Matematický časopis

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Matematický časopis, Vol. 25 (1975), No. 2, 129--133

Persistent URL: http://dml.cz/dmlcz/126955

INTERSECTION GRAPHS OF GRAPHS

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Various types of intersection graphs, i. e. of graphs whose vertex set is a system of some sets and in which two vertices are joined by an edge if and only if the corresponding sets have a non-empty intersection, were studied. Among them there were intersection graphs of abstract algebras — first semigroups [1], then groups [2], lattices [6], semirings [4] etc.; in these graphs the vertex set was the set of all proper subalgebras of the algebra and the condition for the intersection to be non-empty was in some cases a little modified.

Here we shall study the intersection graphs of graphs. Let G be an undirected graph without loops and multiple edges. The intersection graph $I(G)$ of G is the graph whose vertex set is the set of all proper induced subgraphs of G having at least two vertices and in which two vertices are joined by an edge if and only if the corresponding subgraphs of G have a common edge.

An induced (or spanned) subgraph of a graph G is such a subgraph G' of G which contains all edges of G whose end vertices are in G'. Such a subgraph is uniquely determined (induced) by its vertex set and this vertex set can be any subset of the vertex set of G. Therefore every graph contains exactly 2^n induced subgraphs, where n is its number of vertices. One of these subgraphs is the whole graph G, one of them is the empty graph and n of them are one-vertex subgraphs. Thus if $2 \leqq n < \aleph_0$, then $I(G)$ has $2^n - n - 2$ vertices, where n is the number of vertices of G. If the condition to have a common edge were replaced by the condition to have a common vertex, the intersection graph $I(G)$ would not give us any information about the structure of G except for the information about the number of vertices. Two induced subgraphs have a common vertex if and only if their vertex sets (inducing sets) have a common vertex. Therefore the intersection graph of a graph G with n vertices would simply be the intersection graph of the system of all proper subsets of some set of the cardinality n containing at least two elements each; the edges of G would not have any influence on the structure of $I(G)$. Therefore we use the condition of having a common edge. We consider only subgraphs with at least two vertices, because the empty subgraph or a subgraph consisting of one vertex cannot have a common edge with another subgraph of G and the vertex of $I(G)$ corresponding to it would be always isolated.

In this paper we shall prove a theorem on intersection graphs of graphs.

Theorem 1. *Let G be an undirected graph without loops and multiple edges, none of whose connected components is either a star, or a single edge with its end vertices, or an isolated vertex. Then G is uniquely determined by its intersection graph $I(G)$.*

Before proving this theorem we shall prove some lemmas.

Lemma 1. *Let G be an undirected graph without loops and multiple edges and with at least three edges. Let $I(G)$ be its intersection graph. If $I(G)$ is given, then for each vertex of $I(G)$ we can find out how many edges are contained in the induced subgraph of G corresponding to this vertex.*

Proof. A characteristical property of an induced subgraph of G containing no edges is that it has no common edge with any induced subgraph of G and thus it corresponds to an isolated vertex of $I(G)$. A characteristical property of an induced subgraph of G having exactly one edge is that it has a common edge with at least one other induced subgraph and any two induced subgraphs which have a common edge with it have also a common edge with each other. (These assertions can be easily verified by the reader.) For two induced subgraphs having both one edge each it is also easy to find out, whether they both contain the same edge or not; they have the same edge if and only if the corresponding vertices of $I(G)$ are joined by an edge. Now if $k \geqq 2$ is a positive integer, then an induced subgraph of G has exactly k edges, if and only if it has common edges with exactly k pairwise edge-disjoint induced subgraphs which have one edge each.

From the proof of Lemma 1 we see that if e is some edge of G, then the vertices of $I(G)$ corresponding to induced subgraphs of G containing e and no other edge form a clique $K(e)$ of $I(G)$ with the property that any two vertices which are joined with a vertex of this clique are also joined together and with all vertices of this clique; any clique with this property is $K(e)$ for some edge e of G. For $e_1 \neq e_2$ the cliques $K(e_1)$ and $K(e_2)$ are vertex-disjoint. Therefore there is a one-to-one correspondence between the edges of G and such cliques o) $I(G)$. Thus if $I(G)$ is given, we can find all cliques with this property in $I(G)$ and assign an element e to each of them in a one-to-one manner. This element can be considered as an edge of G. Further, we can determine for each vertex of $I(G)$, which edges of G are contained in it; they are exactly those edges which correspond to cliques with the described property.

Lemma 2. *Let G be an undirected graph without loops and multiple edges and with at least three edges. Let the intersection graph $I(G)$ of G be given and let the edges of G be determined in the above described way. Then for any three edges of G we can determine whether they form a triangle or not.*

Proof. Let e_1, e_2, e_3 be three edges of G corresponding to the cliques $K(e_1)$, $K(e_2)$, $K(e_3)$ of $I(G)$. They form a triangle in G if and only if each induced subgraph of G containing two of them contains also the third, i. e. if each vertex of $I(G)$ joined with vertices of two of the cliques $K(e_1)$, $K(e_2)$, $K(e_3)$ is joined also with a vertex of the third clique.

Lemma 3. *Let G be an undirected graph without loops and multiple edges and with at least three edges. Let the intersection graph $I(G)$ of G be given and let the edges of G be determined in the above described way. Then for any three edges of G we can determine whether they form a path of the length* 3 *or not. If they form such a path, we can determine which of them is its inner edge.*

Remark. By a path we mean a simple path whose terminal vertices are distinct.

