Matematický časopis

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Matematický časopis, Vol. 25 (1975), No. 2, 105--110

Persistent URL: http://dml.cz/dmlcz/126952

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Mat. Čas. 25, 1975, No 2, 105—110

FURTHER RELATIONSHIPS BETWEEN DECOMPOSITION THEORIES AND TOPOLOGIES

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In [1] the notion of a decomposition theory was introduced as an abstraction of the family of ordered pairs $(A, \mathscr{A})$ in the context of a topological space X, where A is any subset of X and $\mathscr{A}$ is a partition of $X - A$ into relatively open sets. In [1] a "natural" topology was associated with each decomposition theory, while any topology gives rise to two decomposition theories; however, it was shown that if one goes from a topology to a related decomposition theory, and thence from that decomposition theory to its related topology, the resulting topology may not be related even by inclusion in one direction to the original topology. This, in turn, leads to the conclusion that decomposition theories and topologies are not dual concepts, nor are they „equivalent" notions, at least if one adheres to the definitions presented in [1]. The purpose of the present paper is to inquire more deeply into the relationship between topologies and decomposition theories. During the course of this discussion a new topology will be associated with a decomposition theory which, at least in the presence of a weak separation property, does allow us to speak of a virtual equivalence between a topology and the related decomposition theory.

As in [1], for any topology τ we let Δ_τ denote its associated decomposition theory (allowing for infinite partitions), and for any decomposition theory Δ, τ_Δ will denote the topology derived from Δ as defined in [1]. Generally, the terminology concerning decomposition theories is that of [1].

We begin by discussing two basic separation properties.

Proposition 1: *If Δ is a decomposition theory on a set X such that for any two distinct points x and y of X there is an irreducible element $(A, \mathscr{A})$ of Δ with x and y contained in different members of $\mathscr{A}$, then τ_Δ is T_2.*

Proof: If $(A, \mathscr{A})$ is irreducible, then each member of $\mathscr{A}$ is also a member of τ_Δ.

Proposition 2: *If X, τ is a T_2 topological space and x and y are distinct elements of X, then there is an irreducible element $(A, \mathscr{A}) \in \Delta_\tau$ such that x and y are in different members of $\mathscr{A}$.*

Proof: Since $x \neq y$, there are disjoint open sets U and V with $x \in U$ and $y \in V$. Then $(X - (U \cup V), \{U, V\}) \in \Delta_\tau$ and $X - (U \cup V)$ is closed. Therefore by Proposition 4 of [1], there is an irreducible element $(A, \mathscr{A})$ of Δ_τ such that $(X - (U \cup V), \{U, V\}) \leqslant (A, \mathscr{A})$; and $(A, \mathscr{A})$ has the desired property.

Because of Propositions 1 and 2, the following definition is appropriate.

Definition 1: *A decomposition theory Δ on a set X is said to be T_2 if given any two distinct elements x and y of X, there is an irreducible element $(A, \mathscr{A})$ of Δ such that x and y are contained in different members of $\mathscr{A}$.*

We now consider the property of being T_1. The proofs of the following proposition and its corollary are immediate.

Proposition 3: *A topological space X, τ is T_1 if and only if every two point subset of X is disconnected.*

Corollary: *If X, τ is T_1, then $(X - \{x, y\}, \{\{x\}, \{y\}\}) \in \Delta_\tau$ for each $x, y \in X$.*

Proposition 4: *If a decomposition theory Δ on a set X has the property that $(X - \{x, y\}, \{\{x\}, \{y\}\}) \in \Delta$ for each $x, y \in X$, then τ_Δ is T_1.*

Proof: Suppose x and y are distinct points of X. Then

$$x \in U_x = \cup\{V | x \in V, \ V \in \mathscr{A}, \ (X - \{x, y\}, \{\{x\}, \{y\}\}) \leqslant (A, \mathscr{A})\},$$

but $y \notin U_x$. Since $U_x \in \tau_\Delta$, τ_Δ is T_1.

Propositions 3 and 4 lead to the following definition.

Definition 2: *A decomposition theory Δ on a set X is T_1 if $(X - \{x, y\}, \{\{x\}, \{y\}\}) \in \Delta$ for each $x, y \in X$.*

The decomposition theoretic analog for continuity is given in the next definition.

