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REMARKS ON A NONLINEAR THEORY OF THIN ELASTIC PLATES

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In this paper we shall discuss a certain two-dimensional boundary value problem which arises when we investigate the equilibrium of a built-in plate lying in the plane xy and subjected to a load q perpendicular to the plane xy and to forces $g_1, g_2$ acting in the plane xy. When denoting the axes in a three-dimensional Euclidean space by $x, y, z$ then the $u, v, w$ will denote the displacements parallel to them, respectively. In the paper we shall work in real spaces and with real functions.

Terminology and notation. For simplicity let $\Omega$ be a bounded region in the plane xy with a Lipschitz boundary. Let us denote by $\mathcal{E}(\Omega)$ the space of infinitely many times differentiable functions on $\Omega$ which are continuously prolongable with all their derivatives to $\Omega$. $\mathcal{L}(\Omega)$ are functions of $\mathcal{E}(\Omega)$ with a compact support in $\Omega$. Let $W_p^{(k)}(\Omega)$ be a system of functions having all generalized derivatives up to the $k$-th order integrable with the $p$-th power in $\Omega$. $W_p^{(k)}(\Omega)$ with the norm $|u|^p_{W_p^{(k)}(\Omega)} = \left(\sum_{i=0}^{k} |u^{(i)}|^p_{L_p(\Omega)}\right)^{1/p}$ (addition through all derivatives) is a Banach space. Let the closure of $\mathcal{L}(\Omega)$ in the $W_p^{(k)}(\Omega)$ norm be denoted by $W_p^{(k)}(\Omega)$. In the following we shall write $W_p^{(k)}$ instead of $W_p^{(k)}(\Omega)$.

Let $W = W_p^{(2)} \times W_p^{(1)} \times W_p^{(1)}$ (a Cartesian product of spaces) and let us define for $\tilde{u} = (w, u, v) \in W$ (where $w \in W_p^{(2)}, u \in W_p^{(1)}, v \in W_p^{(1)})$ the norm by

$$||\tilde{u}||_W^2 = |w|_{W_p^{(2)}}^2 + |u|_{W_p^{(1)}}^2 + |v|_{W_p^{(1)}}^2.$$

Put $V = W_p^{(2)} \times W_p^{(1)} \times W_p^{(1)}$. Let $P_1$ be the space of all polynomials of the order $\leq 1$ and $P \subseteq P_1 \times P_1 \times P_1$, $P$ generated by the vectors $(0, 1, 0), (0, 0, 1), (0, y, x)$. That means the polynomials in question are of the type $\tilde{p} = (0, a + \lambda y, b - \lambda x)$. Let us denote by $V/P$ the space of classes $\tilde{u}$ of functions $\tilde{u} \in V$; $\tilde{u}, \tilde{v} \in \tilde{u} \Rightarrow \tilde{u} - \tilde{v} \in P$. The norm in $V/P$ we define as usual

$$||\tilde{u}||_{V/P} = \inf_{\tilde{u} \in V} ||\tilde{u}||_V.$$

Statement 1. $V/P$ with this norm is a Hilbert space (hence $V$ is reflexive)

Proof. Let $V = P + R$ (direct sum). If $\tilde{u} \in V/P$, there is only one element
\( u_r \in R \) such that for any \( \tilde{u} \in \tilde{u} \) there is \( \tilde{u} = \tilde{u}_p + u_r \). In particular, \( u_r \in \tilde{u} \) (because \( \tilde{u} = \tilde{u}_p + u_r \in P \) for \( u \in \tilde{u} \)).

Now it is clear that the scalar product in \( V \mid P \) may be defined in the following way

\[
\langle (\tilde{u}, \tilde{v}) \rangle_P = (u_r, v_r)
\]

and we have \( (\tilde{u}, \tilde{v}) \rangle_P - \inf_{\tilde{u} \in \tilde{u}} \left\| \tilde{u} \right\|_V \inf_{\tilde{v} \in \tilde{v}} \left\| \tilde{v} \right\|_V \left( \left\| \tilde{u}_p \right\|^2 + \left\| u_r \right\|^2 \right) \tilde{u}_r \).

