Juraj Virsik
Concerning Connections on Associated Bundles


Persistent URL: [http://dml.cz/dmlcz/126960](http://dml.cz/dmlcz/126960)

---

**Terms of use:**

© Mathematical Institute of the Slovak Academy of Sciences, 1970

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use.*

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* [http://project.dml.cz](http://project.dml.cz)
CONCERNING CONNECTIONS ON ASSOCIATED BUNDLES

JURAJ VIRSIK, Bratislava

A connection in a principal fibre bundle induces a vertical projection in each of the tangent spaces to each associated fibre bundle, this projection being de facto the covariant differential of sections in this associated bundle. It is even this covariant differentiation which is important in practical computations with connections, e.g. on vector bundles. Suppose there are two fibre bundles associated to the same (or locally the same, cf. below) principal fibre bundle in which a connection is given. Then the two covariant differentiations induced in both associated bundles are in some way mutually related. If there is namely a bundle morphism of these bundles satisfying a general enough condition, then it preserves the covariant differentiations. This follows from the statements of propositions 1 and 1a which allow us to obtain a number of results concerning the behaviour of the covariant differential in concrete cases, usually proved independently (cf. the examples below).

We consider the case of $C^\infty$ differentiability and finite dimension throughout the paper. Suppose $H(B, G)$ to be a principal bundle over $B$ with structure group $G$ and projection $b : H \to B$, and let there be given a (first order) connection in $H$ given by a differentiable distribution $h \to T(H)_h \subset T(H)_h$ on $H$. Here $T(H)$ is the tangent bundle to $H$ and we denote by $T(H)_h$ the vertical subspace of $T(H)_h$, i.e. $X \in T(H)_h \implies (db)X = 0$, where $0$ denotes the zero element in $T(B)_x$. Thus we have for each $h \in H$ the direct decomposition $T(H)_h = T(H)_h^v \oplus T(H)_h^h$ and $T(H)_h^v \approx R_g^* T(H)_h^h$ with $g \in G$. Denote by $V_H$ the canonical vertical projection $V_H : T(H) \to T(H)$ given by the connection.

Suppose now that $E \to E(B, F, G, H)$ is a fibre bundle associated to $H$ and denote $\lambda : H \times F \to E$ and $p : E \to B$ the natural projections. For each $\eta \in T(E)_z$ denote by $T(E)_z \subset T(E)_z$ the subspace of vertical elements, i.e. $\eta \in T(E)_z$.

(d$p$) $\eta = 0_{p\eta}$. Since the projection $\lambda$ satisfies $\lambda(R_g \times L_{g^{-1}})$ for each $g \in G$, where $L_g : F \to F$ denotes the action on the left of the element $g \in G$, we have

$$d\lambda(R_g^* + L_{g^{-1}}^*) - d\lambda . T(H) \oplus T(F) \to T(E).$$
For a fixed $f \in F$ denote $i_f : H \times F \to H \times X$ the natural injection $h \mapsto (h, f)$ and let $\mu_f : \mu_f : H \to E$. Note that $d\mu_f$ takes the element $X \in T(H)_h$ into $X + 0_f \in T(H)_h \oplus T(F)_f$. The linear mapping $d\mu_f$ takes each $T(H)_h$ into $T(E)_x$ where $x = h(f)$ since

\[ (2) \quad \mu_f(h) = h(f). \]

On the other hand $p \mu_f \circ \delta = d\mu_f \circ db$ and this means that $d\mu_f$ takes also $T(H)$ into $T(E)$ whatever $f \in F$ and that $Ker d\mu_f \subset T(H)$.

Now we may define $T(E)_x \cap T(E)_x = T(E)_x$. Note that $\mu_f$ is an isomorphism if restricted to $T(H)_h$, but it need not be either an injection or projection of the whole $T(H)_h$. It might be interesting to find the explicit conditions under which it is.

**Lemma 1.** $T(E)_x \cap T(E)_x = T(E)_x$.

**Proof.** We have $T(E)_x = d\mu_f(T(H)_h)$ for some pair $(h, f) \in \lambda \cdot 1(x)$ and hence $dim T(E)_x = dim T(H)_h + dim T(E)_x$. Suppose $\xi \in T(E)_x \cap T(E)_x$, i.e. $\xi \in (d\mu_f)X$ where $X \in T(H)_h$ and $d\xi \neq 0$. Then $0 = dp(d\mu_f)X = d\delta X = X + T(H)_h$ and hence $X = 0_h$, $\xi = 0_x$.

Remark. We cannot write in general $T(E)_x = d\mu_f(T(H)_h)$, since $d\mu_f$ is an isomorphism if restricted to $T(H)_h$, but it need not be either an injection or projection of the whole $T(H)_h$. It might be interesting to find the explicit conditions under which it is.