Definition 3: *A function $f : X, \Delta \to Y, \Delta'$ is continuous if for each $(A, \mathscr{A}) \in \Delta'$, $(f^{-1}(A), f^{-1}(\mathscr{A})) \in \Delta$, where $f^{-1}(\mathscr{A}) = \{f^{-1}(U) | U \in \mathscr{A}\}$.*

The following proposition follows immediately from Definition 3 and the definition of Δ_τ.

Proposition 5: *If $f : X, \tau \to Y, \tau'$ is continuous, then $f : X, \Delta_\tau \to Y, \Delta_{\tau'}$, is continuous.*

Nevertheless, it is not generally true that if $f : X, \Delta \to Y, \Delta'$ is continuous, then $f : X, \tau_\Delta \to Y, \tau_{\Delta'}$ is also continuous.

Example 1: Let $X = Y = R$, the set of real numbers. Let Δ be the decomposition theory generated by $([-2,2], \{(-\infty, -2), (2, \infty)\})$ and $([-1, 1], \{(-\infty, -1), (1, \infty)\})$, and Δ' be the decomposition theory generated by $([-2, 2], \{(-\infty, -2), (2, \infty)\})$. Then τ_Δ contains $(-\infty, -1)$ and $(1, \infty)$, but not $(-\infty, -2)$ or $(2, \infty)$, while $\tau_{\Delta'}$ does contain these latter two sets.

Therefore if $f: X, \Delta \to Y, \Delta'$ is the identity function, then f is continuous, but $f: X, \tau_\Delta \to Y, \tau_{\Delta'}$ is not continuous.

Most of the standard theorems concerning topological continuity have decomposition theoretic analogs; we now prove several of these.

Definition 4: *A subset $\mathscr{J}$ of a decomposition theory Δ is said to be a basis for Δ if $\Delta_{\mathscr{J}} = \Delta$.*

Definition 5: *Suppose that Δ is a decomposition theory on a set X and $\mathscr{S} \subset \Delta$. We say that $(B, \mathscr{B}) \in \Delta$ is directly derivable from $\mathscr{S}$ if* a) $\mathscr{B} = \{X - B\}$; *or* b) *there is $(B, \mathscr{B}_1) \in \mathscr{S}$ such that $\mathscr{B}_1$ refines $\mathscr{B}$, or* c) *there are $(A_1, \mathscr{A}_1), (A_2, \mathscr{A}_2) \in \mathscr{S}$ such that $(B, \mathscr{B}) = (A_1 \cup A_2, \mathscr{A}_1 \cap \mathscr{A}_2)$.*

Lemma 1: *If $f: X, \Delta \to Y, \Delta'$ and $\mathscr{S} \subset \Delta'$, and for each $(A, \mathscr{A}) \in \mathscr{S}$, $(f^{-1}(A), f^{-1}(\mathscr{A})) \in \Delta$, then if $(B, \mathscr{B})$ is directly derivable from $\mathscr{S}$, then $(f^{-1}(B), f^{-1}(\mathscr{B})) \in \Delta$.*

Proof: If $\mathscr{B} = \{X - B\}$, then the statement is trivial. If there is $(B, \mathscr{B}_1) \in \mathscr{S}$ such that $\mathscr{B}_1$ refines $\mathscr{B}$, then $(f^{-1}(B), f^{-1}(\mathscr{B}_1))$ refines $(f^{-1}(B), f^{-1}(\mathscr{B}))$, which implies that $(f^{-1}(B), f^{-1}(\mathscr{B})) \in \Delta$. Suppose now that there are $(A_1, \mathscr{A}_1)$, $(A_2, \mathscr{A}_2) \in \mathscr{S}$ such that $(B, \mathscr{B}) = (A_1 \cup A_2, \mathscr{A}_1 \cap \mathscr{A}_2)$. Then since $(f^{-1}(A_1) \cup f^{-1}(A_2), f^{-1}(\mathscr{A}_1) \cap f^{-1}(\mathscr{A}_2)) = (f^{-1}(A_1 \cup A_2), f^{-1}(\mathscr{A}_1 \cap \mathscr{A}_2)) = (f^{-1}(B), f^{-1}(\mathscr{B}))$, we have $(f^{-1}(B), f^{-1}(\mathscr{B})) \in \Delta$.

