Ján Plesník

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Since the Tutte theorem on 1-factors [18] was issued, many papers have appeared on this subject or its generalizations [1, 2, 3, 4, 5, 6, 7, 9, 11, 13, 14, 15, 17, 19, 20, 21] (our list is incomplete). One part of the proofs uses hyperprime graphs (e. g. [9, 11, 15, 18]) and another uses the method of alternating paths (e. g. [2, 3, 4, 5, 17, 19]). Gallai [6] was the first, who used a further technique, namely his proof of the Tutte theorem is based on the König-Hall theorem (see [12] and [8] or Ore [16]) on perfect matchings in bipartite graphs (a proof of this kind is also in [1]. In his later paper Gallai mentioned (see [7], the remark on p. 408) a modification of his method suggested by L. Pósa and V. T. Sós. The aim of this paper is to show that if we prefer in the method due to Gallai, Pósa and Sós an induction on the number of lines instead of points, then the Tutte theorem can be proved without using the König-Hall theorem. Such a proof seems to be new, although its separate steps or their modifications are mostly well-known with the exception of one (see step 6 in the sequel) in which we deduce the König-Hall theorem from the Tutte theorem.

The notions used below are based on Harary [10]. The following denotations will be used: A *graph* G is denoted by (V, E), where $V = V(G)$ and $E = E(G)$ are its *point* and *line sets* respectively. If $U \subset V(G)$, then the *induced subgraph* of G on U is denoted by $G(U)$. If $S \subset V(G)$, then by $t_G(S)$ we denote the *number* of *odd components* of the graph $G - S = G(V(G) - S)$ (by an *odd component* we mean one with an odd number of points). By (V_1, V_2, E) we denote a *bipartite graph* G with $V_1 \cap V_2 = \emptyset$, $V(G) = V_1 \cup V_2$, $E(G) = E$, in which there is no line of the form v_1v_2 with $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$. If $A \subset V_1$, then we put $V_2(A) = \{y \mid x \in A, xy \in E\}$.

The Tutte theorem [18]: *A graph G is without 1-factors if and only if there is a set $S \subset V(G)$ with $t_G(S) > |S|$.*

Proof. To prove the part "if" we use the argument due to Tutte [18]: For any $S \subset V(G)$, every perfect matching matches at least one point of each odd component of $G - S$ to a point of S. Therefore, if $t_G(S) > |S|$, then no perfect matching can exist.

To prove the part "only if" it is sufficient to present a proof in the case if $|V(G)|$ is even (otherwise we can put $S = \emptyset$). We shall proceed by induction on the number of lines of G. If $|E(G)| = 0$, the assertion is trivial. Therefore let $|E(G)| \geqslant 1$. Choose an arbitrary line $e \in E(G)$ and form the graph $G' = G - e = (V(G), E(G) - \{e\})$. As G' is without a 1-factor too, there is by induction hypothesis a set $S' \subset V(G') = V(G)$ with $t_{G'}(S') > |S'|$. We shall assume that

(1) $|S'| = \max |M|$, *where the maximum is taken over all* $M \subset V(G)$ *with*

(2) $t_{G'}(M) - |M| = \max\limits_{N \subset V(G)} (t_{G'}(N) - |N|)$.

As $|V(G)|$ is even, $t_{G'}(S') - |S'| \geqslant 2$. Since the adding of e to G' changes no more than two odd components of $G' - S'$, $t_G(S') \geqslant t_{G'}(S') - 2$. Therefore in the case if $t_{G'}(S') - |S'| > 2$, we can put $S = S'$, which gives $t_G(S) - |S| > 0$ as desired. Thus we can assume that

(3) $t_{G'}(S') - |S'| = 2$.

