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REGULARITY AND APPROXIMATION THEOREMS FOR MEASURES AND INTEGRALS

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There are unified theories of measures and integrals (see [1], [2], [5]) studying functions whose domains is a partially ordered set S; if S is a set of sets (ordered by the inclusion), then the measure theory is obtained; if S is a set of real functions (ordered as usually), then the integration theory is obtained. A similar method is used in the present paper where we study regularity and approximation from a general point of view. In the first three sections we present three various problems (regularity, approximation, completion). The general postion leads also to a generalization of the notion of measure. A measure can be studied as a function $\mu : S \to R$, where S is a lattice; of course, S and μ satisfy some further conditions. In the fourth section we study the regularity of a measure on a lattice and in the fifth section the regularity of a measure defined on a logic.

1. Regularity

Let S be a partially ordered set with two binary operations denoted by $+$ and $-$. Moreover, let S be a conditionally σ-complete, σ-continuous lattice, i.e. if $x, y \in S$, $x_n \leqq x_{n+1} \leqq x$, $x_n \in S$ $(n = 1, 2, \ldots)$, then there exists $\bigvee_{n=1}^{\infty} x_n$ and $(\bigvee_{n=1}^{\infty} x_n) \cap y = \bigvee_{n=1}^{\infty} (x_n \cap y)$; and dually. (We shall write $x_n \nearrow \bigvee_{i=1}^{\infty} x_i$, or $x_n \searrow \bigwedge_{i=1}^{\infty} x_i$, resp.) We shall assume

1.1. $(a + b) - (c + d) \leqq (a - c) + (b - d)$ for every $a, b, c, d \in S$.

1.2. $(a - b) - (c - d) \leqq (a - c) + (d - b)$ for every $a, b, c, d \in S$.

1.3. If $a, b, c \in S$, $a \leqq b$, then $c - a \geqq c - b$, $a - c \leqq b - c$.

1.4. If $a, b, c \in S$, $a \leqq b \leqq c$, then $c - a \leqq (c - b) + (b - a)$, $c \leqq (c - b) + b$.

As an example we can present the lattice of all real — valued functions (or all measurable or all integrable functions etc.; $+$ and $-$ are interpreted as usual operations), or more generally a lattice ordered abelian group. Another example is the lattice of all subsets of a set (or all measurable sets; $+$ or $-$, resp. are the set theoretical union, or difference, resp.) or more generally a Boolean ring.

Now let $J : S \to R$ be a function satisfying the following conditions:

1.5. If $a, b \in S$, $a \leqq b$, then $J(a) \leqq J(b)$.

1.6. $J(a + b) \leqq J(a) + J(b)$ for every $a, b \in S$.

1.7. If $x_1, x_2, u_1, u_2 \in S$, $x_1 \leqq x_2$, $x_1 \leqq u_1$, $x_2 \leqq u_2$, then $J((u_1 \cup u_2) - x_2) \leqq J(u_1 - x_1) + J(u_2 - x_2)$.

1.8. If $x_1, x_2, c_1, c_2 \in S$, $x_1 \geqq x_2$, $x_1 \geqq c_1$, $x_2 \geqq c_2$, then $J(x_2 - (c_1 \cap c_2)) \leqq J(x_1 - c_1) + J(x_2 - c_2)$.

1.9. If $a \in S$, $a_n \in S$ $(n = 1, 2, \ldots)$ and $a_n \nearrow a$, or $a_n \searrow a$, resp. then $J(a_n - a) \to 0$, or $J(a - a_n) \to 0$, resp.

Remark. Since $a_n \leqq a$ implies $J(a) \leqq J(a - a_n) + J(a_n)$, we obtain from the $\lim J(a - a_n) = 0$, $\lim J(a_n) = J(a)$. Similarly for non increasing sequences.

Again, J can be interpreted as an integral (linear positive continuous functional defined on a linear lattice) and on the other hand as a measure defined on a ring, or more generally as a subadditive measure (i.e. a function J defined on a ring, $J(\emptyset) = 0$ and satisfying 1.5, 1.6 and 1.9).

Finally we must express regularity in the general case. Let C and U be subsets of S (in the case of a measure J or C, resp., U can be interpreted as a system of compact, or open measurable resp. sets) satisfying the following conditions:

1.10. If $a, b \in C$, then $a + b \in C$, $a \cup b \in C$, $a \cap b \in C$.

1.11. If $a, b \in U$, then $a + b \in U$, $a \cup b \in U$, $a \cap b \in U$.

1.12. If $a \in C$, $b \in U$, then $a - b \in C$, $b - a \in U$.

1.13. To any $a \in S$ there are $c \in C$, $u \in U$ such that $c \leqq a \leqq u$.

1.14. If $c \in S$, $c_n \in C$ $(n = 1, 2, \ldots)$ and $c_n \searrow c$, then $c \in C$.

1.15. If $u \in S$, $u_n \in U$ $(n = 1, 2, \ldots)$ and $u_n \nearrow u$, then $u \in U$.

Theorem 1.1. *Let T be the set of all regular elements, i.e. such elements $x \in S$ that*

$$\inf\{J(u - c);\ u \in U,\ c \in C,\ c \leqq x \leqq u\} = 0.$$

Then T is closed under the operations $+$, $-$. If $x_n \in T$ $(n = 1, 2, \ldots)$ $x_n \nearrow x \in S$ or $x_n \searrow x \in S$, then $x \in T$.

Before proving Theorem 1.1 we want to mention two special cases. The case of a measure (or more generally submeasure) is clear: If S is a δ-ring of sets of finite measure, then the family T of all regular sets is a δ-ring; if moreover S is a σ-algebra, then T is a σ-algebra, too.

Now take the integral. Let S_0 be the set of all simple integrable functions,

C, or U resp. be the set of all integrable limits of all non increasing, or non decreasing, resp. sequences of functions of S_0. It follows from Theorem 1.1 that every integrable function can be approximated by functions belonging to C, or U resp.

Proof of Theorem 1.1. The fact that T is closed under the operations $+$ and $-$ follows from the conditions 1.1, 1.2, 1.5, 1.6, 1.10, 1.11, 1.12.

