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ON THE NON-EXISTENCE OF CERTAIN NEARLY REGULAR PLANAR MAPS WITH TWO EXCEPTIONAL FACES

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If i is an integer greater than 1 we shall say that a face (vertex) of a map is *multi-i-gonal (multi-i-valent)* if the number of its edges is a multiple of i.

A nearly regular planar map of the type $\mathcal{G}(m, k, n)$; $m, k = 2, 3, 4, 5$; $n = 1, 2, \ldots$; is a planar map whose graph is connected, all the vertices are multi-m-valent and all except n of its faces are multi-k-gonal, while the exceptional faces are not multi-k-gonal.

B. Grünbaum and J. Malkevitch have investigated nearly regular planar maps with one and two adjacent exceptional faces (cf. [2, p. 272] or [3]).

In [1], D. W. Crowe investigates 3-valent nearly regular planar maps with $n = 2$ such that the distance between exceptional faces does not exceed 4.

In the present paper we are going to investigate nearly regular planar maps of the type $\mathcal{G}(3, k, 2)$ for $k = 2, 3, 4, 5$ and attempt to answer the question for which pairs of exceptional faces such a map does exist.

The answer is given by the following

Theorem. *A nearly regular planar map of the type $\mathcal{G}(3, k, 2)$ for $k = 2, 3, 4, 5$ with a u-gon and a v-gon as exceptional faces does not exist if and only if*

$k = 3$ *and* $u \equiv v \pmod 3$;

$k = 4$ *and* $u + v \equiv 1 \pmod 2$;

$k = 5, u + v \not\equiv 0 \pmod 5$ *and* $u \not\equiv v \pmod 5$.

Proof. The construction of maps of the types $\mathcal{G}(3, 2, 2)$ and $\mathcal{G}(3, 3, 2)$ for $u + v \equiv 0 \pmod 3$ is described, and the non-existence of maps of the type $\mathcal{G}(3, 3, 2)$ for $u \equiv v \pmod 3$ is proved in [4].

Maps of the type $\mathcal{G}(3, 4, 2)$ for $u + v \equiv 1 \pmod 2$ cannot exist since they violate the condition that the number of oddgonal faces must be even (cf. [2, p. 236]).

The existence of maps of the types $\mathcal{G}(3, 4, 2)$ for $u + v \equiv 0 \pmod 2$ and $\mathcal{G}(3, 5, 2)$ for $u + v \equiv 0 \pmod 5$ or $u \equiv v \pmod 5$ is guaranteed in [1].

It is thus only necessary to prove that no maps of the type $\mathcal{G}(3, 5, 2)$ exist for $u + v \not\equiv 0 \pmod 5$ and $u \not\equiv v \pmod 5$.

Let $u \equiv 1$ or $4 \pmod 5$ and $v = 2$ or $3 \pmod 5$.

Suppose that there is a map of the type $\mathscr{G}(3, 5, 2)$. By repeated reduction as shown in Fig. 1 this map can be transformed into a 3-valent map of the same type. This reduction carries a vertex $\times$ into $\times'$ whose valency is diminished by 3. The faces P, Q, R are carried into the faces P′, Q′, R′ whose number of edges is increased by 5. The number of the edges of all the other faces remains the same (cf. [3, p. 9]). In a 3-valent map obtained in this way we shall use another reduction (to be described later) to bring the *u-gon* and the *v-gon* closer together until we obtain a map which cannot exist on the strength of Grünbaum's statement that *there are no trivalent planar maps whose faces are all multi-5-gonal except the one or two adjacent faces* (cf. [2, p. 272]).