Lemma 2: *Suppose that $\mathscr{J}$ is a basis for Δ and set $\mathscr{S}_0 = \mathscr{J}$, and $\mathscr{S}_{n+1} = \{(A, \mathscr{A}) \in \Delta | (A, \mathscr{A})$ is directly derivable from $\mathscr{S}_n\}$, $n = 0, 1, 2, 3, \ldots$ Then $\Delta = \bigcup_{n=0}^{\infty} \mathscr{S}_n$.*

Proof: Evidently $\bigcup_{n=0}^{\infty} \mathscr{S}_n \subset \Delta_{\mathscr{J}} = \Delta$. Since $\bigcup_{n=0}^{\infty} \mathscr{S}_n$ is itself a decomposition theory which contains $\mathscr{J}$, $\Delta \subset \bigcup_{n=0}^{\infty} \mathscr{S}_n$ which completes the proof.

The following proposition follows immediately from Lemmas 1 and 2.

Proposition 5: *If $f: X, \Delta \to Y, \Delta'$ and $\mathscr{J}$ is a basis for Δ', and if $(f^{-1}(A), f^{-1}(\mathscr{A}))$ is an element of Δ for each $(A, \mathscr{A}) \in \mathscr{J}$, then f is continuous.*

Definition 6: *Let Δ be a decomposition theory on a set X and Y be a subset of X. Then Δ_Y, the decomposition theory induced on Y by Δ, is defined by $\Delta_Y = \{(A \cap Y, \mathscr{A} \cap Y) | (A, \mathscr{A}) \in \Delta\}$, where $\mathscr{A} \cap Y = \{U \cap Y | U \in \mathscr{A}\}$.*

The following proposition is immediately evident.

Proposition 6: *If $Y \subset X$ and $j: Y, \Delta_Y \to X, \Delta$ is injection, then j is continuous.*

Definition 7: *Let* $\{X_i, \Delta_i\}$, $i \in I$, *be a family of decomposition spaces and* $p_i : \mathsf{X}_i X_i \to X_i$ *be the projection into the ith component. Set* $\mathscr{I} = \{(p_i^{-1}(A), p_i^{-1}(\mathscr{A})) | (A, \mathscr{A}) \in \Delta_i, i \in I\}$. *The product decomposition theory on* $\mathsf{X}_i X_i$ *is that which is generated by* $\mathscr{I}$.

The following proposition is now obvious.

Proposition 7: *The product decomposition theory is the coarsest decomposition theory on the product set which makes projections continuous.*

Proposition 8: *If* $f : X, \Delta \to Y, \Delta'$ *and* $g : Y, \Delta' \to W, \Delta''$ *are continuous, then* $g \circ f$: $X, \Delta \to W, \Delta''$ *is also continuous.*

Proposition 9: *Let* $\{S_i, \Delta_i\}$, $i \in I$, *be a family of decomposition spaces and* Δ' *be the product decomposition theory on* $\mathsf{X}_i S_i$. *Suppose* $f : X, \Delta \to \mathsf{X}_i S_i, \Delta'$, *and let* $f_i = p_i \circ f$ *for each* $i \in I$. *Then* f *is continuous if and only if each* f_i *is continuous.*

Proof: If f is continuous, then each f_i is continuous by Propositions 7 and 8. Suppose that each f_i is continuous and $(A, \mathscr{A})$ is a basis element of Δ'. Then $(A, \mathscr{A}) = (p_j^{-1}(B), p_j^{-1}(\mathscr{B}))$ for some $(B, \mathscr{B}) \in \Delta_j$. Therefore $(f^{-1} \circ p_j^{-1}(B), f^{-1} \circ p_j^{-1}(\mathscr{B})) = (f_j^{-1}(B), f_j^{-1}(\mathscr{B})) \in \Delta'$ by the continuity of f_j. Therefore f is continuous by Proposition 5.

Definition 8: *A function* $f : X, \Delta \to Y, \Delta'$ *is said to be a homeomorphism if* f *is one-one, onto, and bicontinuous.*

The proofs of the following lemmas and their consequent proposition are straightforward and parallel the topological analogs.