Let $x_n \in T$, $x_n \nearrow x \in S$, $\varepsilon > 0$. Take $c_n \in C$, $u_n \in U$ such that $c_n \leqq x_n \leqq u_n$ and $J(u_n - x_n) < \varepsilon\, 2^{-n}$, $J(x_n - c_n) < \varepsilon\, 2^{-n}$. If we choose k such that $J(x - - x_k) < \varepsilon/2$, then $c_k \leqq x_k \leqq x$ and according to 1.4, 1.5 and 1.6

$$J(x - c_k) \leqq J(x - x_k) + J(x_k - c_k) < \varepsilon .$$

Put $v_n = \bigcup_{i=1}^{n} u_i$. Then $v_n \in U$ according to 1.11 and

$$J(v_n - x_n) \leqq \sum_{i=1}^{n} J(u_i - x_i) < \varepsilon$$

according to 1.7. According to 1.13 there is $u \in U$, $u \geqq x$. Then (with respect to 1.3, 1.6 and 1.11)

$$J((v_n \cap u) - x_n) < \varepsilon, \quad v_n \cap u \in U, \quad v_n \cap u \geqq x_n .$$

Put $w_n = v_n \cap u \in U$. Since $w_n \leqq w_{n+1}$, $w_n \leqq u$ and S is conditionally complete, there is $w = \bigvee_{n=1}^{\infty} w_n$. According to 1.15 $w \in U$. Since $w_n \nearrow w$, there is m such that

$$J(w - w_m) < \varepsilon .$$

Then

$$J(w - x) \leqq J(w - w_m) + J(w_m - x_m) < 2\varepsilon .$$

Hence to any $\varepsilon > 0$ there are $w \in U$, $c_k \in C$ such that $c_k \leqq x \leqq w$ and

$$J(w - c_k) < 3\varepsilon .$$

Therefore

$$\inf\{J(u - c);\ u \in U,\ c \in C,\ u \geqq x \geqq c\} = 0,$$

i.e. $x \in T$. The dual assertion can be proved analogously.

2. Approximation

Now we shall assume that S is a conditionally σ-complete and distributive lattice. On the other hand no further algebraic structure on S is assumed.

Let $J : S \to R$ be a function satisfying the following conditions:

2.1. If $a, b \in S$, $a \leqq b$, then $J(a) \leqq J(b)$.

2.2. $J(a \cup b) + J(a \cap b) = J(a) + J(b)$ for all $a, b \in S$.

2.3. If $a_n \in S$, $a_n \leqq a_{n+1}$, or $a_n \geqq a_{n+1}$ $(n = 1, 2, \ldots)$, resp. and $\{J(a_n)\}_{n=1}^{\infty}$ is bounded, then there is $a \in S$ such that $a_n \nearrow a$, or $a_n \searrow a$, resp. and $J(a_n) \to J(a)$.

Lemma 2.1. *Let* $a_i,\ b_i \in S$ $(i = 1, 2, \ldots, n)$, $a_1 \leqq a_2 \leqq \ldots \leqq a_n$. *Then*

$$J(a_n \cup (\bigcup_{i=1}^{n} b_i)) - J(a_n \cap (\bigcup_{i=1}^{n} b_i)) \leqq \sum_{i=1}^{n} [J(a_i \cup b_i) - J(a_i \cap b_i)].$$

Proof. We prove the lemma by the induction. Evidently $J(a_1 \cup b_1) - J(a_1 \cap b_1) \leqq J(a_1 \cup b_1) - J(a_1 \cap b_1)$. Let

$$J(a_k \cup (\bigcup_{i=1}^{k} b_i)) - J(a_k \cap (\bigcup_{i=1}^{k} b_i)) \leqq \sum_{i=1}^{k} [J(a_i \cup b_i) - J(a_i \cap b_i)].$$

Then

$$J(a_{k+1} \cup (\bigcup_{i=1}^{k+1} b_i)) - J(a_{k+1} \cap (\bigcup_{i=1}^{k+1} b_i)) =$$

$$= J(a_{k+1} \cup b_{k+1} \cup a_k \cup \bigcup_{i=1}^{k} b_i) - J((a_{k+1} \cap (\bigcup_{i=1}^{k} b_i)) \cup (a_{k+1} \cap b_{k+1})) =$$

$$= J(a_{k+1} \cup b_{k+1}) + J(a_k \cup \bigcup_{i=1}^{k} b_i) - J((a_{k+1} \cup b_{k+1}) \cap (a_k \cup \bigcup_{i=1}^{k} b_i)) -$$

$$- J(a_{k+1} \cap (\bigcup_{i=1}^{k} b_i)) - J(a_{k+1} \cap b_{k+1}) + J(a_{k+1} \cap (\bigcup_{i=1}^{k} b_i) \cap b_{k+1}) \leqq$$

$$\leqq J(a_{k+1} \cup b_{k+1}) - J(a_{k+1} \cap b_{k+1}) + J(a_k \cup \bigcup_{i=1}^{k} b_i) - J(a_k \cap (\bigcup_{i=1}^{k} b_i)) \leqq$$

$$\leqq J(a_{k+1} \cup b_{k+1}) - J(a_{k+1} \cap b_{k+1}) + \sum_{i=1}^{k} [J(a_i \cup b_i) - J(a_i \cap b_i)] =$$

$$= \sum_{i=1}^{k+1} [J(a_i \cup b_i) - J(a_i \cap b_i)].$$

Lemma 2.2. *Let* $a_i, b_i \in S$ $(i = 1, \ldots, n)$, $a_1 \geqq a_2 \geqq \ldots \geqq a_n$. *Then*

$$J(a_n \cup (\bigcap_{i=1}^{n} b_i)) - J(a_n \cap (\bigcap_{i=1}^{n} b_i)) \leqq \sum_{i=1}^{n} [J(a_i \cup b_i) - J(a_i \cap b_i)].$$

