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Matematický časopis, Vol. 25 (1975), No. 2, 173--178

Persistent URL: http://dml.cz/dmlcz/126947

Mat. Čas. 25, 1975, No 2, 173—178

TWO-SIDED BASES OF SEMIGROUPS

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The structure of semigroups, containing one-sided bases is investigated in [1]. The notion of a one-sided base was introduced by Tamura in [4]. The purpose of the present paper is to describe the structure of semigroups containing two-sided bases.

A subset A of a semigroup S is a right (left) base of S if $A \cup SA = S (A \cup AS = S)$, but there exists no proper subset $B \subset A$ for which $B \cup SB = S (B \cup BS = S)$.

Definition 1. *We say that a subset* $A \subset S$ *is a two-sided base of* S, *if* $A \cup SA \cup AS \cup SAS = S$, *but there exists no proper subset* $X \subset A$, $X \neq A$ *such that* $X \cup SX \cup XS \cup SXS = S$.

If $A \subset S$ is a subset of S, then we denote the set $A \cup SA \cup AS \cup SAS$ by $(A)_T$.

A principal two-sided ideal, generated by an element a will be denoted by $(a)_T$, i. e. $(a)_T = a \cup Sa \cup aS \cup SaS$.

Lemma 1. *Let* A *be a two-sided base of* S. *Let* $a, b \in A$. *If* $a \in (Sb \cup bS \cup SbS)$, *then* $a = b$.

Proof. Let $a \in (Sb \cup bS \cup SbS)$ and $a \neq b$. Let us consider the set $B = A - \{a\}$. Then $b \in B$. The relation $a \in (Sb \cup bS \cup SbS)$ implies $(a)_T \subset (Sb \cup bS \cup SbS) \subset (B)_T$, and it follows that $S = (A)_T \subset (B)_T$. But this is a contradiction, because A is a two-sided base.

Now we introduce a quasi-ordering into S, namely $a \leq b$ means $a \cup Sa \cup aS \cup SaS \subset b \cup Sb \cup bS \cup SbS$, thus $(a)_T \subset (b)_T$.

Lemma 2. *Let* A *be a two-sided base of a semigroup* S. *If* $a, b \in A$, $a \neq b$, *then neither* $a \leq b$, *nor* $b \leq a$.

Proof. Let us assume that $a \leq b$, $(a)_T \subset (b)_T$. If there were $a \neq b$, then $a \in (Sb \cup bS \cup SbS)$. Lemma 1 implies that $a = b$.

Theorem 1. *A non-empty subset* A *of a semigroup* S *is a two-sided base of* S *if and only if* A *satisfies the following conditions:*

(1) for any $x \in S$ there exists $a \in A$ such that $x \leq a$.

(2) for any two distinct elements $a, b \in A$ neither $a \leq b$, nor $b \leq a$.

Proof. (a) Let us suppose that (1) and (2) hold for A, let $x \in S$. Then $x \in S$ implies $x \leq a \in A$, i. e. $x \in (a)_T \subset (A)_T$. It follows that $S \subset (A)_T$ so that $S = (A)_T$. It remains to show that A is a minimal subset with the property: $S = (A)_T$. Let $B \subset A$, $B \neq A$ such that $S = (B)_T$. If $a \in A - B$, then there exists $b \in B$ such that $a \in (Sb \cup bS \cup SbS)$. Thus we have: $(a)_T \subset (b)_T$, but this is a contradiction with (2).

(b) Let A be a two-sided base of S, thus $S = (A)_T$. Then if $x \in S$, then $x \in (A)_T$. Then there exists $a \in A$ such that $x \in (a)_T$. This implies $x \leq a$, and so (1) is satisfied and the validity of (2) follows from Lemma 2.

If we define $x \sim y$ iff both $x \leq y$ and $y \leq x$ at the same time, we get the well-known partition of S into the so-called F-classes. If an element a belongs to an F-class, then this F-class will be denoted by F_a.

The condition (2) of Theorem 1 implies that any two elements of a two-sided base A do not belong to the same F-class. In other words: if $a, b \in A$, $a \neq b$, then $F_a \cap F_b = \emptyset$.