Now, let \( q \in L_2(\Omega), g_1 \in L_2(\tilde{\Omega}), g_2 \in L_2(\Omega) \) where by \( \tilde{\Omega} \) we denote the boundary of \( \Omega \).

We shall study the existence of a weak solution of the following system of equations (system which describes the physical problem mentioned at the beginning)

\[
D \Delta^2 w = \frac{\partial^2 w}{\partial x^2} \sigma_x + \frac{\partial^2 w}{\partial y^2} \sigma_y + 2 \frac{\partial^2 w}{\partial x \partial y} \tau + \frac{q}{h}
\]

(1)

\[
\frac{\partial \sigma}{\partial x} + \frac{\partial \tau}{\partial y} = 0,
\]

\[
\frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0,
\]

where

\[
\sigma_x = \frac{E}{1 - \mu^2} \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \mu \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right],
\]

\[
\sigma_y = \frac{E}{1 - \mu^2} \left[ \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 + \mu \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) \right],
\]

\[
\tau = \frac{E}{2(1 + \mu)} \left[ \frac{\partial w}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial w}{\partial y} \right],
\]

\( h \) the plate thickness

\( E \) the compression modulus of elasticity

\( \mu \) the Poisson number

\( D \) the plate stiffness

under the boundary conditions

\[
w \frac{\partial w}{\partial n} = 0, \quad \text{on } \tilde{\Omega},
\]

63
\[
\begin{align*}
\sigma_{x\mu_x} + \tau n_y &= g_1 \\
\tau n_x + \sigma_{y\mu_y} &= g_2
\end{align*}
\] on \(\Omega\),

\(n_x, n_y\) are the components of a normal to \(\partial\Omega\).

Remark. The equations (1) are to be satisfied in the sense of distributions.

The vector \((w, u, v) \in V\) is a weak solution of the given boundary value problem if for any vector \((\bar{w}, \bar{u}, \bar{v}) \in V\) there is

\[
\int_{\Omega} D \left( \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 \bar{w}}{\partial x^2} + 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 \bar{w}}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 \bar{w}}{\partial y^2} \right) \, dx \, dy + \\
\int_{\Omega} \left( \frac{\partial^2 w}{\partial x^2} \frac{\partial x}{\partial x} + \frac{\partial^2 w}{\partial y^2} \frac{\partial y}{\partial y} + 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial x}{\partial x} \frac{\partial y}{\partial y} \right) \bar{w} \, dx \, dy + \\
+ \int_{\Omega} \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \right) \frac{\partial \bar{w}}{\partial x} \, dx \, dy \\
\frac{1}{h} \int_{\Omega} q \bar{w} \, dx \, dy \left( \int_{\Omega} g_1 \bar{u} \, ds \right) \left( \int_{\Omega} g_2 \bar{v} \, ds \right) = 0.
\]

(In general, for the definition of a weak solution see e. g. [1]).

Rearranging the second integral (using integration by parts) we obtain that the vector \(\tilde{\alpha} = (w, u, v) \in V\) is a weak solution of the given problem if the following equation holds for any \(\tilde{\beta} = (\bar{w}, \bar{u}, \bar{v}) \in V\)

\[
F(\alpha) \tilde{\beta} = \int_{\Omega} D \left( \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 \bar{w}}{\partial x^2} + 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 \bar{w}}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 \bar{w}}{\partial y^2} \right) \, dx \, dy + \\
+ \int_{\Omega} \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \right) \frac{\partial \bar{w}}{\partial x} \, dx \, dy \\
\frac{1}{h} \int_{\Omega} q \bar{w} \, dx \, dy \left( \int_{\Omega} g_1 \bar{u} \, ds \right) \left( \int_{\Omega} g_2 \bar{v} \, ds \right) = 0.
\]

It is easy to verify that the operator \(F(\tilde{\alpha}) \in [V \to V^*]\) defined by this equation...
is the potential (see [2]). Hence there exists a functional \( g(\tilde{z}) \) for which the following condition must be satisfied

\[
\nabla g(\tilde{z}) = F(\tilde{z}).
\]