Theorem 2.1. *Let L be a sublattice of the lattice S. Put $M = \{a; a \in S, \forall \varepsilon > 0 \exists b \in L, J(a \cup b) - J(a \cap b) < \varepsilon\}$. Then the set M is monotone, i.e. $a \in S$, $a_n \in M$ $(n = 1, 2, \ldots)$ $a_n \nearrow a$, or $a_n \searrow a$, resp. implies $a \in M$.*

Proof. Let $a_n \nearrow a$. Let $b_n \in L$ be such elements that

$$J(a_n \cup b_n) - J(a_n \cap b_n) < \frac{\varepsilon}{2^n}.$$

Put $c_n = \bigcup_{i=1}^{n} b_i$. Then $c_n \in L$ $(n = 1, 2, \ldots)$ and according to Lemma 2.1 we have

$$J(a_n \cup c_n) - J(a_n \cap c_n) \leqq \sum_{i=1}^{n} [J(a_i \cup b_i) - J(a_i \cap b_i)] < \varepsilon.$$

The sequence $\{c_n\}_{n=1}^{\infty}$ is non decreasing. Moreover

$$J(c_1) \leqq J(c_n) = J(c_n) - J(c_n \cap a_n) + J(c_n \cap a_n) \leqq$$
$$\leqq J(c_n \cup a_n) - J(c_n \cap a_n) + J(a_n) \leqq \varepsilon + J(a),$$

hence $\{J(c_n)\}_{n=1}^{\infty}$ is bounded. Therefore there is $c \in S$ such that $c_n \nearrow c$. Then

$$J(c) = \lim J(c_n),$$
$$J(a \cup c) - J(a \cap c) = \lim [J(a_n \cup c_n) - J(a_n \cap c_n)] \leqq \varepsilon.$$

Now for sufficiently large n it follows

$$J(a \cup c_n) - J(a \cap c_n) = J(a \cup c_n) - J(a) - J(c_n) + J(a \cup c_n) \leqq$$
$$\leqq J(a \cup c) - J(a) - J(c) + J(c) - J(c_n) + J(a \cup c) =$$
$$= J(a \cup c) - J(a \cap c) + J(c) - J(c_n) < 2\varepsilon$$

and $a \in M$. The proof for non increasing sequences is analogous.

Example 2.1. Let S be the set of all integrable functions, L be the set of all simple integrable functions, $J(f) = \int f$. Then all the assumptions 2.1—2.3 are satisfied. Since the monotone set generated by L is S, then (according to Theorem 2.1) to any $\varepsilon > 0$ and any integrable function f there is a simple integrable function g such that

$$\int |f - g| = \int (\max(f, g) - \min(f, g)) = J(f \cup g) - J(f \cap g) < \varepsilon.$$

Example 2.2. Let S be a σ-ring generated by a ring L of subsets of a space X, J be a finite measure on S. Then according to Theorem 2.11 the family M contains the monotone family generated by the ring L and this is (see [4]) S. Hence to any $\varepsilon > 0$ and any $E \in S$ there is $F \in L$ such that

$$J(E \Delta F) = J(E \cup F) - J(E \cap F) < \varepsilon.$$

Remark. Note that in this case we did not obtain a theorem for subadditive measures. Subadditive measures need not satisfy the condition 2.2.

3. Completion

First let H be a conditionally σ-complete lattice, $S \subset H$ a sublattice of H and $J : S \to R$ be a function satisfying the conditions 2.1—2.3. We want to obtain a "complete extension" of J. For this purpose we use the following concept:

Definition 3.1. $\tilde{S} = \{c \in H;\ \exists\, a, b \in S,\ a \leqq c \leqq b,\ J(a) = J(b)\}$.

If $a_1 \leqq c \leqq a_2$, $b_1 \leqq c \leqq b_2$ and $J(a_1) = J(a_2)$, $J(b_1) = J(b_2)$, *then* (*since* $a_2 \geqq b_1$ *and* $b_2 \geqq a_1$) $J(a_1) = J(a_2) \geqq J(b_1) = J(b_2) \geqq J(a_1)$, *hence* $J(a_1) = J(b_1) = J(a_2) = J(b_2)$. *Hence we can introduce the following function:*

Definition 3.2. *Let* $c \in \tilde{S}$, $a, b \in S$, $a \leqq c \leqq b$, $J(a) = J(b)$. *Then we define*

$$\tilde{J}(c) = J(a) = J(b).$$

Theorem 3.1. $\tilde{S}$ *is a lattice.* $\tilde{J}$ *is an extension of* J *satisfying the following conditions:*

3.1. *If* $a, b \in \tilde{S}$, $a \leqq b$, *then* $\tilde{J}(a) \leqq \tilde{J}(b)$.

3.2. $\tilde{J}(a) + \tilde{J}(b) = \tilde{J}(a \cup b) + \tilde{J}(a \cap b)$ *for every* $a, b \in \tilde{S}$.

3.3. *If* $a_n \in \tilde{S}$, $a_n \leqq a_{n+1}$, *or* $a_n \geqq a_{n+1}$ $(n = 1, 2, \ldots)$, *resp. and* $\{\tilde{J}(a_n)\}_{n=1}^{\infty}$ *is bounded, then there is* $a \in \tilde{S}$ *such that* $a_n \nearrow a$ *or* $a_n \searrow a$, *resp. and* $\tilde{J}(a_n) \to \tilde{J}(a)$.

Moreover $\tilde{J}$ *is complete in the following sense: if* $a \leqq b \leqq c$, $a, c \in \tilde{S}$, $b \in H$ $\tilde{J}(a) = \tilde{J}(c)$, *then also* $b \in \tilde{S}$.