Let us ask ourselves, whether a semigroup may contain more than one two-sided base and if yes what is their mutual relation.

Theorem 2. *Let A be a two-sided base of a semigroup S. If there exists at least one F-class generated by an element of A, which contains more than one element, then the semigroup S contains still another two-sided base.*

Proof. Let F_a be an F-class containing more than one element, and let $b \in F_a$, $b \neq a$. Let $A_1 = [(A - \{a\}) \cup \{b\}]$. Evidently, $A \neq A_1$. We are going to show that A_1 is a two-sided base of S. To prove it, it does suffice to show that A_1 satisfies the conditions (1), (2) of Theorem 1. Let $x \in S$. By Theorem 1, there exists an element $c \in A$ such that $x \leq c$. If $c \neq a$, then $c \in A_1$. If $c = a$, then $(c)_T = (b)_T$, $c \neq b$. Then evidently $x \leq b$, and $b \in A_1$. Hence A_1 satisfies the condition (1) of Theorem 1. Let $c_1, c_2 \in A_1$, $c_1 \neq c_2$. Both c_1 and c_2 cannot belong to F_a. Let $c_1 \in F_a$. Then $(c_1)_T = (a)_T$ and $c_2 \in A$. If $c_1 \leq c_2$, then $a \leq c_2$, however this is impossible as $a, c_2 \in A$. Similarly $c_2 \leq c_1$ cannot hold as then it will have to hold $c_2 \leq a$, and it is again impossible by (2) of Theorem 1. If $c_2 \in F_a$, we would proceed similarly. In the case that neither $c_1 \overline{\in} F_a$, nor $c_2 \overline{\in} F_a$, we have $c_1, c_2 \in A$ and the condition (2) of Theorem 1 is satisfied again.

Corollary. *Let A be a two-sided base of S, $a \in A$. If $(x)_T = (a)_T$ for some $x \in S$, $x \neq a$, then x belongs to some two-sided base, which is different from A.*

Theorem 3. *Let A and B be any two two-sided bases of a semigroup S. Then A and B have the same cardinality.*

Proof. Define a mapping φ on A as follows. If $a \in A$, then $\varphi(a) = b$, $b \in B$ if and only if $b \in F_a$. We show that this mapping is defined for every $a \in A$. As B is a two-sided base, then there exists an element $b \in B$ such that $a \leq b$.

Because A is a two-sided base of S also, then for the element $b \in B$ there exists an element $a' \in A$ such that $b \leq a'$. We get $a \leq b \leq a'$. It implies $a \leq a'$, and therefore $a = a'$. However, this implies $(a)_T = (b)_T$, so $b \in F_a$. We show that φ is one-to-one and onto. Let $a_1, a_2 \in A$. If $\varphi(a_1) = \varphi(a_2)$, then $(a_1)_T = (a_2)_T$. The condition (2) of Theorem 1 implies $a_1 = a_2$. It remains to show that φ is onto. If $b \in B$, then there exists $a_1 \in A$ such that $b \leq a_1$. For the same reason, for the element $a_1 \in A$ there exists some $b_1 \in B$ such that $a_1 \leq b_1$. Thus, $b \leq a_1 \leq b_1$, $b_1, b \in B$, therefore by (2) of Theorem 1, $b = b_1$, so $(b)_T = (b_1)_T$ and $(a_1)_T = (b)_T$, i. e. $\varphi(a_1) = b$, for $a_1 \in A$. Therefore, φ is onto.

Simple examples of semigroups show that a two-sided base A of S need not be a subsemigroup (and therefore a two-sided ideal of S either).

Further we show some conditions when a two-sided base of S is a subsemigroup of S.

Remark 1. We can show easily that a two-sided base A of a semigroup S is a two-sided ideal of S if and only if $A = S$.

Theorem 4. *A two-sided base A of a semigroup S is a subsemigroup of S if and only if A consists of one element, which is an idempotent.*

Proof. (a) Let a two-sided base A of a semigroup S be a subsemigroup of S. Then for arbitrary $a, b \in A$, we have $ab \in A$, hence $ab = c$ for some $c \in A$. Therefore, $c \in Sb$. By Lemma 1 $c = b$, and so $ab = b$. However, from the relation $ab = c$ we have $c \in aS$, and again by Lemma 1 we get $c = a$, and so $ab = a$. Both relations $ab = b$, $ab = a$ imply $a = b$.

