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REFLECTORS AND COREFLECTORS ON DIAGRAMS

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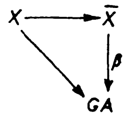
I. INTRODUCTION

In the fall of 1957 the writer began a Ph.D. dissertation under the direction of E. B. Leach investigating what Kan [1] was to call direct and inverse limits and what Freyd [2] was to call left and right roots (or reflections and coreflections). The work was essentially complete by the time Kan's article [1] on adjoint functors appeared in the Transactions during the following year. Due to circumstances beyond the control of the writer there was a delay in the publication of his results and in the meantime some of the results such as the factorization of left roots into differences of products were published independently by other writers [2]. However, since the results in which the dissertation culminates have not to the writer's knowledge yet appeared it seemed to him worthwhile to write them up for publication, adapting for this purpose the elegant language invented by Kan.

The main tool of this paper is the concept suggested by E. B. Leach of a relative reflection: an object $\bar{X}$ in a category $\mathscr{B}$ is a *relative reflection of an object* X *in* $\mathscr{B}$ *with respect to a functor* $G: \mathscr{A} \to \mathscr{B}$ provided there is a morphism $X \to \bar{X}$ in $\mathscr{B}$ satisfying the universal mapping property with respect to GA for all objects A in $\mathscr{A}$. We define a category $\mathscr{D}$ of diagrams over a category $\mathscr{A}$, in which the diagrams are not necessarily of the same form, and imbed $\mathscr{A}$ as a subcategory of $\mathscr{D}$ by means of a functor $J: \mathscr{A} \to \mathscr{D}$. If a diagram D has a *subdiagram functor* $D': \mathscr{K} \to \mathscr{D}$ (see below) and if $L: \mathscr{D} \to \mathscr{A}$ is a reflector [2] then LD' is a relative reflection of D. Since a reflection of relative reflection of D is a reflection of D (and dually for coreflections) a procedure is obtained for the iteration of reflections and coreflections which leads naturally to the investigation of the associativity, "commutativity", and distributivity of reflectors and coreflectors. Categories $\mathscr{D}_I(M)$ of diagrams of the form $D: \mathscr{I} \to \mathscr{A}$ are defined in which for each morphism α in $\mathscr{I}$, $D\alpha$ is constrained to lie in a class $M(\alpha)$ of morphisms in $\mathscr{A}$. Reflectors and coreflectors on these categories are studied. Examples are sums, products, quotients, subobjects, etc. Given functors $\Phi : \mathscr{D}_I(M) \to \mathscr{A}$ and $\Psi : \mathscr{D}_J(M') \to \mathscr{A}$ and a diagram D in $\mathscr{D}_{I\times J}(M \times M')$ we define subdiagram functors $\bar{D}_1 : \mathscr{I} \to \mathscr{D}_J(M')$ and $\bar{D}_2 : \mathscr{J} \to \mathscr{D}_I(M)$ and define the compositions $\Phi\Psi : \mathscr{D}_{I\times J}(M \times M') \to \mathscr{A}$ and $\Psi\Phi : \mathscr{D}_{I\times J}(M \times M') \to \mathscr{A}$ by setting $(\Psi\Phi)D = \Psi(\Phi\bar{D}_2)$ and $(\Phi\Psi)D = \Phi(\Psi\bar{D}_1)$. If $\Psi(\Phi\bar{D}_2)$ is isomorphic to $\Phi(\Psi\bar{D}_1))$ then Φ and Ψ are said to *commute* even though Φ and Ψ have different domains. *Invariance* of reflectors and coreflectors under one another is defined and it is shown that one reflector $\Phi : \mathscr{D}_I(M) \to \mathscr{A}$ commutes with another $\Psi : \mathscr{D}_J(M') \to \mathscr{A}$ provided each is invariant under the other. There is a dual result on coreflectors. If $\Phi : \mathscr{D}_I(M) \to \mathscr{A}$ is a coreflector and $\Psi : \mathscr{D}_J(M') \to \mathscr{A}$ is a reflector such that each is invariant under the other then Φ does not generally commute with Ψ but there exists a natural transformation $\Psi\Phi \to \Phi\Psi$. The latter specializes to the celebrated minimax theorem and may be further specialized to the one-sided distributive law $(x \,.\, y) + (x \,.\, z) < x \,.\, (y + z)$ of lattice theory [3].