Now we shall prove

(4) *Every even component* G'_i *of the graph* $G' - S'$ *has a* 1-*factor.*

In the other case, there is by the induction hypothesis a set $S'_i \subset V(G'_i)$ with $t_{G'_i}(S'_i) - |S'_i| \geqslant 2$. However, then $t_{G'_i}(S' \cup S'_i) - |S' \cup S'_i| \geqslant 4$, contradicting (2) and (3). Analogously we have

(5) *Every odd component* G'_j *of* $G' - S'$ *after the deleting of one arbitrary point* u_j *has a* 1-*factor.*

Otherwise, there is by the induction hypothesis a set $S'_j \subset V(G'_j - u_j)$ with $t_{G'_j - u_j}(S'_j) - |S'_j| \geqslant 2$. Then we have $t_{G'}(S' \cup S'_j \cup \{u_j\}) - |S' \cup S'_j \cup \{u_j\}| = t_{G'}(S') - 1 + t_{G'_j - u_j}(S'_j) - |S'| - |S'_j| - 1 \geqslant 2$. By (2) and (3) the last relation is equality. However, we have a contradiction with (1). Now we are going to prove

(6) *The validity of the induction hypothesis (i. e. the main part of the Tutte theorem) for any graph G with $|E(G)| \leqslant m$ implies the validity of the following assertion (i. e. the main part of the König-Hall theorem) for any bipartite graph $B = (V_1, V_2, F)$ with $|V_1| = |V_2|$ and $|F| \leq m$: If B has no 1-factor, then there is a set $A \subset V_1$ with $|A| > |V_2(A)|$.*

If B has no 1-factor, then there is a set $S = S_1 \cup S_2$ with $S_1 \subset V_1$, $S_2 \subset V_2$ and $t_B(S) > |S|$. We shall assume that S has the maximum cardinality among all sets with these properties. Then every component of $B - S$ consists of a single point as it can be easily seen. If $|V_1 - S_1| > |S_2|$, then we can put $A = V_1 - S_1$ (since $V_2(V_1 - S_1) \subset S_2$, thus $|A| > |S_2| \geqslant |V_2(A)|$). Put $n = |V_1| = |V_2|$. In the case if $|V_1 - S_1| \leqslant |S_2|$ we have $n = |S_1| + |V_1 - S_1| \leq |S_1| + |S_2| = |S|$ and on the other hand $|S| < t_B(S) = |V_1 - S_1| + |V_2 - S_2| \leqslant |S_2| + |V_2 - S_2| = n$, which is a contradiction.

Note that if we have proved the part of König-Hall theorem for any n, then the case $|V_1| \neq |V_2|$ can be derived immediately.

If the line e is incident with at most one odd component of $G' - S'$, then we can put $S = S'$ and obviously we have $t_G(S) - |S| = t_{G'}(S') - |S'| = - 2 > 0$. Thus there remains the possibility

(7) *The line e connects two different odd components of $G' - S'$.*

Then by (3) the graph $G - S'$ has exactly $t_G(S') = t_{G'}(S') - 2$ odd components: $G'_1, G'_2, \ldots, G'_m$, where $m = t_G(S') = |S'|$. Shrink each component G'_i to a single point u_i, $1 \leqslant i \leqslant m$, to form a graph $\tilde{G}$. Consider the maximal bipartite subgraph B of $\tilde{G}$ with $V_1 = V_1(B) = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = - V_2(B) = S'$. If B has a 1-factor, then each component G'_i, $1 \leqslant i \leqslant m$, has a point v_i matched into S'. Since the line e matches two points of the other two odd components of $G' - S'$, thus by (4) and (5) G has a 1-factor, which is impossible. Therefore we have

(8) *There is no perfect matching of V_1 onto V_2 in the graph B.*

This means according to (6) that there is a set $A \subset V_1(B)$ with $|A| > |V_2(A)|$. If we put $S = V_2(A) \subset S'$, then the graph $G - S$ has at least $|A|$ odd components, i. e. $t_G(S) > |S|$. This completes the proof.

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Katedra numerickej matematiky
a matematickej štatistiky
Prírodovedecká fakulta UK
Mlynská dolina
816 31 Bratislava