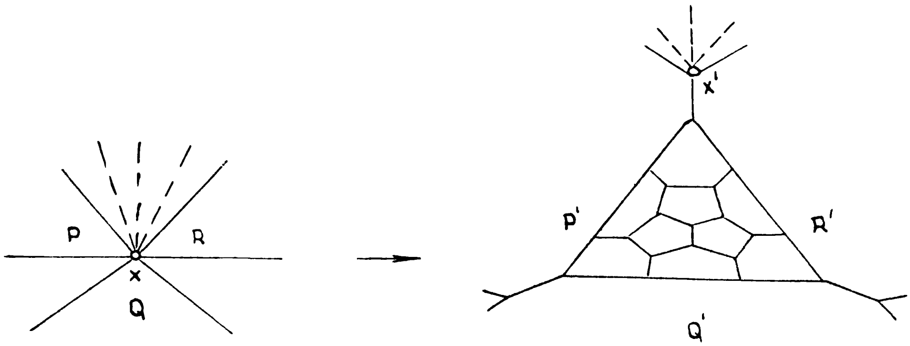

Fig. 1

Let $\mathscr{M}_1$ be a 3-valent planar map of the type $\mathscr{G}(3, 5, 2)$. Choose in $\mathscr{M}_1$ a minimum path connecting the *u-gon* and the *v-gon* (i. e. the shortest path in the graph of this map whose one end vertex is incident with the *u-gon* and the other with the *v-gon*). We shall bring the *u-gon* closer to the *v-gon* along this path. One step of this process depends only on the number of edges which lead to one side of the path (those "nearest" to the *u-gon*) and not on

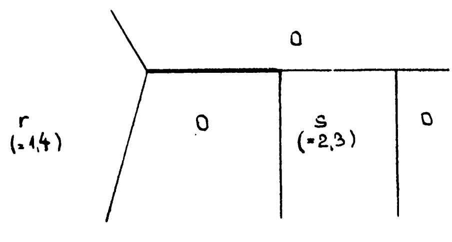

Fig. 2

the number of edges leading to the other side of the path further on, nor on the number of edges leading to any side of the path afterwards.

Suppose that the first $j + 5n$ edges ($j = 1, 2, 3, 4, 5$; n a non-negative integer) branching off the minimum path from the *u-gon* to the *v-gon* and the following k edges ($k \geq 0$) lead to the other side, cf. Fig. 2. In this and the following pictures the heavy line shows the part of the path along which the exceptional faces are brought together. The number i, $i = 0, r, s, t$, is used in the pictures to denote the face whose number of edges is $\equiv i \pmod 5$, where $r - 1, 4$; $s = 0, 2, 3$; $t = 0, 2, 3$. We remark that s and t cannot be both nonzero.

If we have first $1 + 5n$ edges branching off the minimum path, $n \geq 0$, and then k edges to the other side, $k \geq 1$ (i. e. $s = 0$ in Fig. 2), we proceed as follows:

If $u \equiv 4 \pmod 5$, (i. e. $r = 4$ in Fig. 2) we add new edges as in Fig. 3a. Instead of the map $\mathscr{M}_1$ containing *u-gon* we obtain a map $\mathscr{M}_2$ with u_1*-gon*, $u_1 - 1 \pmod 5$. If $u \equiv 1 \pmod 5$ (i. e. $r = 1$ in Fig. 2) a change of the map $\mathscr{M}_1$ into a map $\mathscr{M}_2$ with u_1*-gon*, $u_1 \equiv 4 \pmod 5$, is performed as in Fig. 3b.

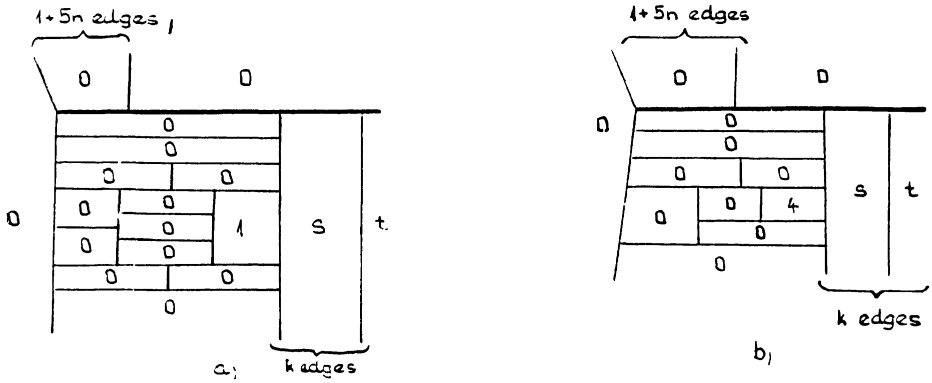

Fig. 3

In both cases we obtain a trivalent map $\mathscr{M}_2$ of the same type as the map $\mathscr{M}_1$ but the distance (the minimum path length) between the exceptional faces is now smaller (with the exception of the case $n = 0$; in that case the next step will shorten the minimum path). In the map $\mathscr{M}_2$ we again select the minimum path connecting the two exceptional faces and repeat the procedure we used on $\mathscr{M}_1$.

