set M. Since $M \subset N$, there are compact sets C_n, for which $M \subset \bigcup_{n=1}^{\infty} C_n$. Since X is locally compact, there is for any n an open set V_n with compact closure $\overline{V}_n$ such that $C_n \subset V_n$. Hence $M \subset \bigcup_{n=1}^{\infty} V_n$, $m^*(V_n) \leqq \leqq m(\overline{V}_n) < \infty$. From Theorem 1,4 there follows the existence of the countable system $\{L_n\}$ of pairwise disjoint sets from L such that $m^* (M - \bigcup_{n=1}^{\infty} L_n) = 0$.

2,3. Definition. *Let Y be a topological space, h be an outer measure defined on the system H of subsets of X, finite and positive on the system $K \subset H$, M be an arbitrary set from H. Then by $\overline{D}_M(x)$ we shall denote the least upper bound of the set*

$$\left\{\lim_{t \in T} \frac{h(M \cap E_t)}{h(E_t)} : E_t \in K,\ \{E_t\} \text{ converges to } x\right\},$$

by $\underline{D}_M(x)$ the greatest lower bound of this set. A system $\{E_t\}_{t \in T}$ with T directed converges to a point $x \in X$, if to any neighbourhood U of x there is such $t_0 \in T$ that for all $t \geqq t_0$ we have $x \in E_t \subset U$. If $\overline{D}_M(x) = \underline{D}_M(x)$, we write $D_M(x) = \overline{D}_M(x)$.

2,4. Lemma. *Let Y be a σ-compact Hausdorff topological space, K be a system of compact subsets of Y covering Y in the Vitali sense, h be a regular Borel measure on the σ-algebra of all Borel subsets of Y, positive on K, m^* be the outer measure induced by m and let $m^* \in \mathscr{D}(K)$.*

Then for every Borel set M of finite measure we have $D_M = \chi_M$ m^*-almost everywhere in Y.

Proof: [4], Corollary 3.

2,5. Theorem.(*) *Let X be a locally compact topological group, m be a regular Haar measure on the system of all Borel sets, m^* be the outer measure induced by m, K be a regular Vitali covering of the space X by compact sets of positive measure.*

Then for any Borel set M of finite measure we have $D_M = \chi_M$ m^*-almost everywhere in X.

Proof. Acording to [3], Theorem 3, § 57, there exists a σ-compact open subgroup Z for which $M \subset Z$. Lemmas 2,2 and 2,4 imply that $D_M = \chi_M$ m^*-almost everywhere on the set Z. Since Z is also closed, we have $D_M = 0$ on $X - Z$.

3. Hausdorff measure. The main result of this paragraph is contained in Theorem 3,3.

3,1. Definition. *Let K be a system of subsets of a topological space Y, h be a set function on the system K, let P be an open covering of a set $E \subset Y$, i. e. the elements of the system are open sets and $\bigcup \{F : F \in P\} \supset E$. Then we put*

$$H\,[K, h, P]\,(E) = \inf \sum_{i=1}^{\infty} h(E_i),$$

where the infimum is taken for all sequences $\{E_i\}$ of sets from K such that $\bigcup_{i=1}^{\infty} E_i \supset$ $\supset E$ and to any E_i there is $P_i \in P$ such that $E_i \subset P_i$. Further we put

$$H[K, h]\,(E) = \sup H\,[K, h, P]\,(E),$$

where the supremum is taken for all open coverings of the set E.

3,2. Note. This definition was given by W. W. Bledsoe and A. P. Morse ([1]). In [1] it is proved apart from other facts that in any metric space $H\,[K, h]$ is identical with the usual Hausdorff construction. To be more exact:

$$H[K, h]\,(E) = \sup \{H_2\,[K, h, r]\,(E) : r > 0\},$$

where

$$H_2\,[K, h, r]\,(E) = \inf \{ \sum_{i=1}^{\infty} h(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i,\ E_i \in K,\ \text{diam}\, E_i < r\}.$$

On the base of results from [1] it may be also proved that in any topological group we have

$$H\,[K, h]\,(E) = \sup \{H_3\,[K, h, U]\,(E) : U \text{ neighbourhood of } 0\},$$

(*) The original formulation of this Theorem contained the assumption of the σ-compactness of the group X. I. Kluvánek called our attention to the fact that this assumption can be omitted.

where

$$H_3\,[K, h, U]\,(E) = \inf \sum_{i=1}^{\infty} h(E_i),$$

and the infimum is taken for all sequences $\{E_i\}$ of sets from K such that $E \subset \bigcup_{i=1}^{\infty} E_i$, $x_i E_i \subset U$ for some $x_i \in X$.

