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A PROBLEM CONCERNING j-PANCYCLIC GRAPHS

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Let G be a finite planar undirected graph with n vertices without loops or multiple edges. (For the notions of the cycle and the length of the cycle, see [1].)

Let n, j be natural numbers such that $n \geqslant 5$ and $3 \leqslant j \leqslant n$.

Let us call a planar graph G with n vertices

a) *j-pancyclic if G contains cycles of every length m, where $3 \leqslant m$ $(\neq j)$ $\leqslant n$;*

b) *pancyclic if G contains cycles of the length m for each m with $3 \leqslant m \leqslant n$.*

Papers [2] and [3] are devoted to the investigation of pancyclic graphs. In [2], a problem concerning *j-pancyclic graphs* is formulated; the problem is solved in the present paper by showing for which n there exists a *j-pancyclic graph* and for which n such a graph does not exist. We shall prove a theorem which solves a problem more general than that proposed in [2].

Theorem. *If $(n, j) \in \{(5, 3), (5, 4), (6, 3), (6, 5)\}$, then there does not exist a j-pancyclic graph G with n vertices. For all other pairs (n, j) a j-pancyclic graph G with n vertices exists.*

Proof. For the (n, j) from the above set the non-existence of a *j-pancyclic graph* is a consequence of the requirement for a cycle with the length 4, 3, 5 or 3, respectively. For other (n, j) we describe a construction of the graph with the above mentioned properties. The construction will be divided into two parts.

I. Let $j \neq 3, n$. Put $s = \left[\dfrac{n}{j-1}\right]$, $r = n - s(j-1)$. We construct a cycle with the length n and call its vertices $v_1, \ldots, v_n$. To this cycle we add the following edges.

(i) If $s \neq 1$, $r \neq j - 2$, we add the edges $\{v_1, v_q\}$ for $3 \leqslant q \leqslant j - 2$ and $\{v_1, v_{tj-t}\}$, where $1 \leqslant t \leqslant s$; in the case of $s \geqslant 3$, we add an edge $\{v_{2j-3}, v_{2j-1}\}$. This graph does not contain a cycle of the length j and contains cycles of the length m, where $3 \leqslant m$ $(\neq j)$ $\leqslant n$. All cycles of the length $m > 3$ contain the vertex v_1. If the graph contains a cycle of the length j, then this cycle must contain the vertex v_2 or v_{tj-t}, where $1 \leqslant t \leqslant s$. In the first case, the edge $\{v_1, v_j\}$ must exist; in the second case, the edge $\{v_1, v_{(t-1)j-(t-1)+1}\}$ or $\{v_1, v_{(t+1)j-(t+1)-1}\}$, but in both cases we get a contradiction. Now it is sufficient to show that there exist cycles of the length m, $3 \leq m\ (\neq j) \leqslant n$. For $3 \leqslant m \leqslant j - 1$ consider the cycle $v_1, v_2, \ldots, v_m, v_1$; for $j + 1 \leqslant m \leqslant 2j - 2$ consider the cycle $v_1, v_{2j-m}, v_{2j-m+1}, v_{2j-m+2}, \ldots, v_{2j-2}, v_1$; for $m = 2j - 1$ consider the cycle $v_1, v_{j-1}, v_j, \ldots, v_{2j-3}, v_{2j-1}, v_{2j}, \ldots, v_{3j-3}, v_1$; for $2j \leqslant m < sj$ put

$$p = \left[\frac{m}{j-1}\right];$$

then the cycle $v_1, v_{(p+1)j-(p+1)-m+2}, \ldots, v_{(p+1)j-(p+1)}, v_1$ is the one we need. For the case $sj \leqslant m \leqslant n - 1$ it is sufficient to take the cycle $v_1, v_{n-m+2}, v_{n-m+1}, \ldots, n_n, v_1$.

(ii) If $s \neq 1$ and $r = j - 2$, then we add the edges $\{v_1, v_q\}$, where $3 \leqslant q \leqslant j - 2$; $\{v_1, v_{tj-t}\}$, where $1 \leqslant t \leqslant s - 1$; further the edge $\{v_{sj-s-1}, v_{sj-s+1}\}$, and, if $s \geqslant 3$, then the edge $\{v_{2j-3}, v_{2j-1}\}$, too.

(iii) In the case when $s = 1$ and $r \neq j - 2$, we add the edges $\{v_1, v_q\}$, where $3 \leqslant q \leqslant j - 1$, $q \neq r + 1$.