The equation \( F(\tilde{z}) \not\equiv 0, \forall \tilde{\beta} \in V \) now implies that we can investigate critical points of \( g(\tilde{z}) \) instead of solving (2). By a calculation it is found that

\[
g(\tilde{z}) = \int_{\Omega} D \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + 2 \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 \right] dx dy +
\]

\[
+ \int_{\Omega} E \left[ \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 + (1 - \mu) \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right] dx dy +
\]

\[
+ \frac{1}{2} \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 dx dy + \int_{\Omega} E \left[ \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right] dx dy +
\]

\[
+ \int_{\Omega} E \left[ \frac{\partial u}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial w}{\partial y} \right] dx dy - \frac{1}{h} \int_{\Omega} w dx dy - \int_{\Omega} g_{1u} d\Omega - \int_{\Omega} g_{3v} d\Omega.
\]

Let us denote the integrals on the right-hand side of (3) by \( J_1, \ldots, J_8 \) respectively so that \( g(\tilde{z}) = \sum_{j=1}^{5} J_j - \sum_{j=6}^{8} J_j \) and let us consider the functional \( f(\tilde{z}) = g(z) + \sum_{j=6}^{8} J_j \).

In [2] it is shown that \( f(\tilde{z}), g(\tilde{z}) \) are weakly lower semicontinuous on \( V \). The functional \( g(\tilde{z}) \) may further be written in the form

\[
g(\tilde{z}) = \int_{\Omega} D \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + 2 \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 \right] dx dy +
\]

\[
+ \int_{\Omega} E \left[ \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right] dx dy +
\]

\[
+ \int_{\Omega} E \left[ \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right] dx dy +
\]

65
In the following, let $g_1, g_2$ be such elements of $L_2(\Omega)$ that
\[ \int_\Omega g_1(a + \lambda y) \, ds = 0, \quad \int_\Omega g_2(b + \lambda x) \, ds = 0. \]

Now let us define the functional $G(\tilde{z})$ in $V P$ as follows for $\tilde{z} \in V P$ putting $G(\tilde{z}) = f(\tilde{z})$, where $\tilde{z} \in V$, $\tilde{z} \in \tilde{z}$ is arbitrary. One can see from the form of the functional $f(\tilde{z})$ that the definition is meaningful.

**Statement 2.** $G(\tilde{z})$ is weakly lower semicontinuous on $V P$.

**Proof.** Let $\tilde{z}_n \rightharpoonup \tilde{z}_0$ in $V P$ (where the symbol $\rightharpoonup$ denotes a weak convergence), i.e. for any $\tilde{u} \in V P$
\[ (\tilde{z}_n, \tilde{u})_{VP} \to (\tilde{z}_0, \tilde{u})_{VP}. \]

According to the definition
\[ (*) \quad (\tilde{z}_n, \tilde{u})_{VP} = (\tilde{z}_0, \tilde{u})_{VP}. \]

Therefore it is sufficient to show
\[ (\tilde{z}_n + \tilde{u}) \to (\tilde{z}_0, \tilde{u}) \text{ for any } \tilde{u} \in V. \]

(Namely, using the weak lower semicontinuity of $f(\tilde{z})$ we obtain the desired result.)

This, however, follows by (*) and by the fact that
\[ \tilde{u} - u_p \to \tilde{u}_r, \quad u_p \in P, \quad u_r \in R, \quad P \subset R. \]

**Remark.** Let us use the notation $U = (u, v)$; by the space $V P$ we may understand the space $W^{(2)}_2 \times (W^{(1)}_2)^2 / P'$, where $P'$ is the space of polynomials of the type $\{a + \lambda y, b - \lambda x\}$. Now, the integral $J_1$ is equivalent to the norm of the element $w$ in $W^{(2)}_2$ (see e.g. [1]); $J_2$ on the other hand is equivalent to the norm of the class $U = (\tilde{u}, \tilde{v})$ in $(W^{(1)}_2)^2 / P$. Here, the inequality $J_1 \leq c \|U\|^2$ is evident and the inequality $J_2 \geq c \|U\|^2$ can be obtained from Korn's inequality (see [3]). We shall therefore write $\|w\|_{W^{(2)}_2}$ instead of $J_1$ and $\|U\|^2$ instead of $J_2$. 