(b) Evidently, a one-element two-sided base of S, which consists of an idempotent is a subsemigroup of S.

Remark 2. Theorems 3 and 4 imply that if a two-sided base of a semigroup S is a subsemigroup of S and therefore a oneelement subsemigroup, then every two-sided base of S is one—element. The question arises whether every two-sided base of S is a subsemigroup. By the following example of a semigroup we can ascertain that this is not true.

Example 1. Let $S = \{a, b, c, d\}$ be a semigroup with the multiplication table:

	a	b	c	d
a	a.	b	a	a
b	b	a	b	b
c	a	b	c	d
d	a	b	d	c

The semigroup S contains two two-sided bases: $A_1 = \{c\}$, $A_2 = \{d\}$ $c^2 = c$, thus A_1 is a subsemigroup, however $d^2 \neq d$, so A_2 is not a subsemigroup.

By $\mathscr{A}$ we shall denote the union of all two-sided bases of a semigroup S.

Theorem 5. *$S - \mathscr{A}$ is either the empty set or a two-sided ideal of the semigroup S.*

Proof. Let $S - \mathscr{A} \neq \emptyset$, let $a \in S - \mathscr{A}$, $x \in S$. To prove the statement it suffices to show that both $xa \in S - \mathscr{A}$ and $ax \in S - \mathscr{A}$. The proof will be done for the first part only, because the other is analogous. Let us assume that $xa \overline{\in} S - \mathscr{A}$. Then $xa \in \mathscr{A}$. It means that xa belongs at least into one two-sided base. Let $xa \in A_i$. Hence $xa = b \in A_i$. It implies $b \in Sa$, $Sb \subset Sa$, $SbS \subset SaS$, and therefore $(b)_T \subset (a)_T$. We show that the relation $(b)_T = (a)_T$ cannot hold. If $(b)_T = (a)_T$, then the Corollary of Theorem 2 implies that $a \in \mathscr{A}$, which is a contradiction with the choice of the element a, because $a \in S - \mathscr{A}$. Therefore $(b)_T \subset (a)_T$, and $(b)_T \neq (a)_T$, thus $b \leq a$. However, A_i is a two-sided base of S. Hence for the element a there exists $b_1 \in A_i$ such that $a \leq b_1$. We have: $b \leq a \leq b_1$, so $b \leq b_1$, but $b, b_1 \in A_i$, $b \neq b_1$, so this is a contradiction with (2) of Theorem 1. Therefore $xa \in S - \mathscr{A}$.

The notion of a maximal proper ideal is used in the same sense as in [2].

The following example of a semigroup shows that $M = S - \mathscr{A}$ need not be a maximal two-sided ideal of S.

Example 2. Let $S = \{a, b, c, d\}$ be a semigroup with the multiplication table:

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	b
d	a	a	b	b

The only two-sided base of S is a subset $A = \{c, d\}$. $S - \mathscr{A} = \{a, b\}$ is an ideal of S, but it is not a maximal one, because $\{a, b, c\}$ is an ideal of S also.

We say that a semigroup S contains a two-sided ideal M^*, if M^* is such a maximal proper two-sided ideal, in which every proper two-sided ideal M of S is contained (see [3]).

Theorem 6. *Let $\emptyset \neq \mathscr{A} \neq S$. Then the following statements for a semigroup S are equivalent.*

(1) *$S - \mathscr{A}$ is maximal proper two-sided ideal of S.*
(2) *For every element $a \in \mathscr{A}$, $\mathscr{A} \subset (a)_T$.*
(3) $S - \mathscr{A} = M^*$.
(4) *Every two-sided base of S is a one-element base.*

Proof. (1) $\Leftrightarrow$ (2). Let $\emptyset \neq S - \mathscr{A}$ be a maximal proper two-sided ideal of S, let $a \in \mathscr{A}$. If $\mathscr{A} \subset (a)_T$ does not hold, then $S - \mathscr{A} \cup (a)_T$ is a proper two-sided ideal of S, and $S - \mathscr{A} \subsetneqq S - \mathscr{A} \cup (a)_T$, and it is contradictary to the assumption that $S - \mathscr{A}$ is a maximal proper two-sided ideal.