Proof. If $a \in \tilde{S}$, then evidently $a \leqq a \leqq a$ and $J(a) = J(a)$, i.e. $a \in \tilde{S}$ and $\tilde{J}(a) = J(a)$. Let $a, b \in \tilde{S}$. Then there are $a_1, a_2, b_1, b_2 \in S$ such that $a_1 \leqq a \leqq a_2$, $b_1 \leqq b \leqq b_2$, $J(a_1) = J(a_2)$ and $J(b_1) = J(b_2)$. Then $a_1 \cup b_1 \in S$, $a_2 \cup b_2 \in S$, $a_1 \cup b_1 \leqq a \cup b \leqq a_2 \cup b_2$ and

$$J(a_1 \cup b_1) = J(a_1) + J(b_1) - J(a_1 \cap b_1) =$$

$$= J(a_2) + J(b_2) - J(a_1 \cap b_1) \geqq J(a_2) + J(b_2) - J(a_2 \cap b_2) =$$

$$= J(a_2 \cup b_2) \geqq J(a_1 \cup b_1),$$

hence $J(a_1 \cup b_1) = J(a_2 \cup b_2)$ i.e. $a \cup b \in \tilde{S}$. Similarly it can be proved $a \cap b \in \tilde{S}$. Moreover,

$$\tilde{J}(a) + \tilde{J}(b) = J(a_1) + J(b_1) = J(a_1 \cup b_1) + J(a_1 \cap b_1) =$$

$$= \tilde{J}(a \cup b) + \tilde{J}(a \cap b),$$

i.e. 3.2 holds. If $a \leqq b$, then $a_1 \leqq a \leqq b \leqq b_2$, hence $\tilde{J}(a) = J(a_1) \leqq J(b_2) = \tilde{J}(b)$ and also 3.1 is satisfied.

Let $a_n \in \tilde{S}$, $a_n \leqq a_{n+1}$ and $\{\tilde{J}(a_n)\}_{n=1}^{\infty}$ is bounded. Then there are $b_n, c_n \in S$ such that $b_n \leqq a_n \leqq c_n$ and $J(c_n) = J(b_n)$. Put $d_n = \bigcup_{i=1}^{n} b_i$, $e_n = \bigcup_{i=1}^{n} c_i$. Then $d_n, e_n \in S$, $d_n \leqq a_n \leqq c_n$, $d_n \leqq d_{n+1}$, $e_n \leqq e_{n+1}$ $(n = 1, 2, \ldots)$ and $J(d_n) = J(e_n) = \tilde{J}(a_n)$, $\{J(d_n)\}_{n=1}^{\infty}$, $\{J(e_n)\}_{n=1}^{\infty}$ are bounded hence there are $d = \bigvee_{n=1}^{\infty} d_n$, $e = \bigvee_{n=1}^{\infty} e_n$ and $J(d) = \lim J(d_n)$, $J(e) = \lim J(e_n)$. Since $a_n \leqq e_n \leqq e$ $(n = 1, 2, \ldots)$ and H is conditionally σ-complete, there exists $a = \bigvee_{n=1}^{\infty} a_n \in H$. Moreover,

$$d = \bigvee_{n=1}^{\infty} d_n \leqq \bigvee_{n=1}^{\infty} a_n = a \leqq \bigvee_{n=1}^{\infty} e_n = e$$

and

$$J(d) = \lim J(d_n) = \lim J(e_n) = J(e).$$

Therefore $a \in \tilde{S}$ and

$$\tilde{J}(a) = J(c) = \lim J(e_n) = \lim J(a_n).$$

The dual assertion can be proved similarly.

Let finally $a \leqq b \leqq c$, $a, c \in \tilde{S}$, $b \in H$, $\tilde{J}(a) = \tilde{J}(c)$. Then there are $a_1, a_2, c_1, c_2 \in S$ such that $a_1 \leqq a \leqq a_2$, $c_1 \leqq c \leqq c_2$ and $J(a_1) = J(a_2)$, $J(c_1) = J(c_2)$. It follows $a_1 \leqq b \leqq c_2$ and

$$J(a_1) = \tilde{J}(a) = \tilde{J}(c) = J(c_2),$$

hence $b \in \tilde{S}$.

Now we shall assume similarly as in section 1 that two binary operations, $+$ and $-$, are given on H satisfying the following conditions:

3.4. If $a_1 \leqq a_2$ and $b_1 \leqq b_2$, then $a_1 + b_1 \leqq a_2 + b_2$ and $(b_2 + a_2) - (a_1 + b_1) \leqq (b_2 - b_1) + (a_2 - a_1)$.

3.5. If $a_1 \leqq a_2$ and $b_1 \leqq b_2$, then $b_1 - a_2 \leqq b_2 - a_1$ and $(b_2 - a_1) - (b_1 - a_2) \leqq (b_2 - b_1) + (a_2 - a_1)$.

Further let J satisfy the following additional property:

3.6. If $b \leqq a$, $a, b \in S$, then $a - b \in S$ and $J(a) = J(b) + J(a - b)$.

3.7. If $a, b \in S$, then $a + b \in S$ and $J(a + b) \leqq J(a) + J(b)$.

Theorem 3.2. *Let S be closed under the operations $+$, $-$ and H, or J resp., satisfy the conditions* 3.4—3.7. *Then S is closed under the operations $+$ and $-$.*

Moreover $\tilde{J}(a + b) \leqq \tilde{J}(a) + \tilde{J}(b)$ *for every* $a, b \in \tilde{S}$ *and if* $b \leqq a$, *then* $\tilde{J}(a) = \tilde{J}(b) + \tilde{J}(a - b)$.

Proof. Let $a, b \in \tilde{S}$, $a_1, a_2, b_1, b_2 \in S$, $a_1 \leqq a \leqq a_2$, $b_1 \leqq b \leqq b_2$, $J(a_1) = J(a_2)$, $J(b_1) = J(b_2)$. Then

$$a_1 + b_1 \leqq a + b \leqq a_2 + b_2, \quad a_1 - b_2 \leqq a - b \leqq a_2 - b_1$$

and

$$0 \leqq J(a_2 + b_2) - J(a_1 + b_1) = J((a_2 + b_2) - (a_1 + b_1)) \leqq$$
$$\leqq J((a_2 - a_1) + (b_2 - b_1)) \leqq J(a_2 - a_1) + J(b_2 - b_1) =$$
$$= J(a_2) - J(a_1) + J(b_2) - J(b_1) = 0.$$