66
Theorem 1. There is

$$\liminf_{\tilde{u} \to \infty} \frac{G(\tilde{u})}{|\tilde{u}|} = c > 0.$$ 

Proof. From formula (3) we obtain

$$G(\tilde{u}) = w^{\frac{p}{p+2}}_m \tilde{U}^{\frac{p}{p+1}}_m R(\tilde{u}),$$

where $R(\tilde{u}) > J_4$ (because $J_3 \geq 0$). From formula (4) we obtain

$$G(\tilde{u}) = w^{\frac{p}{p+2}}_m k(\tilde{u}),$$

where $k(\tilde{u}) = 0$. Let us estimate $I_4 + I_5$ using Schwartz's inequality. (Note that $w \in L_1$ and that $w w^c = c w^{\frac{p}{p+2}}$ where $c$ does not depend on $w$; these facts follow from the Sobolev imbedding theorems.)

Let $\tilde{u} \in \tilde{u}$ be arbitrary. Then

$$J_4 \leq c_1( w^{\frac{p}{p+2}}_m + v^{\frac{p}{p+2}}_m) \leq c_1(\|U\| \|w\|^{\frac{p}{p+2}}_m),$$

$$J_5 \leq c_2( w^{\frac{p}{p+2}}_m - v^{\frac{p}{p+2}}_m) \leq c_2(\|U\| \|w\|^{\frac{p}{p+2}}_m),$$

so that we have

$$J_4 + J_5 \leq c |U| \|w\|^{\frac{p}{p+2}}_m \text{ for any } \tilde{u} \in \tilde{u},$$

i.e.,

$$J_4, J_5 \leq c |\tilde{U}|_{V_1}[\|w\|^{\frac{p}{p+2}}_m].$$

Let now $\tilde{u} \not\in P$ for $r > 0$ (we can consider $r > 1$).

$$G(\tilde{u}) \geq r^2 c |\tilde{U}|_{V_1} \|w\|^{\frac{p}{p+2}}_m \geq r^2 c r \|w\|^{\frac{p}{p+2}}_m \geq r \alpha$$

for those $\tilde{u}$ satisfying $|w|^{\frac{p}{p+2}}_m \leq (r - \alpha)/c$ (we can choose a convenient $\sigma > 0$). In this case we can see that

$$G(\tilde{u}) \geq \alpha.$$ 

If $w \not\in \mathcal{A}$, using formula (6) we obtain an estimate

$$G(\tilde{u}) \geq r \alpha k(\tilde{u}) \geq r - \alpha,$$

so that

$$G(\tilde{u}) \geq r \alpha c \geq \frac{1}{c},$$

$$\frac{1}{c} \geq \frac{1}{c} \alpha \geq \frac{1}{c} \alpha 1 - \alpha \geq \frac{1}{c}.$$
In any case we have
\[ G(\tilde{u}) \geq \min \left( \alpha, \frac{1 - \alpha}{c} \right). \]

**Theorem 2.** If \( g_1 - g_2 = 0 \), then for any \( q \in L_2(\Omega) \) there exists a solution of the problem in question.

**Proof.** When writing \( \frac{1}{h} \int_{\Omega} qw \, d\Omega = \langle w, q \rangle \) it is sufficient to prove \n
\[ \lim_{|q| \to \infty} \inf \left( G(\tilde{u}) - \langle w, q \rangle \right) = + \infty \]

because \( G(\tilde{u}) - \langle w, q \rangle \) is a lower weakly semicontinuous functional in a reflexive Banach space \( V|P \), thus by (*) it has an absolute minimum on \( V P \) and the point that minimizes \( G(\tilde{u}) - \langle w, q \rangle \) is a solution of the given boundary value problem with \( g_1 = g_2 = 0 \) (see e.g. [4]). Let us prove (*).