Lemma 3: *If* $\mathscr{I}$ *is a basis for* Δ *and* $Y \subset X$, *then* $\mathscr{I} \cap Y = \{(A \cap Y, \mathscr{A} \cap \cap Y) | (A, \mathscr{A}) \in \Delta\}$ *is a basis for* Δ_Y.

Lemma 4: *If* $f : X, \Delta \to Y, \Delta'$ *is continuous and* $W \subset X$, *then* $f|W : W, \Delta_W \to Y, \Delta'$ *is continuous.*

Proposition 10: *Let* $\{S_i, \Delta_i\}$, $i \in I$, *be a family of decomposition spaces,* $j \in I$, *and* $x_i \in S_i$, $i \neq j$. *Set* $Y = \mathsf{X}_i W_i$, *where* $W_i = \{x_i\}$, $i \neq j$, *and* $W_j = S_j$. *Then* $h : S_j, \Delta_j \to Y, \Delta_Y$ *is a homeomorphism.*

Even though most of the analogs of theorems concerning topological continuity are true for decomposition theories, Example 1 indicates that we do not have a strict correspondence between topologies and decomposition theories, at least we do not have such a correspondence if we associate a topology with a decomposition theory in the manner prescribed in [1]. The discussion which follows shows that there is another way though by which a topology can be associated with a decomposition theory by which topologies and decomposition theories are more closely related.

Definition 9: *If $x \in X$, $A \subset X$, and U and V are subsets of X, we say that x and A are (U, V)-separated if $x \in U$, $A \subset V$, and $U \cap V = \emptyset$.*

Definition 10: *Let X, Δ be a decomposition space. For $A \subset X$ define* $\mathrm{Cl}\emptyset = \emptyset$, *and* $\mathrm{Cl}A = \{x \in X \mid$ *x cannot be (U, V)-separated from A for any U and V contained in $\mathscr{B}$ for any $(B, \mathscr{B}) \in \Delta\}$ if $A \neq \emptyset$.*

Proposition 11: Cl *as defined above is a closure operator for a topology on X; we denote this topology by $\tilde{\tau}_\Delta$.*

Proof: Clearly, $\mathrm{Cl}\emptyset = \emptyset$ and $A \subset \mathrm{Cl}A$; also $\mathrm{Cl}A \subset \mathrm{Cl}(\mathrm{Cl}A)$. If $y \notin \mathrm{Cl}A$, then y can be (U, V)-separated from A with $y \in U$, $A \subset V$ and U and V both members of $\mathscr{B}$ for some $(B, \mathscr{B}) \in \Delta$. This, in turn, implies that $\mathrm{Cl}A \subset V$ and y is (U, V)-separated from $\mathrm{Cl}A$; thus, $y \notin \mathrm{Cl}(\mathrm{Cl}A)$. Hence $\mathrm{Cl}A = \mathrm{Cl}(\mathrm{Cl}A)$. Now since $S \subset T$ means $\mathrm{Cl}S \subset \mathrm{Cl}T$ for any subsets S and T of X, it follows that $\mathrm{Cl}A \cup \mathrm{Cl}B \subset \mathrm{Cl}(A \cup B)$ for any subsets A and B of X. But if x can be separated from $A \cup B$, then x can be separated from both A and B; hence $\mathrm{Cl}(A \cup B) \subset \mathrm{Cl}A \cup \mathrm{Cl}B$. Therefore $\mathrm{Cl}(A \cup B) = \mathrm{Cl}A \cup \mathrm{Cl}B$ This completes the proof that Cl is a closure operator for a topology on X.

Definition 11: *Let X, Δ be a decomposition space*; *set* $g(A) = \{x \in X - A \mid$ *x cannot be (U, V)-separated from A for $U, V \in \mathscr{B}$ for some $(B, \mathscr{B}) \in \Delta\}$.*

Then $\mathrm{Cl}A = A \cup g(A)$ and the following proposition is trivially true.

Proposition 12: *If X, Δ is a decomposition space and $A \subset X$, then* $\mathrm{Cl}A = A$ *if and only if $g(A) = \emptyset$.*

The following proposition tells us that for a topological space X, τ, $\tilde{\tau}_{\Delta_\tau}$ is generally more closely related to τ than is τ_{Δ_τ}.