First it is clear that $d\lambda$ is always a projection. Let us find its kernel. We make use of the following well-known statement: „Let $\Phi : M \to X$ be a differentiable projection of manifolds which is of maximal rank at $m \in M$. Then the points of $M$ in which $\Phi$ takes the constant value $n \Phi(m)$ in a sufficiently small neighbourhood of $m$ form an imbedded submanifold $W \subset M$, $n \subset W$.

The tangent space $T(W)_m \subset T(M)_m$ is exactly the kernel of $d\Phi$ at $m \in M$.” Hence the kernel of $d\lambda$ at $(h, f)$ is the tangent space of a submanifold $W \subset H \times F$ consisting of pairs $(hg, g^{-1}f)$, $g \in G$. In other words, it consists of exactly all the tangent vectors to the curves $t \mapsto (hg(t), g(t)^{-1}f)$ at $t = 0$ where $t \mapsto g(t)$ is a curve in $G$, $g(0) = e$ and $e$ is the identity of $G$. Denote by $\iota : T(G)_e \to T(H)_h$ the canonical isomorphism of the Lie algebra of $G$ onto the tangent space to the fibre in $H$ at $h$. Further denote $j_f : T(G)_e \to T(F)$ the canonical projection of the Lie algebra of $G$ onto the tangent space at $f$ to the orbit of $G$ on $F$ containing $f$. Note that $j_f$ is an injection iff $G$ acts freely on this orbit. Now using these notations we see easily that $Ker d\lambda$ consist of exactly all the elements

$$\iota(h(Z)) + j_f d\sigma(Z) \quad \iota(h(Z)) \quad j_f(Z),$$

where $Z$ is any element of $T(G)_e$ and $\sigma : \theta \to \theta^1$, i.e. $d\sigma(Z) = Z$.
These results yield the required characterization of the mapping $d\mu_f$. We see namely that $d\mu_f$ is injective iff there does not exist such $Z \in T(G)_e$ that
\[ j_f(Z) = 0, Z \neq 0, \text{ i.e. iff } G \text{ acts freely on the orbit containing } f \in F. \]
Since $d\Sigma$ is always projective we there are for each $\xi \in T(E)_x$ elements $X \in T(H)_h$ and $Y \in T(F)_f$ such that $\xi = d\Sigma(X + Y)$. All the elements in $T(H)_h \oplus T(F)_f$, which are taken into $\xi$, are of the form $X + \iota_h(Z) + Y j_f(Z)$. The mapping $d\mu_f$ is projective iff one can find such $Z \in T(G)_e$ that $Y j_f(Z)$ for any $Y \in T(F)_f$. Hence $d\mu_f : T(H)_h \to T(E)_x$ is projective iff $f \in F$ is an interior point of the corresponding orbit of $G$ on $F$.

Returning to the general situation. Lemma 1 gives for each $\xi \in T(E)_x$ the unique decomposition
\[ \xi = d\mu_f(X) + \eta, \]
such that $d\mu_f(\eta) = 0$ and $V_H X = 0$. Thus denoting by $V_E : T(E) \to T(E)$ the induced projection $V_E(\xi) = \eta$, we have
\[ V_E d\mu_f = d\mu_f V_H. \]

We proceed now as follows. Let $H = B(B, G)$ and $\tilde{H} = \tilde{B}(B, \tilde{G})$ be principal fibre bundles over $B$ and let $\varphi : H \to \tilde{H}$ be a homomorphism of them. Let in $H$ and $H$ connections be given with the corresponding vertical projections satisfying
\[ V_H d\varphi = d\varphi V_H. \]
If $E = E(B, F, G, H)$ and $\tilde{E} = E(\tilde{B}, \tilde{F}, \tilde{G}, \tilde{H})$ are fibre bundles associated to $H$ and $\tilde{H}$ respectively, call a bundle morphism $\gamma : E \to \tilde{E}$ ($\gamma_0$, $\varphi$)-typed if there exists a differentiable mapping $\gamma_0 : F \to \tilde{F}$ such that for each $\hat{h} \in H$, $\hat{f} \in F$ the relation
\[ \gamma(h(f)) = \varphi(h)(\gamma_0(f)) \]
holds. If we introduce analogously the mappings $\tilde{\mu}_f : \tilde{H} \to \tilde{E}$ for each $\hat{f} \in \tilde{F}$ we get according to (2) the equivalent to (6) form
\[ \gamma \mu_f(\hat{h}) \equiv \tilde{\mu}_{\gamma_0(\hat{h})}(\varphi(\hat{h})), \]
which implies
\[ d\gamma d\mu_f = d\tilde{\mu}_{\gamma_0(\hat{f})} d\varphi. \]