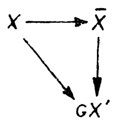
II. RELATIVE REFLECTIONS

Definition. *Let $G: \mathscr{A} \to \mathscr{B}$ be a functor and let $X \to \bar{X}$ be a morphism in $\mathscr{B}$. Suppose that for any morphism $X \to GA$ in which A is an object in $\mathscr{A}$ there exists a unique morphism $\bar{X} \xrightarrow{\beta} GA$ such that*



commutes. Then $X \to \bar{X}$ is a relative reflection and $\bar{X}$ is a relative reflection of X (both with respect to G). When there exists an object X' and a unique morphism $X' \xrightarrow{\alpha} A$ in $\mathscr{A}$ such that $\beta = G\alpha$ then the relative reflection is absolute: in this case X' is called a reflection of X and $X \to GX'$ is called a *reflection (both with respect to G).*

Relative reflections with respect to a functor $G: \mathscr{A} \to \mathscr{B}$ form a subcategory of $\mathscr{B}$. A reflection of a relative reflection of an object X is a reflection of X. Furthermore if $X \to \bar{X}$ is a relative reflection and $X \to GX'$ is a reflection then there is a unique morphism $\bar{X} \to GX'$ such that (1) *the diagram*



commutes and (2) *the morphism $\bar{X} \to GX'$ is a reflection.*

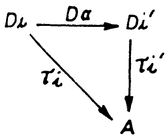
Recall that a diagram D over a category $\mathscr{A}$ is a functor $D\colon \mathscr{I} \to \mathscr{A}$ in which $\mathscr{I}$ is a small category [1]. If $D\colon \mathscr{I} \to \mathscr{A}$ and $D'\colon \mathscr{I}' \to \mathscr{A}$ are diagrams, then a *mapping* $\tau\colon D \to D'$ consists of a functor $\tau_1\colon \mathscr{I} \to \mathscr{I}'$ onto $\mathscr{I}'$, together with a natural transformation $\tau_2\colon D \to D'\tau_1$. We will write $\tau = (\tau_1, \tau_2)$. (If τ_1 is obvious we will write only τ_2 instead of (τ_1, τ_2).) A morphism equal to $\tau_2 i$ for some i in $\mathscr{I}$ is called a *component* of the mapping.

Definition. *If $\sigma\colon F \to F'$ is a natural transformation between functors $F, F'\colon \mathscr{X} \to \mathscr{B}$ and if $G\colon \mathscr{C} \to \mathscr{X}$ is a functor then $\sigma G\colon FG \to F'G$ is the natural transformation defined by $(\sigma G)c = \sigma(Gc)$ for each object c in $\mathscr{C}$.*

Definition. *If $\tau'\colon D' \to D''$ is another mapping then $\tau'\tau\colon D \to D''$ is defined by $(\tau'\tau)_1 = \tau_1'\tau_1$ and $(\tau'\tau)_2 = (\tau_2'\tau_1)\tau_2$. Consequently we obtain the category $\mathscr{D}$ of diagrams over $\mathscr{A}$.*

There is an obvious imbedding functor $J\colon \mathscr{A} \to \mathscr{D}$ and under this imbedding we may regard $\mathscr{A}$ as a subcategory of $\mathscr{D}$. It follows that a reflector [2] $F\colon \mathscr{D} \to \mathscr{A}$ is a (direct) limit functor. By a *reflection of a diagram D over $\mathscr{A}$ we mean a reflection of D with respect to J*. We will usually suppress mention of J and identify objects A and morphisms α in $\mathscr{A}$ with their corresponding diagrams JA and $J\alpha$.

A mapping $\tau\colon D \to A$ in which A is an object of $\mathscr{A}$ consists of a family of morphisms $\tau i\colon Di \to A$ indexed by objects in $\mathscr{I}$ such that for each morphism $\alpha\colon i \to i'$ in $\mathscr{I}$ the diagram

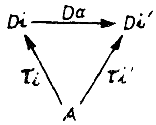


commutes.