Similarly

$$0 \leqq J(a_2 - b_1) - J(a_1 - b_2) = J((a_2 - b_1) - (a_1 - b_2)) \leqq$$
$$\leqq J((a_2 - a_1) + (b_2 - b_1)) \leqq J(a_2 - a_1) + J(b_2 - b_1) = 0.$$

Further

$$\tilde{J}(a + b) = J(a_1 + b_1) \leqq J(a_1) + J(b_1) = \tilde{J}(a) + \tilde{J}(b).$$

Finally, if $b \leqq a$, then

$$\tilde{J}(a) = J(a_2) = J(b_1) + J(a_2 - b_1) = \tilde{J}(b) + \tilde{J}(a - b).$$

Example 3.1. Let H be the set of all finite measurable functions, $S \subset H$ be a linear lattice of integrable functions satisfying together with the integral $J(f) = \int f$ the conditions 2.1—2.3; moreover, J is linear. Then evidently $\tilde{S}$ is a linear lattice and $\tilde{J}$ is linear too. Hence we get from a "good integration theory" another, which is moreover complete.

Example 3.2. Let H be the family of all subsets of a space X, $S \subset H$ be a σ-algebra, J be a finite measure on S. Then $\tilde{S}$ is a σ-algebra, $\tilde{J}$ is a measure on S and J is complete.

4. Measures on lattices

Now we shall study the regularity of measures on lattices. A measure on a lattice S with the least element O is a function $\mu : S \to R \cup \{\infty\}$ satisfying the following three conditions:

4.1. If $x_n \nearrow x$, $x_n \in S$ $(n = 1, 2, \ldots)$, $x \in S$, then $\lim \mu(x_n) = \mu(x)$.

4.2. $\mu(x) + \mu(y) = \mu(x \cup y) + \mu(x \cap y)$ for every $x, y \in S$.

4.3. $\mu(O) = 0$ and $\mu(x) \geqq 0$ for every $x \in S$.

If S is a σ-complete, modular, complemented lattice, then μ is a measure if and only if (see [6] theorem 4) μ satisfies 4.3 and

4.4. $\mu(\bigvee_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty} \mu(a_n)$ for every disjoint sequence $\{a_n\}_{n=1}^{\infty}$ of elements of S. A sequence $\{a_n\}_{n=1}^{\infty}$ is called disjoint if for any disjoint sets α, β of indices we have $\bigvee_{i\in\alpha} x_i \cap \bigvee_{j\in\beta} x_j = 0$. We shall need also some further properties of measures on lattices.

Lemma 4.1. *Let μ be a measure on a modular, complemented lattice S, a, $b \in S$, $a \leqq b$. Then*

$$\mu(b) = \mu(a) + \mu(b \cap a')$$

for every complement a' of a.

Proof. If $a \leqq b$ and a' is a complement of a, then

$$a \cup (b \cap a') = b \cap (a \cup a') = b \cap 1 = b,$$

hence (according to 4.2 and 4.3)

$$\mu(b) = \mu(a \cup (b \cap a')) = \mu(a) + \mu(b \cap a').$$

Lemma 4.2. *If S is a complemented lattice and μ is a probability measure (i.e. $\mu(1) = 1$), then $\mu(a') = 1 - \mu(a)$.*

A lattice S is called σ-continuous if $a_n \in S$, $a \in S$, $b \in S$, $a_n \nearrow a$ implies $a_n \cap b \nearrow a \cap b$; and dually.

Lemma 4.3. *Let μ be a measure on a modular, complemented σ-continuous lattice S. Let $a_n \in S$, $\mu(a_n) < \infty$ $(n = 1, 2, \ldots)$, $a \in S$, $a_n \searrow a$. Then*

$$\mu(a) = \lim \mu(a_n).$$

Proof. Let a' be any complement of a. Recall the following lemma from [3] (lemma 1): If $c \leqq b \leqq a$, c' is a complement of c, $c' \geqq a'$, then there is a complement b' of b such that $c' \geqq b' \geqq a'$. Therefore there exist such complements a_n' of a_n $(n = 1, 2, \ldots)$ that $a_n' \nearrow a'$. Further $a_1 \cap a_n' \nearrow a_1 \cap a'$ since S is σ-continuous. According to Lemma 4.1 we obtain

$$\mu(a_1) - \mu(a) = \mu(a_1 \cap a') = \lim \mu(a_1 \cap a_n') =$$

$$= \lim (\mu(a_1) - \mu(a_n)) = \mu(a_1) - \lim \mu(a_n),$$

hence

$$\mu(a) = \lim \mu(a_n).$$

Definition 4.1. *Let U, C be non — empty subsets of a lattice S, μ be a measure on S. An element $a \in S$ is called (C, U)-regular (or shortly regular), if*

$$\mu(a) = \inf\{\mu(u);\ u \in U,\ u \geqq a\} =$$
$$= \sup\{\mu(c);\ c \in C,\ c \leqq a\}.$$

Theorem 4.1. *Let S be a lattice, C, $U \subset S$ and $x, y \in C$ (or $x, y \in U$ resp.) implies $x \cup y \in C$ (or $x \cup y \in U$ resp.). Then the joint $a \cup b$ of two regular elements $a, b \in S$ is also a regular element.*

Proof. First let $\mu(a) < \infty$, $\mu(b) < \infty$. Then to any $\varepsilon > 0$ there are c, $d \in C$ and $u, v \in U$ such that

$$c \leqq a \leqq u,\ d \leqq b \leqq v,\ \mu(u) - \mu(c) < \varepsilon,\ \mu(v) - \mu(d) < \varepsilon.$$

Then $c \cup d \leqq a \cup b \leqq u \cup v$, $c \cup d \in C$, $u \cup v \in U$ and

$$\mu(a \cup b) - \mu(c \cup d) = \mu(a) + \mu(b) - \mu(a \cap b) - \mu(c) - \mu(d) +$$
$$+ \mu(c \cap d) = \mu(a) - \mu(c) + \mu(b) - \mu(d) + \mu(c \cap d) - \mu(a \cap b) < 2\varepsilon$$

since $a \cap b \geqq c \cap d$. Similarly

$$\mu(u \cup v) = \mu(u) + \mu(v) - \mu(u \cap v) \leqq$$
$$\leqq \mu(u) + \mu(v) - \mu(a \cap b) \leqq$$
$$\leqq \mu(a) + \mu(b) - \mu(a \cap b) + 2\varepsilon = \mu(a \cup b) + 2\varepsilon.$$