**Proposition 1.** Let $\gamma : E(B, F, G, H) \to E(\tilde{B}, \tilde{F}, \tilde{G}, \tilde{H})$ be a ($\gamma_0$, $\varphi$)-typed bundle morphism and let there be given connections in $H$ and $\tilde{H}$ subject to (5). Then the associated vertical projections satisfy
\[ V_{\tilde{E}} d\gamma = d\gamma V_E. \]
Proof. First note that both sides of (9) take the tangent space at $h(f) \in E$ into the tangent space at $\gamma(\gamma_0(f))$ according to (6). We have by (3) the relations $dy V_E(\xi)$ and $V_E dy(\xi) = V_E dy(\eta)$ and thus $V_E dy(\eta)$ is an inclusion map) then (5) means that the connection in $\bar{H}$ is reducible to that in $H$. Applying Proposition 1 to $F \to \hat{F}$

The assumptions of this proposition can be modified in the following way:

The bundle morphism $\gamma : E(B, F, G, H) \to E(B, \hat{F}, \hat{G}, \hat{H})$ is called $(\gamma_0, \bar{\gamma})$-antityped if $\bar{\gamma} : \bar{H} \to H$ is a covering extension of the bundle $H$ (i.e., $\bar{\gamma}$ is a homomorphism of the principal bundles over $B$ inducing isomorphisms $d\bar{\gamma} : T(H) \to T(H)_{\psi(h)}$ and $\gamma_0$ being given as before satisfies for each $\hat{h} \in \bar{H}, f \in F$

(6a) \[ \gamma(\bar{\gamma}(\hat{h}) f) = \hat{h}(\gamma_0(f)) \]

which is again equivalent to

(7a) \[ \gamma \mu J \bar{\gamma}(\hat{h}) \hat{\mu}_{\gamma_0(f)}(\hat{h}) \]

implying

(8a) \[ d\gamma d\mu f \hat{\mu}_{\gamma_0(f)} d\bar{\gamma}^1. \]

Note that a covering extension $\bar{\gamma}$ of $H$ assigns to each connection in $H$ a connection in $\bar{H}$ according to

(5a) \[ V_{\bar{\gamma}} - d\bar{\gamma}^1 V_H d\bar{\gamma}. \]

Proposition 1a. Let $\gamma : E(B, F, G, H) \to \hat{E}(B, \hat{F}, \hat{G}, \hat{H})$ be a $(\gamma_0, \bar{\gamma})$-antityped bundle morphism and let there be given connections in $H$ and $\hat{H}$ subject to (5a) Then the associated vertical projections satisfy (9).

The proof runs analogously to that of Proposition 1.

If now $\psi : B \to E$ is a local section in $E$ then its covariant differential is usually defined as

(10) \[ \nabla \psi = V_E(d\psi) \]

or if $Y \in T(B)_a$, where $\psi$ is defined in a neighbourhood of $a \in B$

(11) \[ \nabla Y \psi = (\nabla \psi)(Y) V_E(d\psi)(Y). \]

We give the analogous meaning to $\hat{\nabla}$ associated with $\hat{E}$ and get (9) in the form

(12) \[ \hat{\nabla} Y(\gamma \psi) = d\gamma \nabla Y \psi. \]

Note that if $\psi : H \to \bar{H}$ is a reduction of the structure group $\hat{G}$ to its subgroup $G$ (i.e., $H \subset \bar{H}$ and $\psi$ is the inclusion map) then (5) means that the connection in $\bar{H}$ is reducible to that in $H$. Applying Proposition 1 to $F \to \hat{F}$
implying \( E = \hat{E} \), and \( \gamma \) \( \text{id}_E \) we refine the coincidence \( \nabla \nabla \) of both the induced covariant differentials. On the other hand if especially \( H = \hat{H} \) and \( \varphi \) is the identity, then the condition (6) or (6a) on \( \gamma \) expresses the fact that the induced mappings \( \gamma_x : E_x \to \hat{E}_x \) commute with the action of the associated groupoid (c.f. [1] or [2]) on \( E \) and \( \hat{E} \).

This leads directly to a number of applications but first let us generalize Proposition 1 to the case of several fibre bundles where then \( \gamma \) may play the rôle of a "multiplication" connected with some structure (e.g. the tensor multiplication connected with the tensor product of vector bundles, cf. below).