For any \( K > 0 \) we shall find \( R > 0 \) such that for \( \|\tilde{u}\| \geq R \)

\[ G(\tilde{u}) - \langle w, q \rangle \geq K. \]

We have \( \|\tilde{u}\|^2 = \|w\|^2 + \|\tilde{U}\|^2 \); let \( r_1 \geq \max (2\|q\|, \frac{K}{|q|}) \).

For \( \|w\| \geq \|\tilde{w}\| \geq r_1 \) using formula (6) we obtain

\[ G(\tilde{u}) - \langle w, q \rangle \geq \|w\|^2 - \|\tilde{w}\|^2 + k(\tilde{u}) - \|w\|^2 \|\tilde{u}\| L_4(\Omega) \geq |w| (|w| - |q|) \geq r_1 r_1 - |q| \geq K. \]

If \( \|w\| \leq r_1 \), then using (5) we obtain

\[ G(\tilde{u}) - \langle w, q \rangle \geq \|\tilde{u}\|^2 - \|w\| |q| - c \|\tilde{U}\| \|w\|^2 \geq \|\tilde{U}\|^2 - r_1 |q| - c \|\tilde{U}\| r_1 \geq \|\tilde{U}\| (\|\tilde{U}\| - cr_1^2) - r_1 |q|. \]

If we now choose \( r_2 > 0 \) such that

\[ r_2 (r_2 - Cr_1^2) - r_1 |q| \geq K, \]

then for \( \|\tilde{U}\| \geq r_2 \) we have \( G(\tilde{u}) - \langle w, q \rangle \geq K \).

Finally put \( R^2 = r_1^2 + r_2^2 \); then for \( \|\tilde{u}\| \geq R \) there is \( \|\tilde{U}\|^2 \geq R^2 - \|w\|^2 \) and for \( \|w\|^2 \geq r_1^2 \) the relation (7) is true; for \( \|w\|^2 \leq r_1^2 \) we have \( \|\tilde{U}\|^2 \geq r_1^2 + r_2^2 - r_1^2 \) and \( r_2^2 \), so that (7) is true again, what was to be proved.

In the following considerations we shall study a wider problem:

Let \( H \) be a Hilbert space and such that \( V|P \) is a subspace of \( H \). Let \( F \) be a bounded linear functional on \( H \). Instead of the symbol \( \tilde{u} \) we shall simply
write \( u \) and instead of \( G(\tilde{u}) \) we shall write \( f(\tilde{u}) \) (according to the definition of \( G(\tilde{u}) \)).

We shall put

\[
M_s = \{ F \in H^*; \lim_{\| \tilde{u} \|_P \to \infty} \inf (f(\tilde{u})) (\tilde{u}, F) = + \infty \};
\]

\[
M_t = \{ F \in H^*; \lim_{\| \tilde{u} \|_P \to \infty} \inf (f(\tilde{u})) (\tilde{u}, F) = c \neq \pm \infty \};
\]

\[
M_t = \{ F \in H^*; \lim_{\| \tilde{u} \|_P \to \infty} \inf (f(\tilde{u})) (\tilde{u}, F) = - \infty \}.
\]

We shall show that \( M_s \neq \emptyset \); for \( F \in M_s \) there exists an absolute minimum of the functional \( f(\tilde{u}) - (\tilde{u}, F) \), hence a solution of certain boundary value problem as it follows from the foregoing considerations (especially from the proof of Theorem 2.)

**Theorem 3.** The set \( M \) is convex.

**Proof.** Let \( F_1, F_2 \in M_s \); then for \( F = (1 - \lambda)F_1 + \lambda F_2 (0 < \lambda < 1) \) we have

\[
f(\tilde{u}) (\tilde{u}, F) = f(\tilde{u}) - (1 - \lambda) f(\tilde{u}) (\tilde{u}, F_1) - \lambda f(\tilde{u}) (\tilde{u}, F_2) =
\]

\[
(1 - \lambda) f(\tilde{u}) (\tilde{u}, F_1) + \lambda f(\tilde{u}) (\tilde{u}, F_2),
\]

hence \( F \in M_s \).