Proposition 13: *If X, τ is a T_1 topological space, then $W \subset X$ is τ-closed if and only if $g(W) = \emptyset$, where $g(W)$ is computed with respect to Δ_τ. Therefore if X, τ is T_1, then $\tilde{\tau}_{\Delta_\tau} = \tau$.*

Proof: Assume first that W is τ-closed and $x \notin W$. Then $(X - \{W \cup \cup\{x\}\}, \{W, \{x\}\})$ is a member of Δ_τ which shows that $x \notin g(W)$. Therefore $g(W) = \emptyset$.

Next suppose that $g(W) = \emptyset$ and $x \notin W$; we will show that x has a neighborhood in $X - W$. There exists $(B, \mathscr{B}) \in \Delta_\tau$ such that $x \in U$, $W \subset V$, and $U \neq V$, for certain $U, V \in \mathscr{B}$. Then $U = U' \cap (X - B)$ and $V = V' \cap \cap (X - B)$, where U' and V' are members of τ. Therefore since $U' \cap W = \emptyset$, U' is a neighborhood of x which is contained in $X - W$.

The author originally believed that τ_Δ was the coarsest topology for which $\Delta \subset \Delta_{\tau_\Delta}$. The next propositions show that this is not generally the case.

Proposition 14: *For any decomposition space X, Δ, $\tilde{\tau}_\Delta \subset \tau_\Delta$.*

Proof: Suppose that W is $\tilde{\tau}_\Delta$-closed and $x \notin W$. Then there is $(A, \mathscr{A}) \in \Delta$, $U, V \in \mathscr{A}$, $U \neq V$, such that x and W are (U, V)-separated. We may therefore use U to obtain a subbasic element of τ_Δ which does not meet W. This, in turn, implies that x has a τ_Δ-neighborhood in $X - W$; hence W is τ_Δ-closed.

Proposition 15: *If Δ is a decomposition theory on X, then $\Delta \subset \tilde{\tau}_{\Delta_\tau}$.*

Proof: Suppose that $(A, \mathscr{A}) \in \Delta$; we must show that if $U \in \mathscr{A}$, then U is relatively open in $X - A$ with respect to $\tilde{\tau}_{\Delta_\tau}$. Set $H = \cup\{V \in \mathscr{A} \mid U \neq V\}$. Then $\mathrm{Cl}H \subset X - U$; hence $X - \mathrm{Cl}H$ contains $U \cap (X - A)$, where the closure is taken in $\tilde{\tau}_\Delta$. Therefore U is relatively open in $X - A$.

The next proposition is further evidence that $\tilde{\tau}_\Delta$ is a better topology to associate with a decomposition theory than is τ_Δ.

Proposition 16: *If $f : X, \Delta \to Y, \Delta'$ is continuous, then so is $f : X, \tilde{\tau}_\Delta \to Y, \tilde{\tau}_{\Delta'}$.* (Cf. Example 1 and the discussion preceding it.)

Proof: Suppose that F is a $\tau_{\Delta'}$-closed subset of Y and $x \notin f^{-1}(F) \dot{\subset} X$. We may use some member $(A, \mathscr{A})$ of Δ' to separate $f(x)$ and F. But we may then use $(f^{-1}(A), f^{-1}(\mathscr{A})) \in \Delta$ to separate x and $f^{-1}(F)$, which implies that $f^{-1}(F)$ is closed and, therefore, that f is continuous.

From Propositions 13 and 14 we have:

Proposition 17: *If X, τ is a T_1 topological space, then $\tau = \tilde{\tau}_{\Delta_\tau} \subset \tau_{\Delta_\tau}$.*

We conclude with an example of a T_1 topological space for which $\tau_{\Delta_\tau} \neq \tilde{\tau}_{\Delta_\tau}$.

Example 2: Let R be the set of real numbers and let τ consist of all complements of finite subsets of R, $\emptyset$, and $(-\infty, 0)$ Then $\tilde{\tau}_{\Delta_\tau} = \tau$, but τ_{Δ_τ} consists only of $\emptyset$ and the complements of finite subsets of R.

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Received May 31, 1973

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