The indices \( i \) run always from 1 to the integer \( A \geq 1 \). Let there be given connections in the principal bundles \( H_i(B_i, G_i) \) by means of the differentiable distributions \( h_i \to T(H_i)_{h_i} = T(H_i)_{h_i} \). Let

\[
(13) \quad V_{H_i} : T(H_i) \to T(H_i)
\]

be the vertical projections related to these connections. Then \( H = H_1 \times H_2 \times \ldots \times H_A \) is a principal fibre bundle and \( (h_1, \ldots, h_A) \to T(H_1)_{h_1} \oplus \ldots \oplus T(H_A)_{h_A} \) is a differentiable distribution which defines a connection in \( H \). Here and in the following we identify as usual the tangent space \( T(H)_{h_1, \ldots, h_A} \), with \( T(H_1)_{h_1} \oplus \ldots \oplus T(H_A)_{h_A} \). The vertical projection related to this connection satisfies

\[
(14) \quad V_H = V_{H_1} + \ldots + V_{H_A},
\]

where \( V_H(X) = 0 \) unless \( X \in T(H_i) \).

If \( \hat{E} = \hat{E}(B_1, \ldots, B_A, \hat{F}, G_1 \times \ldots \times G_A, H) \) is a fibre bundle associated to \( H \), we get a projection \( V_{\hat{E}} \) as above and it satisfies

\[
(15) \quad V_{\hat{E}} \, d\hat{\mu}_j \, d\hat{\mu}_j \, (V_H + \ldots + V_{H_A})
\]

for each \( j \in \hat{E} \). At the same time suppose that there are fibre bundles \( E_i = E_i(B_i, F_i, G_i, H_i) \) associated to \( H_i \) and denote again by \( \mu_i : H_i \to E_i \), \( f_i \in F_i \), the associated mappings. The product \( E = E_1 \times \ldots \times E_A \) is a fibre bundle over \( B_1 \times \ldots \times B_A \) with the above defined associated groupoid \( H \). A bundle morphism \( \gamma : E_1 \times \ldots \times E_A \to \hat{E} \) is called \( \gamma_0 \)-typed if there is a differentiable mapping \( \gamma_0 : F_1 \times \ldots \times F_A \to \hat{F} \) such that for each \( (h_1, \ldots, h_A) \in H \) and \( (f_1, \ldots, f_A) \in F_1 \times \ldots \times F_A \) the relation

\[
(16) \quad \gamma(h_1(f_1), \ldots, h_A(f_A)) \quad (h_1 \times \ldots \times h_A) \gamma_0(f_1, \ldots, f_A)
\]

holds, which is again equivalent to

\[
(17) \quad \gamma(\mu_1(h_1), \ldots, \mu_A(h_A)) \quad \mu_{\gamma_0(f_1, \ldots, f_A)}(h_1, \ldots, h_A),
\]
implying

\[ (18) \quad d\gamma(d\mu_1 + \ldots + d\mu_A) = d\mu_{\gamma_{\mathfrak{r}}(1,\ldots,F_0)} \cdot \]

Using this relation we prove analogously to Proposition 1

**Proposition 2.** Let \( \gamma : E_1(B_1, F_1, G_1, H_1) \times \ldots \times E_A(B_A, F_A, G_A, H_A) \rightarrow E(B_1 \times \ldots \times B_A, \hat{F}, G_1 \times \ldots \times G_A, H_1 \times \ldots \times H_A) \) be a \( \gamma_0 \)-typed bundle morphism and let there be given a connection in each \( H_i \) (i \( = 1, \ldots, A \)). Then the associated vertical projections satisfy

\[ (19) \quad V_{\hat{E}} d\gamma (d\gamma(V_{E_1} + \ldots + V_{E_A})) . \]

Thus we have generalized the situation of the preceding propositions only for \( \varphi = \text{id}_H \) since the more general case is then obtained by combining Proposition 2 with Proposition 1 or 1a.

If now \( \psi_i : B \rightarrow E_i \) are local sections in \( E_i \), then \( \gamma(\psi_1 \times \ldots \times \psi_A) \) is a local section in \( E_1 \times \ldots \times E_A \) and (19) takes again the more usual form

\[ (20) \quad (\nabla_{\psi_1} \ldots \nabla_{\psi_A}) \gamma(\psi_1 \times \ldots \times \psi_A) = d\gamma(\nabla_{\psi_1} + \ldots + \nabla_{\psi_A}) . \]

Consider the special situation with \( B_1 = \ldots = B_A = B \). It is now natural to investigate beside \( I = I_1 \times \ldots \times I_A \) the principal bundle \( \Box H(B, G_1 \times \ldots \times G_A) \), which is the "restriction of \( H \) to the diagonal in \( B \times \ldots \times B" \). Denoting by \( \iota_H : \Box H \rightarrow H \) the natural injection, we see easily that the connections in \( H_i \) define a connection in \( \Box H \), where the corresponding vertical projection satisfies

\[ (21) \quad d\iota_H V_{\Box H} = (V_{\Box H} + \ldots + V_{\Box H}) d\iota_H . \]

Suppose now that there is a fibre bundle associated to \( \Box H \). There exists then always a fibre bundle \( \hat{E}(B \times \ldots \times B, \hat{F}, G_1 \times \ldots \times G_A, H) \) associated to \( H \) such that the bundle associated to \( \Box H \) is the "restricted" bundle \( \hat{E} \) defined analogously to \( \Box H \). Denote again \( \iota_{\hat{E}} : \Box \hat{E} \rightarrow \hat{E} \) the natural injection.