A category $\mathscr{A}$ has an opposite category $\mathscr{A}^{\mathrm{op}}$ in which the objects and morphisms of $\mathscr{A}$ are the objects and morphisms of $\mathscr{A}^{\mathrm{op}}$ but $\hom_{\mathscr{A}}(A, B) = \hom_{\mathscr{A}}^{\mathrm{op}}(B, A)$ [4]. Moreover the product $\alpha \circ \beta$ of morphisms α, β in $\mathscr{A}^{\mathrm{op}}$ is defined by $\alpha \circ \beta = \beta\alpha$ whenever $\beta\alpha$ is a product in $\mathscr{A}$.

If $D\colon \mathscr{I} \to \mathscr{A}$ and $D'\colon \mathscr{I}' \to \mathscr{A}$ are diagrams then a *comapping* $\tau\colon D \to D'$ is a functor $\tau_1\colon \mathscr{I}' \to \mathscr{I}$ together with a natural transformation $\tau_2 : D\tau_1 \to D'$. In effect a comapping has the same definition as a mapping except that domain and range are interchanged and components of τ from $\mathscr{A}$ are replaced by morphisms from $\mathscr{A}^{\mathrm{op}}$. The *converse category $\mathscr{D}^*$ of diagrams* is the category whose objects are diagrams and whose morphisms are comappings. Coreflections are defined using comappings.

A comapping $\tau: A \to D$ in which A is an object in $\mathscr{A}$ and $D: \mathscr{I} \to \mathscr{A}$ is a diagram consists of a family of morphisms $\tau i: A \to Di$ indexed by objects in $\mathscr{I}$ such that for each morphism $\alpha: i \to i'$ in $\mathscr{I}$ the diagram



commutes.

Definition. *Let $P: \mathscr{I} \to \mathscr{K}$ be a functor. A functor induced category $P^{-1}(\mathscr{K})$ is defined as follows: the objects are diagrams $E_K: \mathscr{I}_K \to \mathscr{I}$ one for each object K in $\mathscr{K}$ and there corresponds to each morphism $\beta: K \to K'$ in $\mathscr{K}$ a mapping $(F_\beta, \sigma_\beta): E_K \to E_{K'}$. The functor induced category satisfies the properties:*

(1) *A morphism α in $\mathscr{I}$ is an image under E_K if and only if $P\alpha = e_K$.*

(2) *Each morphism in $\mathscr{I}$ is the product of factors α such that α is either an image under one of the E_K or a component of a mapping (F_β, σ_β) such that $P\alpha = \beta$.*

(3) *E_K is an imbedding, i. e. is one-to-one into.*

(4) *(F_β, σ_β) is an identity if and only if β is an identity.*

(5) $(F_{\beta_2}, \sigma_{\beta_2})(F_{\beta_1}, \sigma_{\beta_1}) = (F_{\beta_2\beta_1}, \sigma_{\beta_2\beta_1})$.

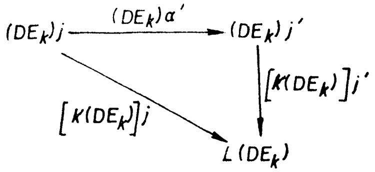
If such a category $P^{-1}(\mathscr{K})$ exists then the mapping $P^{-1}: \mathscr{K} \to P^{-1}(\mathscr{K})$, defined by $P^{-1}K = E_K$ for each object K in $\mathscr{K}$ and $P^{-1}\beta = (F_\beta, \sigma_\beta)$ for each morphism β in $\mathscr{K}$, is by (4) and (5) a functor. The functor P induces a factorization of $\mathscr{I}$ into subcategories and morphisms between the subcategories. Thus $P^{-1}: \mathscr{K} \to P^{-1}(\mathscr{K})$ may be called a *factorization* of $\mathscr{I}$.