If there are $2 + 5n$, $3 + 5n$, $4 + 5n$ or $5 + 5n$ edges ($n \geq 0$) branching off the minimum path and then k edges to the other side, $k \geq 1$, the correct procedure is shown in Figs. 4, 5, 6 and 7, respectively, with the "a" figures

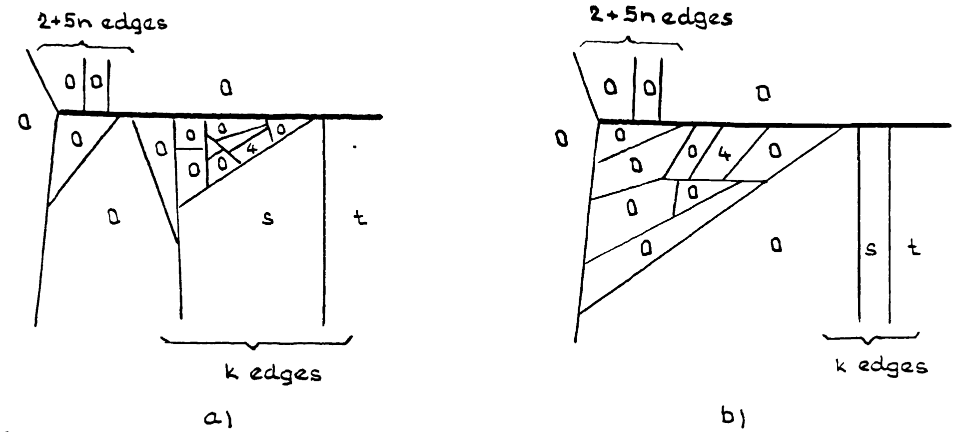

Fig. 4

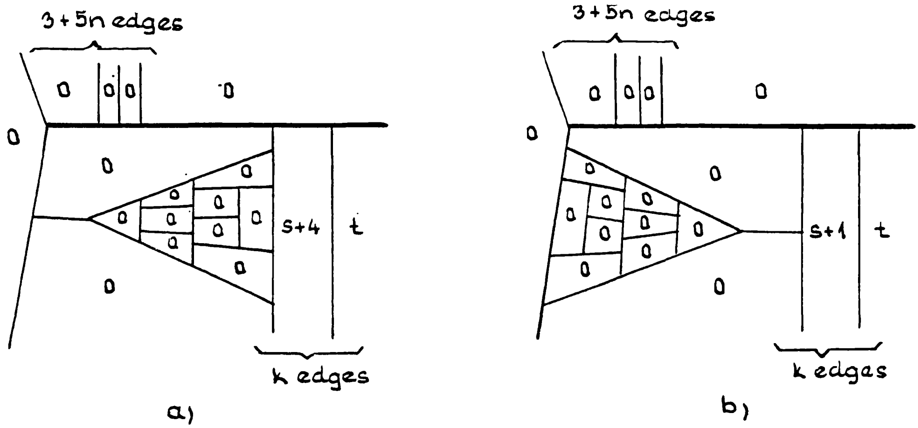

Fig. 5

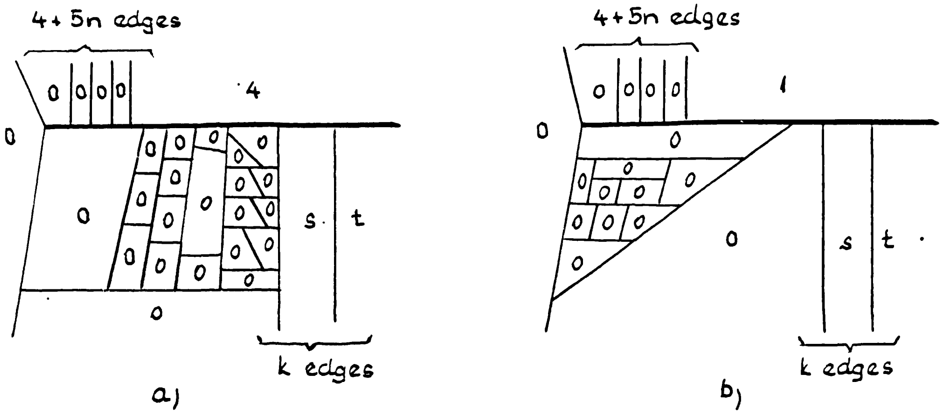

Fig. 6

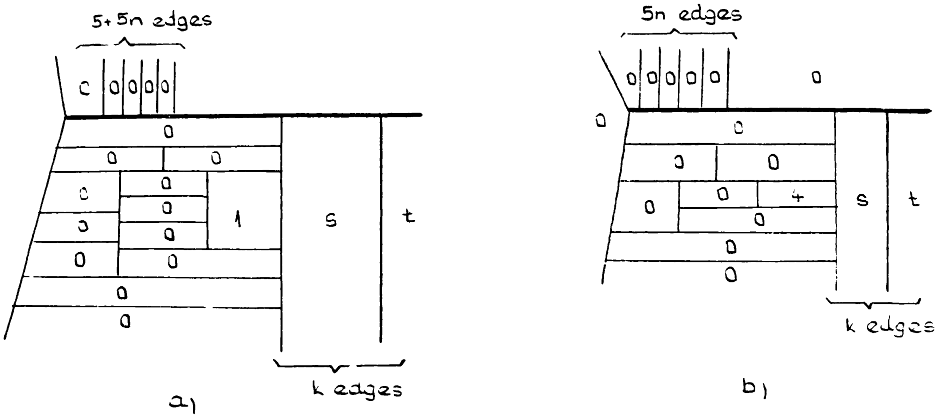

Fig. 7

illustrating the case $u \equiv 4 \pmod 5$, the "b" figures the case $u \equiv 1 \pmod 5$ (compare these figures with Fig. 2 for $j = 2, 3, 4, 5$).

If $k = 0$, we can proceed in the same way as for $k \geq 1$, taking into consideration the fact that in this case there is $s = 2$ or 3 and $t = 0$ in Fig. 2. This cannot happen if $k \geq 1$.

In all these cases we obtain either a trivalent map with only one exceptional face (if $k = 0, j = 3$ — see Fig. 5) or a trivalent planar map with two adjacent exceptional faces (if $k \geq 1$, $j = 3$ and $t = 2, 3$ or $k = 0$ and $j = 1, 4, 5$ or $k = 0, j = 2$ and $r = 4$), or a trivalent planar