**Lemma 2.** The vertical projections in \( T(\hat{E}) \) and \( T(\Box E) \) satisfy

\[ (22) \quad d\iota_{\hat{E}} V_{\Box \hat{E}} = V_{\hat{E}} d\iota_{\hat{E}} . \]

**Proof.** For a fixed \( \tilde{e} \in \hat{E} \) define again the natural maps \( \iota_{\tilde{e}} : H \rightarrow \hat{E} \)

The relation

\[ (23) \quad \iota_{\tilde{e}} \Box \iota_{\tilde{e}} = \iota_{\tilde{e}} \iota_H \]

follows immediately from the very definition of \( \Box H \) and \( \Box \hat{E} \). We have by (15)

\[ V_{\hat{E}} d\iota_{\tilde{e}} V_{\Box \hat{E}} = V_{\hat{E}} d(\Box \iota_{\tilde{e}}) \]

which implies by (23) and (21)

\[ d\iota_{\hat{E}} V_{\hat{E}} d(\Box \iota_{\tilde{e}}) = d\iota_{\tilde{e}} d\iota_H V_{\Box \hat{E}} d\iota_{\tilde{e}} d\iota_H V_{\hat{E}} d\iota_{\tilde{e}} d\iota_H V_{\hat{E}} d(\Box \iota_{\tilde{e}}) . \]
\[ d_{E} V_{E}(\eta) \], since \( d_{E} \) takes clearly elements tangent to fibres in \( T(\square E) \) into elements tangent to fibres in \( T(E) \). Combining these two results we obtain (22) since according to (3) each element \( \xi \in T(\square E) \) can be decomposed into \( \xi \) according to (3) each element \( \xi \in T(\square E) \) can be decomposed into \( \xi \) according to (3) each element \( \xi \in T(\square E) \) can be decomposed into \( \xi \) according to (3) each element \( \xi \in T(\square E) \) can be decomposed into \( \xi \)

If now \( \gamma : E_{1} \times \cdots \times E_{A} \to \tilde{E} \) is a bundle morphism there exists a unique bundle morphism \( \gamma_{\square} : \square(E_{1} \times \cdots \times E_{A}) \to \square \tilde{E} \) satisfying

\begin{equation}
\gamma_{\square} = \iota_{E} \gamma_{\square},
\end{equation}

where \( \iota_{E} : \square(E_{1} \times \cdots \times E_{A}) \to E_{1} \times \cdots \times E_{A} \) is defined analogously as \( \iota_{E} \).

From the above and Lemma 2 we obtain at once the

**Corollary.** If in the assumptions of Proposition 2 we have \( B_{1} = \cdots = B_{A} = B \) and \( \gamma \) is defined by (24) then

\begin{equation}
d_{E} V_{E} d_{E} \gamma_{\square} = d\gamma(V_{E_{1}} + \cdots + V_{E_{A}}) d_{E}
\end{equation}

or, by means of covariant differentials,

\begin{equation}
\nabla_{Y}(\gamma_{\square}(\psi_{1} \cdots \square \psi_{A})) = d\gamma_{\square} d_{E}^{1} (\nabla_{Y} \psi_{1} + \cdots + \nabla_{Y} \psi_{A}),
\end{equation}

where \( \psi_{i} \) are local sections in \( E_{i} \) defined in neighbourhoods of \( x \in B \), \( Y \in T(B)_{x} \) and \( \psi_{1} \square \cdots \square \psi_{A} \) is the natural section in \( \square(E_{1} \times \cdots \times E_{A}) \) given by the sections \( \psi_{i} \). Note that (26) is well defined since \( \nabla_{Y} \psi_{1} + \cdots + \nabla_{Y} \psi_{A} \in T(E_{1})_{x} \oplus \cdots \oplus T(E_{A})_{x} = d\iota_{E}(T(\square(E_{1} \times \cdots \times E_{A}))_{y}). \)

If \( E(B, F, G, H) \) is a vector bundle then there is a canonical identification of \( T(E)_{z} \) with \( E_{p(z)} \) which takes \( 0_{z} \in T(E)_{z} \) into \( 0 \in E_{p(z)} \). If \( \gamma : E \to \tilde{E} \) is a vector bundle morphism, then \( d\gamma | T(E)_{z} = \gamma|_{E_{p(z)}} \). Thus in this case (12) can be written in the form

\begin{equation}
\nabla_{Y}(\gamma \psi) = \gamma(\nabla_{Y} \psi),
\end{equation}

since \( \nabla_{Y} \psi \) belongs now to \( E_{x} \) if \( Y \in T(B)_{x} \) and it is the usual ,,linear“ covariant differential.