Let for any $a \in \mathscr{A}$ be $\mathscr{A} \subset (a)_T$. Theorem 5 implies that $S - \mathscr{A}$ is an ideal of S. Let $S - \mathscr{A} \subsetneqq M \subsetneqq S$, where M is an ideal of S. Then $M \cap \mathscr{A} \neq \emptyset$. Let $c \in M \cap \mathscr{A}$, thus $c \in M$, $c \in \mathscr{A}$. $c \in M$ implies $Sc \subset SM \subset M$, $cS \subset \subset MS \subset M$, $ScS \subset SMS \subset SM \subset M$. Therefore, $M \supset S - \mathscr{A} \cup (c)_T = S$, and so $M = S$ because $\mathscr{A} \subset (c)_T$ and it is a contradiction with $M \subsetneqq S$.

(3) $\Leftrightarrow$ (4). Let $S - \mathscr{A} = M^*$. We know that if $S - \mathscr{A}$ is a maximal proper two-sided ideal, then for every $a \in \mathscr{A}$, $\mathscr{A} \subset (a)_T$ holds. We show that if $S - \mathscr{A} = M^*$, then every two-sided base of S is a one-element one. At first we show that for any $a \in \mathscr{A}$, $S - \mathscr{A} \subset (a)_T$. If the last relation does not hold, then $(a)_T$ is a proper two-sided ideal of S, distinct from $S - \mathscr{A}$, which is a contradiction to the assumption. Thus, $S - \mathscr{A} \subset (a)_T$ and at the same time $\mathscr{A} \subset (a)_T$. Both relations imply $S \subset (a)_T$, so, $S = (a)_T$. Therefore $\{a\}$ is a two-sided base of S and because a is an arbitrary element of $\mathscr{A}$, then each two-sided base is a one-element base.

Let every two-sided base of S be a one-element base, and so, for any $a \in \mathscr{A}$, $(a)_T = S$ holds. We show that $S - \mathscr{A} = M^*$. The statement that $S - \mathscr{A}$ is a maximal proper two-sided ideal follows from the proof (1) $\Leftrightarrow$ (2). It remains to show that every two-sided ideal of S is contained in $S - \mathscr{A}$. Let T be a two-sided ideal of S, which is not contained in $S - \mathscr{A}$. Then $\mathscr{A} \cap T \neq \emptyset$. If $x \in \mathscr{A} \cap T$, then $x \in \mathscr{A}$, $x \in T$. It follows that $Sx \subset ST \subset T$, $xS \subset TS \subset T$, $SxS \subset ST \subset T$. Thus $T \supset (x)_T = S$, therefore $T = S$ and the proof is complete.

(1) $\Leftrightarrow$ (3). Let $S - \mathscr{A}$ be a maximal proper two-sided ideal of S. We have to show that $S - \mathscr{A} = M^*$, thus every two-sided proper ideal of S is contained in $S - \mathscr{A}$. Let us suppose that an ideal M is not contained in $S - \mathscr{A}$, thus $M \not\subset S - \mathscr{A}$. Then M must have the following form: $M = \mathscr{A} \cup X$, where $X \subset S - \mathscr{A}$. The ideal M can be expressed as a union of principal two-sided ideals, generated both by elements of $\mathscr{A}$ and by elements of $X = = S - \mathscr{A} \cap M$. According to the condition (1) of Theorem 1 we know that every principal two-sided ideal generated by an element of $S - \mathscr{A}$ is contained in a principal two-sided ideal, generated by some element of $\mathscr{A}$. We have that the union of all principal ideals, generated by the elements of $\mathscr{A}$ contains both $\mathscr{A}$ and $S - \mathscr{A}$, thus $M = S$. We get that if $S - \mathscr{A}$ is a maximal proper two-sided ideal of S, then each two-sided ideal which is not contained in $S - \mathscr{A}$ is equal to S. Hence $S - \mathscr{A} = M^*$.

If $S - \mathscr{A} = M^*$, then evidently $S - \mathscr{A}$ is a maximal two-sided ideal of S.

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Received December 7, 1973

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