If now, e.g. $\mu(a) = \infty$, then

$$\mu(a \cup b) = \infty = \{\sup \mu(c);\ c \in C,\ c \leqq a\} \leqq$$
$$\leqq \sup\{\mu(c);\ c \in C,\ c \leqq a \cup b\}$$

and

$$\mu(a \cup b) = \infty = \inf\{\mu(u);\ u \in U,\ u \geqq a \cup b\}$$

since $u \geqq a \cup b \geqq a$ implies $\infty \leqq \mu(a) \leqq \mu(u)$.

Theorem 4.2. *Let S be a complemented lattice. Let C, $U \subset S$ fulfil the following property: If $c \in C$, $u \in U$, c' is a complement of c, u' is a complement of u, then $c \cap u' \in C$, $u \cap c' \in U$. Let μ be a probability measure on S. Then the following implication holds: If a, b are regular elements and b' is a complement of b, then $a \cap b'$ is also a regular element.*

Proof. To any $\varepsilon > 0$ there exist $c, d \in C$, $u, v \in U$ such that

$$c \leqq a \leqq u,\ d \leqq b \leqq v,\ \mu(u) - \mu(c) < \varepsilon,\ \mu(v) - \mu(d) < \varepsilon.$$

Choose such complements v' of v and d' of d that $v' \leqq b' \leqq d'$. Then

$$\mu(a \cap b') - \mu(c \cap v') = \mu(a) + \mu(b') - \mu(a \cup b') - \mu(c) - \mu(v') +$$

$$+ \mu(c \cup v') = \mu(a) - \mu(c) + 1 - \mu(b) - (1 - \mu(v)) + \mu(c \cup v') -$$
$$- \mu(a \cup b') < 2\varepsilon$$

since $c \cup v' \leqq a \cup b'$ and hence $\mu(c \cup v') \leqq \mu(a \cup b')$. Similarly

$$\mu(u \cap d') - \mu(a \cap b') = \mu(u) + \mu(d') - \mu(u \cup d') - \mu(a) - \mu(b') + \mu(a \cup$$
$$\cup b') = \mu(u) - \mu(a) + 1 - \mu(d) - (1 - \mu(b)) + \mu(a \cup b') - \mu(u \cup d') < 2\varepsilon$$

since $a \cup b' \leqq u \cup d'$ and hence $\mu(a \cup b') \leqq \mu(u \cup d')$.

Lemma 4.4. *Let S be an arbitrary lattice, $a_i \in S$, $u_i \in S$, $u_i \geqq a_i$, $\mu(a_i) < \infty$ $(i = 1, \ldots, n)$, $a_1 \leqq a_2 \leqq \ldots \leqq a_n$. Then*

$$\mu(\bigcup_{i=1}^{n} u_i) - \mu(a_n) \leqq \sum_{i=1}^{n} (\mu(u_i) - \mu(a_i)).$$

Proof. We prove the inequality by induction.

$$\mu(\bigcup_{i=1}^{n+1} u_i) - \mu(a_{n+1}) = \mu(\bigcup_{i=1}^{n} u_i) + \mu(u_{n+1}) -$$
$$- \mu((\bigcup_{i=1}^{n} u_i) \cap u_{n+1}) - \mu(a_{n+1}).$$

But $a_{n-1} \leqq u_{n+1}$, $\bigcup_{i=1}^{n} a_i \leqq \bigcup_{i=1}^{n} u_i$ implies $a_n = a_{n+1} \cap (\bigcup_{i=1}^{n} a_i) \leqq u_{n+1} \cap (\bigcup_{i=1}^{n} u_i)$, hence

$$\mu(\bigcup_{i=1}^{n+1} u_i) - \mu(a_{n+1}) \leqq \mu(\bigcup_{i=1}^{n} u_i) + \mu(u_{n+1}) - \mu(a_{n+1}) - \mu(a_n) \leqq$$
$$\leqq \sum_{i=1}^{n} (\mu(u_i) - \mu(a_i)) + \mu(u_{n+1}) - \mu(a_{n+1}) = \sum_{i=1}^{n+1} (\mu(u_i) - \mu(a_i)).$$

Definition 4.2. *Let $U \subset S$, μ be a measure on S. We say that an outer $a \in S$ is outer regular if $\mu(a) = \inf\{\mu(u);\ a \leqq u,\ u \in U\}$.*

Theorem 4.3. *Let S be a σ-complete lattice, $U \subset S$ and $u_i \in S$ $(i = 1, 2, \ldots) \Rightarrow$ $\Rightarrow \bigcup_{i=1}^{\infty} u_i \in S$ and $\bigcup_{i=1}^{n} u_i \in S$ $(n = 1, 2, \ldots)$. Let μ be a measure on S and let $\{a_n\}_{n=1}^{\infty}$ be a seguence of inner regular elements, $a_n \nearrow a$. Then a is also an outer regular element.*

Proof. If $\mu(a_n) = \infty$ for some n, then $\mu(a) \geqq \mu(a_n) = \infty$ and $u \geqq a \geqq a_n$ implies $\mu(u) = \infty$. Now let $\mu(a_n) < \infty$ $(n = 1, 2, \ldots)$, $\varepsilon > 0$. Then there are $u_n \geqq a_n$, $u_n \in U$ and

$$\mu(u_n) - \mu(a_n) < \frac{\varepsilon}{2^n} \quad (n = 1, 2, \ldots).$$

Put $u = \bigvee_{n=1}^{\infty} u_n$, $w_n = \bigvee_{i=1}^{n} v_i$ $(n = 1, 2, \ldots)$. Then $u \in U$, $w_n \in U$ $(n = 1, 2, \ldots)$ and according to Lemma 4.3

$$\mu(w_n) - \mu(a_n) \leqq \sum_{i=1}^{n} (\mu(u_i) - \mu(a_i)) < \varepsilon .$$