For \( F \in H^* \), \( |F| = 1 \) we define a real-valued function corresponding to the chosen \( H \) in the following way:

\[
\lambda_H(F) = \sup \{ \sigma; \sigma F \in M_s \}.
\]

Then \( \lambda_H(F) > 0 \) (from this it is clear that \( M_s \neq \emptyset \)). Namely, by Theorem 1 the existence of such \( R > 0 \) follows that for \( \| \tilde{u} \|_P > R \) we have \( f(\tilde{u}) \geq \alpha \tilde{u} \| \tilde{u} \|_P \) (for some \( \alpha > 0 \)) so that all right-hand sides with a sufficiently small norm belong to the \( M_s \). From this there also follows an existence of a neighbourhood of zero at \( H^* \), the whole belonging to the \( M_s \).

\[
\left( \text{Really, } f(\tilde{u}) (\tilde{u}, F) \geq \alpha \tilde{u} \| \tilde{u} \|_P \quad \text{and} \quad \| \tilde{u} \|_P \| F \|_H; \quad \left\{ F, \| F \|_H \leq \frac{\alpha}{2c} \right\} \subset M_s \right).
\]

We shall prove several theorems concerning \( \lambda_H(F) \).

**Theorem 4.** If \( \sigma > \lambda_H(F) \), then \( \sigma F \in M_1 \) (so that for \( \sigma > \lambda_H(F) \) there is no absolute minimum of \( f(\tilde{u}) - \sigma(\tilde{u}, F) \)).

**Proof.** Let \( \sigma > \lambda_H(F) \). If \( \lim_{|\tilde{u}| \to \infty} \inf (f(\tilde{u}) - \sigma(\tilde{u}, F)) = c \neq \pm \infty \), then for \( \sigma > \sigma_1 > \lambda_H(F) \) there should be \( \lim_{|\tilde{u}| \to \infty} \inf (f(\tilde{u}) - \sigma_1(\tilde{u}, F)) = K \neq \pm \infty \). Really, \( \lim_{|\tilde{u}| \to \infty} (f(\tilde{u}) - \sigma_1(\tilde{u}, F)) \) cannot hold because \( \sigma_1 > \lambda_H(F) \) and if \( \lim_{|\tilde{u}| \to \infty} (f(\tilde{u}) - \sigma(\tilde{u}, F)) = - \infty \) then
\[
\infty = \liminf_{|u| \to \infty} (f(u)) \geq \liminf_{|u| \to \infty} (f(u)) \quad c \equiv \infty
\]

Moreover, we have
\[
f(u) \sigma_1(u, F) = (f(u) - \sigma(u, F)) \frac{\sigma_1}{\sigma} + f(u) \begin{pmatrix} 1 & \sigma_1 \\ \sigma & \sigma \end{pmatrix}
\]
and
\[
\liminf_{|u| \to \infty} (f(u)) \sigma_1(u, F) \geq \sigma_1 \liminf_{|u| \to \infty} \left[(f(u)) \sigma(u, F) \sigma \begin{pmatrix} 1 & \sigma_1 \\ \sigma & \sigma \end{pmatrix}\right]
\]
which means that
\[
\pm \infty + K \geq \sigma_1 C + \infty \begin{pmatrix} 0 \in M_s \to \liminf_{|u| \to \infty} f(u) \begin{pmatrix} 1 & \sigma_1 \\ \sigma & \sigma \end{pmatrix} \infty \end{pmatrix}
\]
which is a contradiction.