Proof. Let e_1, e_2, e_3 be three edges of G. They form a path of the length 3 with the inner edge e_2 if and only if there exists an induced subgraph of G which contains e_1, e_2 and not e_3, an induced subgraph of G which contains e_2, e_3 and not e_1, but each induced subgraph of G containing e_1 and e_3 contains also e_2. The determination can be done again with help of the cliques $K(e_1)$, $K(e_2)$, $K(e_3)$.

Lemma 4. *Let G be a graph without loops and multiple edges, none of whose connected components is either a star, or a single edge with its terminal edges. Let e_1, e_2 be two edges of G which neither belong both to the same triangle, nor both to the same path of the length* 3. *Then e_1, e_2 are adjacent if and only if any path of the length* 3 *containing e_1 is transformed by deleting e_1 and adding e_2 to another path of the length* 3 *with the same inner edge.*

Proof. If the graph G satisfies the assumption, any edge of it belongs either to some triangle, or to some path of the length 3. Let e_1, e_2 have a common end vertex v, let v_1 (or v_2) be the end vertex of e_1 (or e_2, respectively) different from v. Since e_1, e_2 neither belong both to the same triangle, nor both to the same path of the length 3, the vertices v_1, v_2 have the degree 1. Therefore none of the edges e_1, e_2 belongs to a triangle. Thus there exist paths of the length 3 containing e_1 or e_2, but neither e_1, nor e_2 can be an inner edge of such a path. If P is a path of the length 3 containing e_1 and f is the inner edge of P, then v is the common end vertex of f and e_1 and by deleting e_1 and adding e_2 we obtain a path P' which has also the length 3 and f is its inner edge.

Now let h_1, h_2 be two edges of G with the property that any path of the length 3 containing h_1 is transformed by deleting h_1 and adding h_2 to another path of the length 3 with the same inner edge. Then neither h_1, nor h_2 can be the inner edge of any path of the length 3, because by deleting one of them and adding the other we could not obtain a path with the same inner edge. Let R be a path of the length 3 containing h_1; in R the edge h_1 is a terminal edge. Let the inner edge of R be k; then k and h_1 have a common end vertex w. The assumption implies that h_2 must have also a common end vertex w' with k. If $w' \neq w$, then it is also a common end vertex of h_2 and the other terminal edge of R and by deleting h_1 and adding h_2 from R we obtain a star with three edges and not a path. Thus $w' = w$ and this is the common end vertex of h_1 and h_2.

Remark. Two edges are called adjacent if they have a common end vertex.

Proof of Theorem 1. Let $I(G)$ be given. We reconstruct the edge set of G as described above. For any triple of edges we find out whether they form a triangle (Lemma 2) or a path of the length 3 (Lemma 3); if they form a path of the length 3, we find out, which of them is the inner edge of it. Now let us have two edges e_1, e_2 of G. If they both belong to the same triangle, they are adjacent. If they both belong to the same path of the length 3, they are adjacent, if and only if one of them is the inner edge of this path. If they neither both belong to the same triangle, nor both to the same path of the length 3, we find out whether they are adjacent or not by using Lemma 4. Thus we obtain the set of edges of G and the relation of adjacency on it. If G is without isolated vertices and no connected component of G is a triangle or a star with three edges, the graph G can be uniquely reconstructed according to Whitney's theorem [5, 7], which was generalized for infinite graphs by H. A. Jung [3]. If a graph contains connected components which are stars with three edges or triangles, according to Whitney's theorem all connected components which are not of this kind can be reconstructed and, having a connected component of this kind, we can recognize that it is of this kind, but we cannot distinguish, whether it is a triangle, or a star with three edges. But in our case we have excluded stars and determined all triangles, therefore we can determine G.

Conjecture. *Let G be a finite undirected graph without loops and multiple edges and with at least four vertices. Then G is uniquely determined by its intersection graph $I(G)$.*

If we wish to prove this conjecture we can consider the number of the independent subsets of a graph as a possible way, because isolated vertices of $I(G)$ correspond to independent subsets of the vertex set of G.

Figs. 1 and 2 we see two finite graphs G_1, G_2 not satisfying the assumption of Theorem 1. In Figs. 3 and 4 we see the graphs $I(G_1)$ and $I(G_2)$; they are non-isomorphic.

The situation changes substantially, if we construct infinite graphs G_1' and G_2' by adding $\aleph_0$ isolated vertices to the graphs G_1 and G_2, respectively. We have $I(G_1') \cong I(G_2')$. This graph consists of three cliques K_1, K_2, K_3, each of which has $\mathfrak{c}$ vertices ($\mathfrak{c}$ is the power of continuum), of all edges joining a vertex of K_3 with a vertex of K_1 or K_2, and of c isolated vertices. Thus we have proved a theorem.

Fig. 1

Fig. 2

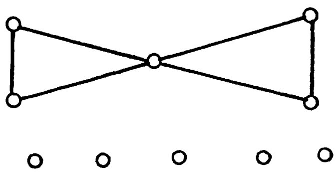

Fig. 3

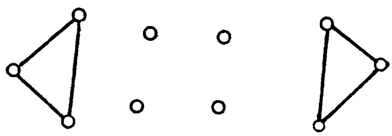

Fig. 4

Theorem 2. *There exist two infinite undirected graphs without loops and multiple edges which are not isomorphic, but whose intersection graphs are isomorphic.*

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Received April 7, 1972

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