Since $w_n \nearrow u$, $a_n \nearrow a$, we have

$$\mu(u) = \lim \mu(w_n) \leqq \lim \mu(a_n) + \varepsilon = \mu(a) + \varepsilon .$$

Theorem 4.4. *Let S be a modular, complemented, σ-continuous lattice. Let C, $U \subset S$, U be closed under finite and countable supremums, C be closed under finite and countable infimums. Let μ be a finite measure on S. Then the set M of all regular elements is monotone, i.e. $a_n \nearrow a$ (or $a_n \searrow a$ resp.), $a_n \in M$ $(n = 1, 2, \ldots)$, $a \in S$ implies $a \in M$.*

Proof. We study only the case of $a_n \nearrow a$. In the second case the situation is similar. We know that a is *outer* regular; we have to prove

$$\mu(a) = \sup \{\mu(c);\ c \leqq a,\ c \in C\}.$$

But

$$\mu(a) = \lim \mu(a_n) .$$

If $\mu(a) < \infty$, then to any $\varepsilon > 0$ there is such n, that

$$\mu(a) < \mu(a_n) + \varepsilon .$$

Since a_n is regular there is $c \in C$ such that $c \leqq a_n \leqq a$ and

$$\mu(a_n) < \mu(c) + \varepsilon ,$$

hence

$$\mu(a) < \mu(c) + 2\varepsilon .$$

If $\mu(a) = \infty$, then to any n_0 there is a_n such that $\mu(a_n) > n_0$ and therefore there is $c \in C$, $c \leqq a_n \leqq a$ such that $\mu(c) > n_0$. It follows that $\sup \{\mu(c);\ c \leqq a,\ c \in C\} = \infty$.

Now we can form a closed theory of the Halmos type (see [4]). What did we assume about C and U?

4.5. C and U are sublattices of S.

4.6. If $c \in C$, $u \in U$ and c' or u' resp. is a complement of c, or u resp., then $c \cap u' \in C$ and $u \cap c' \in U$.

4.7. If $c_n \in C$, or $u_n \in U$ $(n = 1, 2, \ldots)$ resp., then $\bigwedge_{n=1}^{\infty} c_n \in C$, or $\bigvee_{n=1}^{\infty} u_n \in U$, resp.

Now we add also the following condition

4.8. To any $c \in C$ there are $u_n \in U$ $(n = 1, 2, \ldots)$ such that $c = \bigwedge_{n=1}^{\infty} u_n$.

Definition 4.3. *Let S be a lattice, μ be a measure on S, C, $U \subset S$. μ is called a regular measure if every element of S is regular.*

Definition 4.4. *Let S be a complemented σ-complete lattice, $C \subset S$. We shall say that $D \subset S$ is generated by C if D is the least lattice over C with the following two properties:*

1. *If $a, b \in D$, b' is a complement of b, then $a \cap b' \in D$.*
2. *If $a_n \in D$ $(n = 1, 2, \ldots)$, $a_n \nearrow a$ or $a_n \searrow a$, then $a \in D$.*

Remark. It is possible to define a (lattice)-ring as a lattice D satisfying the condition 1 (see [5]). In our case D is the smallest monotone ring over C. It is proved in [5] (Lemma 1) that the smallest monotone ring over C coincides with the smallest σ-ring over C i.e. the smallest σ-complete ring over C. The assertion has been generalized for relatively complemented lattices in [3] (Theorem 3).

Theorem 4.5. *Let S be modular, complemented, σ-continuous, σ-complete lattice. Let C, $U \subset S$ be sets satisfying the conditions 4.5—4.8. Let S be generated by C. Then every finite measure on S is regular.*

Proof. Put $M = \{a \in S;\ a \text{ is regular}\}$. According to 4.8 $C \subset M$. Now it is sufficient to prove that M is a lattice satisfying the conditions 1 and 2. If $a, b \in M$, then $a \cup b \in M$ according to Theorem 4.1. Analogously it can be proved that $a \cap b \in M$. The conditions 1 and 2 follows from Theorems 4.2—4.4.

5. Measures on logics

A partially ordered set L with the least element O and the greatest element 1 is called a logic if there is a one-to-one mapping $\perp : L \to L$ such that the following properties are fulfilled:

5.1. $(a^{\perp})^{\perp} = a$ for all $a \in L$.

5.2. If $a, b \in L$, $a < b$, then $b^{\perp} < a^{\perp}$.

5.3. $a \cap a^{\perp} = 0$ for all $a \in L$.

5.4. $a \cup a^{\perp} = 1$ for all $a \in L$.

5.5. If $a, b \in L$, $a \leqq b$, then there is $c \in L$ such that $a + c = b$ (i.e. $c \leqq a^{\perp}$ and $a \cup c = b$).

5.6. If $a_i \in L$ $(i = 1, 2, \dots)$ and $a_i \leqq a_k^\perp$ for $i \neq k$, then $\bigvee_{i=1}^{\infty} a_i$ exists.

In the last case we shall write $\sum_{i=1}^{\infty} a_i = \bigvee_{i=1}^{\infty} a_i$. If $a \leqq b^\perp$, then $b \leqq a^\perp$; the elements a, b are called orthogonal and we write $a \perp b$. If $a \leqq b$, then $b = a \cup (b \cap a^\perp)$. Finally we shall write $a \leftrightarrow b$ if there are $a_1, b_1, c \in L$ such that $a_1 \perp b_1$, $a_1 \perp c$, $b_1 \perp c$ and $a = a_1 + c$, $b = b_1 + c$. If $a \leftrightarrow b$, then $a = (a \cap b) + (a \cap b^\perp)$. (In paper [7] the elements a, b for which $a \leftrightarrow b$ are called compatible; in the book [8] such elements are called simultaneously verifiable.)

A measure on a logic L is a function $\mu : L \to R$ such that

5.7. $\mu \geqq 0$ and $\mu(0) = 0$.