Definition. *Let $D: \mathscr{I} \to \mathscr{A}$ be a diagram and let $\bar{D}: P^{-1}(\mathscr{K}) \to \mathscr{D}$ be a functor such that $\bar{D}(E_K) = DE_K$ and $\bar{D}(F_\beta, \sigma_\beta) = (F_\beta, D\sigma_\beta)$. Thus $\bar{D}$ is a restriction of D to the subcategories of $\mathscr{I}$ in $P^{-1}(\mathscr{K})$ and to the mappings between the subcategories. Let $D' = \bar{D}P^{-1}: \mathscr{K} \to \mathscr{D}$ then $D'K = \bar{D}P^{-1}K = DE_K: \mathscr{I}_K \to \mathscr{A}$ is a diagram for each object K in $\mathscr{K}$. D' is called a subdiagram functor of D and the functor P is a projection functor of D.*

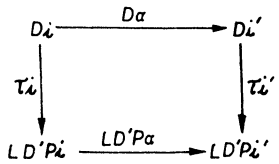
Theorem 1. *Suppose $L: \mathscr{D} \to \mathscr{A}$ is a reflector and $\varkappa: E_{\mathscr{D}} \to L$ is the natural transformation induced by L,* [1] (called a *front adjunction* by Mac Lane [4]) *Let $D: \mathscr{I} \to \mathscr{A}$ be a diagram and let $D': \mathscr{K} \to \mathscr{D}$ be a subdiagram functor of D. If $P: \mathscr{I} \to \mathscr{K}$ is a projection functor of D'* and *$\tau: D \to LD'P$ is the transformation such that $\tau E_k = \varkappa(D'k)$ for each object k in $\mathscr{K}$ then $(P, \tau): D \to LD'$ is a relative reflection. (By the object $L(DE_k)$ of $\mathscr{A}$, the range-object of $\varkappa(D'k)$, is here meant the diagram $JL(DE_k)$ where $J: \mathscr{A} \to \mathscr{D}$ is the chosen imbedding of $\mathscr{A}$ in $\mathscr{D}$.)*

Proof. In order to establish that the transformation $\tau: D \to LD'P$ is natural let $\alpha: i \to i'$ be a morphism in $\mathscr{I}$. There are two cases to consider.

Case 1. There exists a morphism $\alpha' : j \to j'$ in $\mathscr{I}_k$ such that $E_k\alpha' = \alpha$ for some object k in $\mathscr{K}$. It follows that $(D'k)\alpha' = (DE_k)\alpha' = D\alpha$ and since the natural transformation $\varkappa(DE_k) : DE_k \to L(DE_k)$ may be regarded as a mapping, the diagram

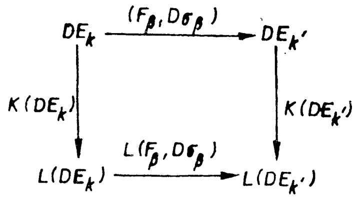


commutes. But $L(DE_k) = L(D'k) = L(D'Pi) = L(D'Pi')$ and $P\alpha = e_k$ and consequently the diagram

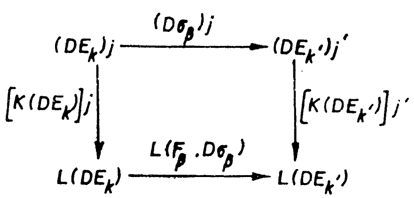


commutes.

Case 2. $P\alpha = \beta$ and α is a component of (F_β, σ_β), i. e. there exists j such that $\sigma_\beta j = \alpha$. Let k and k' be objects in $\mathscr{K}$ such that $E_k j = i$ and $E_{k'} j' = i'$. The diagram



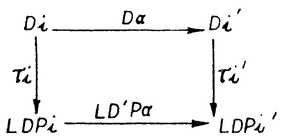
commutes and consequently for each j in $\mathscr{I}_k$ the diagram



commutes. But

$$\begin{aligned} L(DE_k) &= L(D'k) &&= LD'Pi \\ L(DE_{k'}) &= L(D'k') &&= LD'Pi', \\ LD'P\alpha &= LD'\beta &&= L(F_\beta, D\sigma_\beta). \end{aligned}$$

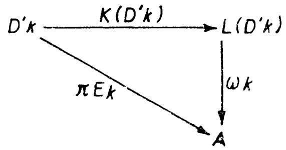
and therefore the diagram



commutes.

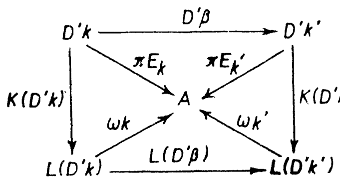
Since all morphisms α in $\mathscr{I}$ are products of morphisms of the types in cases 1 and 2 it follows that τ is natural.