map $\mathscr{M}_2$ of the same type as $\mathscr{M}_1$ with the distance between the exceptional faces smaller than in $\mathscr{M}_1$. The first two maps contradict the statement of B. Grünbaum which we have already mentioned. To $\mathscr{M}_2$ we can apply the same procedure used previously on $\mathscr{M}_1$.

The only remaining possibility is that the minimum path connecting the *u-gon* and the *v-gon* in $\mathscr{M}_1$ consists of just one edge (see Fig. 8). The procedure for this case is the same as in the case $k = 0, j = 5$. (Compare Fig. 8, Fig. 2 and Fig. 7.) The result is a trivalent planar map with adjacent exceptional faces which contradicts B. Grünbaum's result.

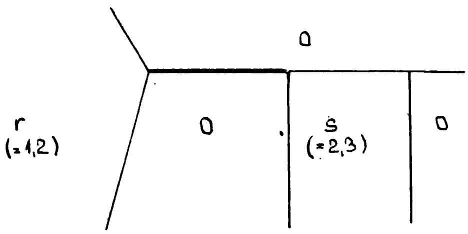

Fig. 8

We have described a procedure which, after a finite number of steps, would change any map of the type $\mathscr{G}(3, 5, 2)$ with $u + v \not\equiv 0 \pmod 5$ and $u \not\equiv v \pmod 5$ into a map whose existence is impossible according to the result due to B. Grünbaum and mentioned above.

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