**Theorem 5.** Function \( \lambda_H(F) \) is continuous.

**Proof.** At first let \( F_0 \) be such that \( \lambda_H(F_0) < + \infty \). Let \( \varepsilon > 0 \) be arbitrary (but fixed). By the preceding there is \( r : 0 < r < \lambda_H(F_0) \) such that \( D \) \( \{F; \ F_H < \leq r\} \subset M_s \). Let \( s \) take a cone \( C \) \( \{aF; a \geq 0, F \in D - \lambda_H(F_0)F_0\} \lambda_H(F_0)F_0 \) so that \( C \) is a convex cone with a vertex at the point \( \lambda_H(F_0)F_0 \) and containing all the points of \( D \). From the convexity of \( M_s \) it follows that the points of the type \( 2\lambda_H(F_0)F_0 + (1 - \alpha)F, F \in D, \alpha \in (0, 1) \) belong to \( M_s \). Furthermore, let \( K \) be another cone, \( K \) \( \{aF; a \geq 0, F \in D - \lambda_H(F_0)F_0\} + \lambda_H(F_0)F_0 \). One can easily see that \( V \) \( \{aF; a \geq 0, F \in K \cap \{F; F \lambda_H(F_0)F_0 - \varepsilon\} \) is a convex cone with a vertex at the origin and \( F_0 \in \text{Int} \ V \). Now, the set \( \text{Int} \ V \cap \{F; \ |F| < 1\} \) is the neighbourhood of \( F_0 \) we were looking for.

Certainly, let \( F \in \text{Int} A: \lambda_H(F_0)F \) lies on the ray \( aF, a > 0 \). We must show that \( \lambda_H(F)F \) lies in the \( \varepsilon \)-neighbourhood of \( \lambda_H(F_0)F_0 \). But it is clear that \( \lambda_H(F) < \lambda_H(F_0) \) \( \varepsilon \) is impossible (by the definition of \( \lambda_H(F) \) and for all the interior points of \( M \) \( \{aF; 1 \geq a \geq 0, F \in D - \lambda_H(F_0)F_0\} + \lambda_H(F_0)F_0 \) belong to \( M_s \) and in the case of \( \lambda_H(F)F \) \( \varepsilon \) the point \( \lambda_H(F_0)F \) would lay in \( \text{Int} \ M \) and having \( \lambda_H(F) > \lambda_H(F_0) + \varepsilon \) we can easily find that on the ray \( aF_0, a \geq 0 \) there is a point \( \sigma F_0 \in M_s \) with \( \sigma > \lambda_H(F_0) \), which is a contradiction.

Now let \( \lambda_H(F_0) \) \( \infty \). Choose \( R > 0 \) and consider a „cone“ \( K \) \( \{aF; 1 \geq a \geq 0, F \in D - 2RF_0\} + 2RF_0 \). It is clear that \( \text{Int} K \subset M_s \). Now for all \( F \in V \cap \{F; |F| < 1\} \) we have \( \lambda_H(F) > R \), where \( V \) \( \{aF; a \geq 0, F \in \{x; |x| > R\} \cap K\} \) and \( V \cap \{F; |F| < 1\} \) is the neighbourhood we were looking for.
Finally we shall mention another property of the function $\lambda_H(F)$. Let $B \subset H$ and let us suppose that the identical imbedding $H \to B$ is totally continuous. Let $B_n \subset B$, closed subspaces of $B$ ($n = 1, 2, \ldots$) such that $\lim_{n \to \infty} B_n = B$ (i.e. $\forall v \in B \exists v_n \in B_n [\lim_{n \to \infty} v = v_n \in B]$).

The following theorem is true

**Theorem 6.** If we denote by $D_n = \{F \in B^*; v \in B_n \to Fv = 0\}$ then

$$\lim \inf_{n \to \infty} \lambda_{B^*}(F) = \infty$$

**Proof.** It is sufficient to prove that

$$\lim_{n \to \infty} \sup_{F \in D_n} F_{H^r} = 0.$$

Let us suppose that this does not hold. Then there exists such an $F_n \in D_n$ that $F_n \in H^r$ for some $\varepsilon > 0$. Let $v \in B$ be arbitrary; then $F_n v = F_n v_n + F_n (v - v_n) > 0$ so that $F_n \to 0$ in $B^*$. But the identical imbedding $B^* \to H^r$ is totally continuous hence $F_n \to 0$ in $H^r$ so that $F_n \to 0$, which is a contradiction.

**REFERENCES**


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