Suppose now that \( E_{1}, \cdots E_{A}, \tilde{E} \) appearing in (20) and (26) are vector bundles and let \( \gamma : E_{1} \times \cdots \times E_{A} \to \tilde{E} \) be multilinear on each fibre. This means that if we fix any point \( (z_{1}, \ldots z_{A}) \in E_{1} \times \cdots \times E_{A} \), then all the mappings \( \gamma_{i} : E_{i} \to \tilde{E} \) defined by \( \gamma_{i}(z) - \gamma(z_{1}, \ldots z_{i+1}, \ldots z_{A}) \) are vector bundle morphisms. From this and the above remark we obtain (20) in the form

\begin{equation}
(\nabla_{Y_{1}} + \cdots + \nabla_{y_{A}}) \gamma(\psi_{1} \times \cdots \times \psi_{A}) = \sum_{i=1}^{A} \gamma(\psi_{1} \times \cdots \times \psi_{i-1} \times \nabla_{Y_{i}} \psi_{i} \times \psi_{i+1} \times \cdots \times \psi_{A})
\end{equation}

and (26) in the form
Example 1. Let $E_1, \ldots, E_A$ be vector bundles and let $\hat{F} \to F_1, \ldots, F_A$ be vector bundles. Then the associated space $\hat{E}$ can be written as $E_1 \otimes \ldots \otimes E_A$ although in practice where $B_1, \ldots, B_A$ is almost always supposed it is the bundle $\square \hat{E}$ which is denoted by this symbol and called the tensor product of $E_i$. The fibres of $\hat{E}$ (and consequently of $\square \hat{E}$) consist of tensor products of the fibres of $E_1, \ldots, E_A$ at the corresponding points. Denoting $\gamma : E_1 \otimes \ldots \otimes E_A \to \hat{F}$ the bundle morphism that assigns to $(z_1, \ldots, z_A)$ the element $z_1 \otimes \ldots \otimes z_A$, we see that it is $\otimes$-typed, where $\otimes : F_1 \times \ldots \times F_A \to F_1 \otimes \ldots \otimes F_A$ is the natural tensor multiplication. In this way we can write — either (28) or (29) in the form

\[
(\nabla_{\gamma_1} \ldots \nabla_{\gamma_A}) (\psi_1 \otimes \ldots \otimes \psi_A) \sum_{i=1}^{A} \psi_1 \times \ldots \times i^{(i)} \nabla_{\gamma_i} \psi_i \times \ldots \times \psi_A.
\]

or

\[
\nabla_{\gamma}(\psi_1 \otimes \ldots \otimes \psi_A) \sum_{i=1}^{A} \psi_1 \otimes \ldots \otimes i^{(i)} \nabla_{\gamma_i} \psi_i \otimes \ldots \otimes \psi_A.
\]

which are well-known formulas and show that the $\nabla_{\gamma}$ corresponding to various vector bundles behave as differentiation with respect to the tensor product. If $E_1 \otimes \ldots \otimes E_A \to \hat{E}$, then it shows that $\nabla_{\gamma}$ is a derivation of the algebra which is the sheaf of germs of local sections in the tensor algebra bundle over $E$. Here and in the following we use, as is common, the same symbol $\nabla$ for all covariant differentiations induced by a fixed connection in a fixed principal fibre bundle.

Note that this is in fact the only interesting example of the application of Proposition 2 to vector bundles. Namely if $\gamma : E_1 \times \ldots \times E_A \to \hat{E}$ is multilinear where $\hat{E}$ is now an arbitrary vector bundle over $B_1 \times \ldots \times B_A$, then it can be always split to a mapping $E_1 \times \ldots \times E_A \to E_1 \otimes \ldots \otimes E_A$ of the above example and a vector bundle morphism $\tilde{\gamma} : E_1 \otimes \ldots \otimes E_A \to \hat{E}$, and if $\gamma$ is $\gamma_0$-typed then $\tilde{\gamma}$ is $(\tilde{\gamma}_0, \text{id})$-typed, where $\gamma_0$ is given by the commutative diagram in the category of vector spaces

\[
\begin{array}{ccc}
F_1 \times \ldots \times F_A & \to & F_1 \otimes \ldots \otimes F_A \\
\gamma_0 & \downarrow & \tilde{\gamma}_0 \\
\hat{F} & \leftarrow & \hat{E}
\end{array}
\]