3,3. Theorem. *Let m be a regular Haar measure on any locally compact topological group X, m^* be the outer measure induced by m. Let K be a regular Vitali covering of a σ-compact Borel open set $U \subset X$. Suppose that to any set $F \subset U$ of a measure zero, any number $\delta > 0$ and any open covering P of the set F, there is a sequence $\{F_i\}$ of sets from K such that*

$$\sum_{i=1}^{\infty} m(F_i) < \delta,\ F \subset \bigcup_{i=1}^{\infty} F_i$$

and $F_i \subset P_i$ for some $P_i \in P$ $(i = 1, 2, \ldots)$.

Then

$$m^*(E) = H\,[K, m]\,(E)$$

for every set $E \subset U$.

Proof. Clearly $m^*(E) \leqq H\,[K, m]\,(E)$. Let E be a Borel set, $E \subset U$, $m(E) < \infty$. Let $\delta > 0$ be an arbitrary number. Since m is a regular measure, there is an open Borel set V such that $E \subset V \subset U$ and

$$m(E) + \delta > m(V). \tag{5}$$

Let P be any open covering of the set V. Denote by L the system of all sets from K which are subsets of V and subsets of sets from the covering P. The system L is a regular Vitali covering of the set V. Similarly as in Lemma 2,2 we prove the existence of open sets V_k for which $m^*(V_k) < \infty$ and $V \subset \bigcup_{k-1}^{\infty} V_k$. Hence according to Theorem 1,4 there exists a sequence $\{E_i\}$ of pairwise disjoint sets from L such that

$$m(V - \bigcup_{i=1}^{\infty} E_i) = 0. \tag{6}$$

By the assumption there exists to the set $F = V - \bigcup_{i=1}^{\infty} E_i$ a sequence $\{F_i\}$ of sets from L such that to any F_i there is $P_i \in P$ containing F_i and

$$\sum_{i=1}^{\infty} m(F_i) < \delta,\ F \subset \bigcup_{i=1}^{\infty} F_i. \tag{7}$$

Put $M = \{E_i\} \cup \{F_i\}$. This system covers the set V and for any $F \in M$ there is $R \in P$ such that $F \subset R$. Hence

$$(8) \qquad \sum_{i=1}^{\infty} m(E_i) + \sum_{i=1}^{\infty} m(F_i) \geqq \sum_{F \in M} m(F) \geqq H\,[K, m, P]\,(E).$$

From the relations (5) — (8) we get

$$m(E) + \delta > m(V) = m(V - \bigcup_{i=1}^{\infty} E_i) + m(\bigcup_{i=1}^{\infty} E_i) =$$

$$= \sum_{i=1}^{\infty} m(E_i) > \sum_{i=1}^{\infty} m(E_i) + \sum_{i=1}^{\infty} m(F_i) - \delta \geqq$$

$$\geqq H\,[K, m, P]\,(E) - \delta$$

for any $\delta > 0$. Hence

$$m(E) \geqq H\,[K, m, P]\,(E)$$

for any open covering P of the set V. Hence

$$m(E) \geqq H\,[K, m]\,(E),$$

whenever E is a Borel set of finite measure.

Let $E \subset U$ be an arbitrary set. The equality $m^*(E) = H\,[K, m]\,(E)$ evidently holds if $m^*(E) = \infty$. Let $m^*(E) < \infty$, $\delta > 0$ be an arbitrary number. Let us put a Borel set F such that $E \subset F$ and

$$m^*(E) + \delta > m(F).$$

Then, for $m(F) = H\,[K, m]\,(F)$, the following holds

$$m^*(E) + \delta > H\,[K, m]\,(F) \geqq H\,[K, m]\,(E)$$

for any $\delta > 0$, wherefrom the assertion of Theorem 3,3 easily follows.

3,4. Note. The classical result of the Lebesgue and Hausdorff measures follows immediately from Theorem 3,3. It suffices to put K equal to the system of all closed spheres in the n-space.

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