5.8. If $a_i \in L$ $(i = 1, 2, \dots)$, $a_i \perp a_j$ $(i \neq j)$ then $\mu(\sum_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} \mu(a_i)$.

For proving the regularity theorem we shall use the following properties of the given sets C, $U \subset L$.

5.9. If $c \in C$, $u \in U$, then $c^\perp \in U$, $u^\perp \in C$.

5.10. If $c_1, c_2 \in C$, $c_1 \perp c_2$, then $c_1 + c_2$ exists and $c_1 + c_2 \in C$.

5.11. If $u_i \in U$ $(i = 1, 2, \dots)$, then $\bigvee_{i=1}^{\infty} u_i$ exists and $\bigvee_{i=1}^{\infty} u_i \in U$.

5.12. If $d \in C$, $v \in U$ and $d \leqq v$, then $v \cap d^\perp \in U$.

5.13. If $d \in C$, $v \in U$, then $d \leftrightarrow v$ and $d \cap v^\perp \in C$.

Theorem 5.1. *The set M of all regular elements of L (i.e. such elements $a \in L$ that*

$$\mu(a) = \inf \{\mu(u);\ u \geqq a,\ u \in U\} =$$

$$= \sup \{\mu(c);\ c \leqq a,\ c \in C\})$$

is a sublogic of the logic L.

Proof. First we prove that $a \in M$ implies $a^\perp \in M$. Let ε be an arbitrary positive number. Take $c \in C$ such that $c \leqq a$ and $\mu(a) - \varepsilon < \mu(c)$. Then

$$\mu(1) - \mu(a) - \varepsilon > \mu(1) - \mu(c),$$

i.e.

$$\mu(a^\perp) - \varepsilon > \mu(c^\perp) \geqq \mu(a^\perp)$$

since $a^\perp \leqq c^\perp$. Since $c^\perp \in U$ (see 5.9) we have

$$\mu(a^\perp) = \inf \{\mu(u);\ u \in U,\ u \geqq a^\perp\},$$

hence $a^\perp$ is outer regular. Similarly it can be proved that $a^\perp$ is inner regular.

Now let $a_i \in M$, $a_i \leqq a_k^\perp (i \neq k)$. Take $c_i \leqq a_i$, $c_i \in C$ such that

$$\mu(a_i) - \frac{\varepsilon}{2^i} < \mu(c_i).$$

Then

$$\mu(\sum_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} \mu(a_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(a_i),$$

hence there is n such that

$$\mu(\sum_{i=1}^{\infty} a_i) - \varepsilon < \sum_{i=1}^{n} \mu(a_i) < \sum_{i=1}^{n} \mu(c_i) + \varepsilon = \mu(\sum_{i=1}^{n} c_i) + \varepsilon$$

and we proved (see 5.10) that $\sum_{i=1}^{\infty} a_i$ is inner regular. Take now $u_i \in U$ such that $u_i \geqq a_i$ and

$$\mu(a_i) + \frac{\varepsilon}{2^i} > \mu(u_i).$$

Then (see 5.11)

$$\mu(\sum_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} \mu(a_i) \geqq \sum_{i=1}^{\infty} \mu(u_i) - \varepsilon \geqq \mu(\bigvee_{i=1}^{\infty} u_i) - \varepsilon$$

and we see that $\sum_{i=1}^{\infty} a_i$ is also outer regular.

Finally let $a \leqq b$, a, $b \in M$, $c = b \cap a^{\perp}$. We want to prove that $c \in M$. First take $d \in C$, $v \in U$ such that $d \leqq a$, $b \leqq v$ and

$$\mu(a) - \varepsilon < \mu(d), \ \mu(b) + \varepsilon > \mu(v).$$

Put $k = v \cap d^{\perp}$. Then $v = d + k$, $k = v \cap d^{\perp} \geqq b \cap a^{\perp} = c$, $k \in U$ (see 5.12) and

$$\mu(k) = \mu(v) - \mu(d) < \mu(b) - \mu(a) + 2\varepsilon = \mu(c) + 2\varepsilon$$

hence c is outer regular. Further take $f \in C$, $u \in U$ such that $f \leqq b$, $a \leqq u$, $f \in C$, $u \in U$ and

$$\mu(b) - \varepsilon < \mu(f), \ \mu(a) + \varepsilon > \mu(u).$$

Since f, u are compatible (see 5.13), we have $f = f \cap u^{\perp} + f \cap u$, hence

$$\mu(f) = \mu(f \cap u^{\perp}) + \mu(f \cap u) \leqq \mu(f \cap u^{\perp}) + \mu(u)$$

and therefore

$$\mu(c) = \mu(b \cap a^{\perp}) = \mu(b) - \mu(a) < \mu(f) - \mu(u) + 2\varepsilon \leqq$$

$$\leqq \mu(f \cap u^{\perp}) + 2\varepsilon.$$

Finally $c = b \cap a^{\perp} \geqq f \cap u^{\perp}$, $f \cap u^{\perp} \in C$, hence c is also inner regular. i.e. $c \in M$.

REFERENCES

[1] ALFSEN, E. M.: Order theoretic foundations of integration. Math. Ann. *149*, 1963 419—461.

[2] FUTÁŠ, E.: Extension of continuous functionals. Mat. Čas. *21*, 1971, 191—198.

[3] GYÖRFFY, L. — RIEČAN, B.: On the extension of measures in relatively complemented lattices. Mat. Čas. *23*, 1973, 158—163.

[4] HALMOS, P. R.: Measure Theory. New York 1950.

[5] RIEČAN, B.: О непрерывном продолжении монотонных функционалов некоторого типа. Mat.-fyz. čas. *15*, 1965, 116—125.

[6] RIEČAN, B.: On the extension of a measure on lattices. Mat. Čas. *19*, 1969, 44—49.

[7] VARADAJAN, V. S.: Probability in physics and a theorem on simultaneous observability. Communs Pure and Appl. Math. *15*, 1962, 189—217.

[8] VARADAJAN, V. S.: Geometry of Quantum Theory. Vol. 1. New York 1967.

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