Thus in the following since the covariant differential is really interesting mostly in the linear case, we shall give some examples of the application of the first propositions.
Example 2. Let $E$ be a vector bundle and let $\gamma_0 : \otimes E \to \wedge E$ (or $\gamma_0 : \otimes F \to \wedge F$) be the natural antisymmetrization (or symmetrization) operator. Then it is clearly $(\gamma_0, \text{id})$-typed, where $\gamma_0 : \otimes F \to \wedge F$ (or $\gamma_0 : \otimes F \to \wedge F$) is the antisymmetrization (or symmetrization) projection of the fibre types. We obtain, combining (27) with (31), the formulas

(32) $\nabla_Y (\psi_1 \wedge \ldots \wedge \psi_A) = \sum_{i=1}^A \psi_1 \wedge \ldots \wedge \nabla_Y \psi_i \wedge \ldots \wedge \psi_A$

(33) $\nabla_Y (\psi_1 \circ \ldots \circ \psi_A) = \sum_{i=1}^A \psi_1 \circ \ldots \circ \nabla_Y \psi_i \circ \ldots \circ \psi_A$

Example 3. Let $E = E(B, F, G, H)$ be again a vector bundle and $E_q^p(B, F, G, H)$ its $p$-times contravariant and $q$-times covariant tensor powers. Thus $F_q^p F (F^p)$. Let $C C_b : E_q^p \to E_q^p F^p, H)$ be a fixed contraction bundle morphism, where the contraction acts upon the $a$-th contravariant and $b$ th covariant indices. The relation (27) shows then that the contraction commutes with $\nabla_Y$. Especially for $p = q = 1$ we derive in this way the relation

(34) $\nabla_Y \eta, \xi = \eta, \nabla_Y \xi, Y(\eta, \xi)$,

where $\xi$ and $\eta$ are local sections in $E$ and $F^\ast$, respectively, over a neighbourhood of $x$, where $Y \in T(B)_x$.

In general, if $E(B, F, G, H)$ and $E(B, \hat{F}, G, H)$ are vector bundles and $\gamma_0 : F \to \hat{F}$ is a homomorphism which commutes with the action of $G$, it induces in a natural way a $(\gamma_0, \text{id})$-typed bundle morphism $\gamma : E \to \hat{E}$ commuting with the covariant differentiation. Conversely, if $\gamma : E \to \hat{E}$ is a $(\gamma_0, \text{id})$ typed bundle morphism, then $\gamma_0$ commutes necessarily with the action of $G$. On the other hand in this linear case a bundle homomorphism $\gamma$ is a section in $E^\ast \hat{E}$, which is again a vector bundle associated to $H$.

Lemma 3. $\gamma$ is a typed bundle morphism iff $\nabla_Y \gamma = 0$ for any $Y \in T(B)$ and any connection in $H$.

Proof. In fact, applying Proposition 2 and especially (27) and (31) to the natural pairing $(E^\ast \times \hat{E}) \otimes E \to \hat{E}$, we get for any local section $\xi$ in $E$

$$\nabla_Y (\gamma(\xi)) = (\nabla_Y \gamma)(\xi) + \gamma(\nabla_Y \xi).$$

But the left hand side is equal to the second term of the right hand side if $\gamma$ is typed. On the other hand $\nabla_Y \gamma = 0$ implies $\nabla_Y (\gamma(\xi)) = \gamma(\nabla_Y \xi)$ for any local section $\xi$, which implies (7) as one can see from the proof of Proposition 1.

We apply this fact to the following

Example 4. Let $E(B, F, G, H)$ be a vector bundle and $\gamma_0 : F \to F^\ast$ an isomorphism which commutes with $G$. Then the corresponding $\gamma : E \to E^\ast$...
defines a metric in $E$ which is autoparalell under any connection in $H$. If $G$ operates effectively on $F$, dim $F = n$, then the existence of such a $\gamma_0$ implies, of course, that $G$ is isomorphic with a subgroup of $O(n)$. Conversely, any autoparalell metric in $E$ can be obtained from such a $\gamma_0$ commuting with the action of $G$.

Example 5. Let $H = H(B, SO(n))$ be a principal bundle over $B$ with the structure group $SO(n)$, $n = 2v, v \geq 2$ (especially let $H$ be the principal bundle of oriented orthonormal frames of a Riemann manifold). The group $SO(n)$ operates naturally on $C^n$ and also on the Clifford algebra $\bullet C^n$ of $C^n$. In this way we obtain the vector bundles $E(B, C^n, SO(n), H)$ associated to $H$ (especially the complexified tangent bundle to $B$), and $\bullet E(B, \bullet C^n, SO(n), H)$ (especially what one may call the Clifford bundle of the oriented Riemann structure on $B$), where the fibres of $\bullet E$ are the Clifford algebras of the corresponding fibres of $E$ endowed with the quadratic form induced by $H$. Let there be given a connection in $H$. The injection $E \rightarrow \bullet E$ is clearly typed and thus Proposition 1 shows that the restriction of the covariant differentiation to $E$ via $\bullet E$ coincides with the one defined directly on $E$. More generally, from a situation analogous to that in Example 2 we obtain the formula

$$\nabla_Y(\psi_1 \bullet \cdots \bullet \psi_A) = \sum_{i=1}^{A} \psi_1 \bullet \cdots \bullet \nabla_Y \psi_i \bullet \cdots \bullet \psi_A.$$