Now let π: $D \to A$ be an arbitrary mapping into $\mathscr{A}$. For each object k in $\mathscr{K}$ there is a unique morphism ωk such that

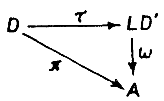


commutes.

The transformation ω: $LD' \to A$ is a mapping because if β: $k \to k'$ is a morphism in $\mathscr{K}$ then $LD'\beta$ is the unique morphism such that



commutes. Since $\varkappa(D'k) = \tau E_k$ it follows from diagram (1) that $\omega : LD' \to A$ is tne unique mapping such that



commutes.

Theorem 1 can be readily applied to proving known theorems such as the associativity of sums and products and that reflections of diagrams may be factored into a summation followed by a "quotient" and that coreflections may be factored into a multiplication followed by a "difference". (This may be seen from the following.)

Definition. *$\hat{\mathscr{A}}$ is the largest discrete subcategory of the category $\mathscr{A}$. (A discrete category* [2] *is a category whose only morphisms are identities.)*

Definition. *If D: $\mathscr{I} \to \mathscr{A}$ is a diagram and E: $\hat{\mathscr{I}} \to \mathscr{I}$ is an injection then $\hat{D}$: $\hat{\mathscr{I}} \to \mathscr{A}$ is the family defined by $\hat{D} = DE$.* (A family $(Di)_{i \in I}$ is a diagram D: $\mathscr{I} \to \mathscr{A}$ in which $\mathscr{I}$ is a discrete category). *Thus $\hat{D}$ is the largest family in D.*

We finally show that every diagram D: $\mathscr{I} \to \mathscr{A}$ has a subdiagram functor D': $\mathscr{K} \to \mathscr{D}$ of the form Δ: $\hat{D} \to \hat{D}$ in which Δ is a family of mappings. Such a functor may be called an *object-map factorization*.

Let E: $\hat{\mathscr{I}} \to \mathscr{I}$ be an injection and for each morphism α: $i \to i'$ in $\mathscr{I}$ define $(F_\alpha, \sigma_\alpha)$: $E \to E$ as a mapping having α as a component and whose other components are identity morphisms. Let $\mathscr{K}$ be the category whose only object is E and whose morphisms are mappings $(F_\alpha, \sigma_\alpha)$: $E \to E$. Define $Pi = E$ and $P\alpha = (F_\alpha, \sigma_\alpha)$ for each object i and each morphism α in $\mathscr{I}$. Define D': $\mathscr{K} \to \mathscr{D}$ by setting $D'E = \hat{D}$ and $D'P\alpha = (F_\alpha, D\sigma_\alpha)$: $\hat{D} \to \hat{D}$. Then D is of the form Δ: $\hat{D} \to \hat{D}$ in which Δ is a family of mappings $D'P\alpha$ each of which consists of $D\alpha$ together with identity morphisms.

III. "COMMUTATIVITY" OF REFLECTORS AND COREFLECTORS

In the category of R-modules in which R is a commutative ring with identity the direct sum functor Φ has as its domain families of modules and the quotient functor Ψ has as its domain pairs of modules of which one is a submodule of the other. It is known that "the quotient of the sums is the sum of the quotients" so that in a sense Φ "commutes" with Ψ although Φ and Ψ have different domains. In this section we characterize such functors in the cases they may be regarded as reflectors and coreflectors and define the composition of such functors relative to which they commute.

Given a diagram D: $\mathscr{I} \times \mathscr{J} \to \mathscr{A}$ there correspond subdiagram functors of the forms D_1: $\mathscr{I} \to \mathscr{D}$ and D_2: $\mathscr{J} \to \mathscr{D}$. A theorem on the commutativity of reflectors and coreflectors will be proved by applying Theorem 1 to these functors.