(iv) If $s = 1$ and $r = j - 2$, then $n = 2j - 3$. Let $n \geqslant 11$. In this case we add the edges $\{v_1, v_3\}$, $\{v_1, v_{j-2}\}$ and the edges $\{v_1, v_{j+q}\}$, where $2 \leqslant q \leqslant j - 5$. The situation in the cases of $n = 7$ and $n = 9$ is illustrated in Fig. 1a and Fig. 1b, respectively.

It is possible to verify the non-existence of a cycle of the length j and the existence of cycles with a length different from j in a similar way as in (i).

II. In this part we shall describe the construction for $j = 3$ and $j = n$.

Let $j = 3$. Construct a cycle of the length n consisting of the vertices $v_1, v_2, \ldots, v_n$. If n is an odd number, $n \geqslant 11$, add the edges $\{v_1, v_4\}$, $\{v_1, v_7\}$, $\{v_2, v_6\}$ and the edges $\{v_{3+q}, v_{n-q}\}$, where $0 \leqslant q \leqslant \left(\frac{n-1}{2} - 5\right)$. If $n = 7,9$, see Fig. 2a, 2b. If n is even, $n \geqslant 12$, add the edges $\{v_1, v_5\}$, $\{v_1, v_8\}$, $\{v_2, v_7\}$ and the edges $\{v_{9+q}, v_{n-q}\}$, where $0 \leqslant q \leqslant \left(\frac{n}{2} - 6\right)$. For the cases of $n = 8,10$ see Fig. 3a and 3b, respectively. It is easy to verify that this graph satisfies the conditions of the Theorem for $j = 3$.

Let $j = n$. Construct a cycle with the length $n - 1$ and call its vertices $v_1, v_2, \ldots, v_{n-1}$; the n-th vertex not belonging to the cycle will be called v_n. Add the edges $\{v_1, v_q\}$, where $3 \leqslant q \leqslant n$, $q \neq n - 1$. The existence of cycles of the length m, $3 \leqslant m \leqslant n - 1$ and the non-existence of a cycle of the length n is evident.

This completes the proof of the theorem.

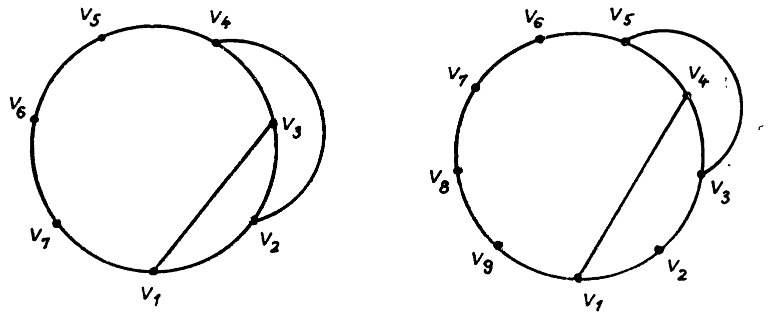

Fig. 1a, b

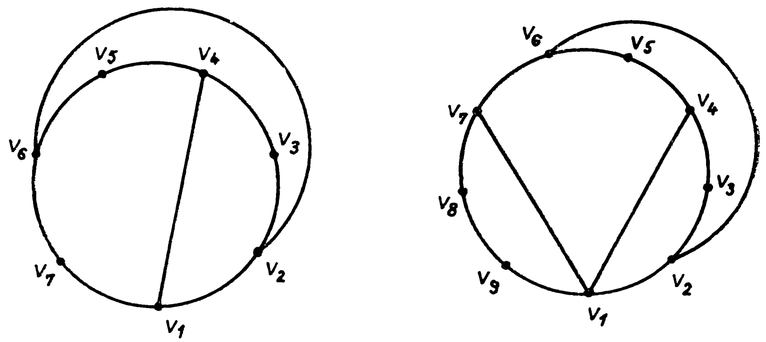

Fig. 2a, b

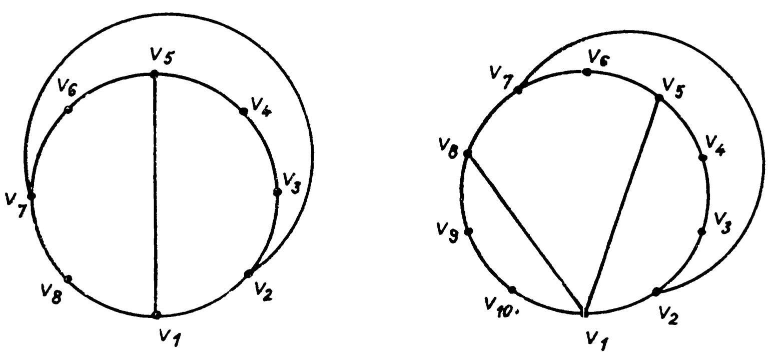

Fig. 3a, b

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