Let now $\hat{H}(B, Spin(n))$ be such that $q : \hat{H} \rightarrow H$ is a covering extension of $H$, where $Spin(n)$ is the reduced Clifford group associated with the natural metric in $C^n$ (c.f. [3]), i.e. it is the subgroup of all elements $A \in GL(2^r, C)$ satisfying

$$A\gamma_\alpha A^{-1} = A^\dagger_{\alpha}, \det(A^\dagger_{\alpha}) = 1, \gamma(A)A = I,$$

where $\gamma_\alpha \in GL(2^r, C)$ are matrices for $\alpha = 1, \ldots n$ satisfying

$$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2\delta_{\alpha\beta} I,$$

$I \in GL(2^r, C)$ is the unity matrix and $\gamma$ is the main antiautomorphism of the Clifford algebra $\bullet C^n$ (c.f. [3]), which is identified with the algebra of all $2^r \times 2^r$ complex matrices, i.e. elements of $(C^2^r)^* \otimes C^2^r$. It is well known that $\text{Ker}q \cap \hat{H}_x$ consists of exactly two points for each $x \in B$, since $Spin(n)$ is a covering group of $SO(n)$, the projection being given by $A \rightarrow (A^\dagger_{\alpha})$ in (36). Let $E(B, C^n, SO(n), H)$ be the vector bundle above and let $\mathcal{S} = \mathcal{S}(B, C^2, Spin(n), H)$ be the vector bundle associated to $\hat{H}$ if $Spin(n)$ operates naturally (effectively) on $C^2$. The elements of $\mathcal{S}$ may be called spinors over $E$ and they correspond to the usual spinors on the oriented even dimensional Riemann manifold $B$ in the mentioned above special case. In this case the elements of $\hat{H}$ are the spinframes (cf. [4], [5]). There is a unique connection in $\hat{H}$ induced by the connection in $H$ and hence a covariant differentiation in e.g. $\mathcal{S}^* \times \mathcal{S}$. Let $\gamma_0 : \bullet C^n \rightarrow (C^2^r)^* \otimes C^2^r$ be the natural isomorphism of algebras which
takes the vectors \(e_n (a^1, \ldots n)\) of the canonical frame in \(C^n\) into the matrices \(\gamma_\alpha\) of (35). The isomorphism \(\varphi_0\) satisfies

\[
\hat{h}(\varphi_0(\xi)) - \hat{h} \Lambda (\varphi_0(\varphi_0(\Lambda \xi)))
\]

for each \(\xi \in \bullet C^n\), \(\Lambda \in \text{Spin}(n)\), \(\hat{h} \in \hat{H}\), where \(\varphi_0 : \Lambda \to (\Lambda \xi)\) is the covering homomorphism \(\text{Spin}(n) \to SO(n)\). In fact, since \(\Lambda \in \text{Spin}(n)\), \(\eta \in (C^\omega)^* \subset C^\omega\) implies \(\Lambda(\eta) = \Lambda \eta A^1\), this is for \(\xi \in C^n\) equivalent to (35). Further \(\varphi_0\) is an isomorphism of algebras and from there we conclude that (37) holds for any \(\xi \in \bullet C^n\). But (37) means that there is a unique \((\varphi_0, \varphi)\)-antityped bundle isomorphism \(\gamma : \bullet E \to \mathcal{F}^* \otimes \mathcal{F}\) which, according to Proposition 1a, takes the covariant differentiations in \(\bullet E \) and \(\mathcal{F}^* \otimes \mathcal{F}\) one into the other. In other words one can identify the Clifford algebra bundle of \(E\) with the bundle of \((1,1)\) spinors over \(E\) and consequently inject the tensor algebra bundle of \(E\) into the tensor algebra bundle of \(\mathcal{F}\) — including the covariant differentiations on them induced from any connection in \(H\). We can express this also by saying, that the covariant differentiation in the bundle of spinors over \(E\) is a "correct" extension of the differentiation on \(E\), a result again well-known at least in the special case of spinors on \(B\).

It is now clear how one could immediately obtain other results regarding the behaviour of the covariant differential of spintensors by applying Proposition 1a to other antityped bundle morphisms.

REFERENCES


Received January 5, 1968.

Matematický ústav
Slovenskej akadémie vied
Bratislava

37