Let E_j: $\mathscr{I} \to \mathscr{I} \times \mathscr{J}$ be the imbedding functor defined by $E_j i = (i, j)$, $E_j \alpha = (\alpha, e_j)$ and let E^i: $\mathscr{J} \to \mathscr{I} \times \mathscr{J}$ be the imbedding functor defined by $E^i j = (i, j)$, $E^i = (e_i, \beta)$ for objects i in $\mathscr{I}$, j in $\mathscr{J}$ and morphisms α in $\mathscr{I}$, β in $\mathscr{J}$. Since the E^i are all of the same form, a mapping or a comapping $E^i \to$

$\to E^{i'}$ may be regarded as a natural transformation. For each morphism $\alpha: i \to i'$ in $\mathscr{I}$ let $\sigma^{\alpha}: E^i \to E^{i'}$ be the transformation defined by $(\sigma^{\alpha})j = (\alpha, e_j)$ for each object j in $\mathscr{J}$. To show that σ^{α} is natural let $\beta: j \to j'$ be a morphism in $\mathscr{J}$. Then $E^i\beta = (e_i, \beta)$ and $E^{i'}\beta = (e_{i'}, \beta)$ and the diagram

$$\begin{array}{ccc} (i,j) & \xrightarrow{(e_i,\beta)} & (i,j') \\ {\scriptstyle (\alpha,e_j)}\downarrow & & \downarrow{\scriptstyle (\alpha,e_{j'})} \\ (i',j) & \xrightarrow{(e_{i'},\beta)} & (i',j') \end{array}$$

commutes. The natural transformation $\sigma^{\alpha}: E^i \to E^{i'}$ induces a natural transformation $D\sigma^{\alpha}: DE^i \to DE^{i'}$. Define a functor $D_1: \mathscr{I} \to \mathscr{D}$ by setting $D_1 i = DE^i$ and $D_1\alpha = D\sigma^{\alpha}$ for each object i and each morphism α in $\mathscr{I}$.

A similar procedure generates a functor $D_2: \mathscr{J} \to \mathscr{D}$. For each morphism β: $j \to j'$ in $\mathscr{J}$ and for each object i in $\mathscr{I}$ we set $\sigma_{\beta}\, i = (e_i, \beta)$ and then define $D_2\beta = D\sigma_{\beta}$ and set $D_2 j = DE_j$ for each object j in $\mathscr{J}$.

If $\alpha: i \to i'$ is a morphism in $\mathscr{I}$ and $\beta: j \to j'$ is a morphism in $\mathscr{J}$ then

$$\begin{aligned} (\alpha, \beta) &= (\alpha, e_{j'}) \circ (e_i, \beta) \\ &= E_{j'}\alpha \circ \sigma_{\beta} i \\ &= \sigma^{\alpha} j' \circ E^i\beta. \end{aligned}$$

It follows that each morphism (α, β) in $\mathscr{I} \times \mathscr{J}$ is the product of two factors one of which is an image under $E_{j'}$ and the other is a component of one the mappings $E_j \to E_{j'}$, and furthermore one of them is an image under E^i and the other is a component of a mapping $E^i \to E^{i'}$. Consequently $D_1: \mathscr{I} \to \mathscr{D}$ and $D_2: \mathscr{J} \to \mathscr{D}$ are subdiagram functors.

Corresponding to the subdiagram functors are the projection functors P_1: $\mathscr{I} \times \mathscr{J} \to \mathscr{I}$ and $P_2: \mathscr{I} \times \mathscr{J} \to \mathscr{J}$. It follows from Theorem 1 and its dual that if $L: \mathscr{D} \to \mathscr{A}$ is a reflector or a coreflector then $L(LD_1)$ is isomorphic to $L(LD_2)$. We now apply this result to the study of the commutativity of reflectors and coreflectors.

Definition. *Let $\mathscr{I}$ be a small category, let $\mathscr{A}$ be a category and let $\mathscr{D}_I$ be the category of diagrams of the form $D: \mathscr{I} \to \mathscr{A}$. For every morphism α in $\mathscr{I}$ let $M(\alpha)$ be a class of morphisms in $\mathscr{A}$ such that if α is an identity then $M(\alpha)$ is the class of identity morphisms in $\mathscr{A}$. Define $\mathscr{D}_I(M)$ as the full subcategory of $\mathscr{D}_I$ such that $D\alpha$ belongs to $M(\alpha)$ for every morphism α in $\mathscr{I}$.*

Examples. 1. If $\mathscr{I}$ is a discrete category then $\mathscr{D}_I(M) = \mathscr{D}_I$ and a reflector $\mathscr{D}_I(M) \to \mathscr{A}$ is a sum functor and a coreflector $\mathscr{D}_I(M) \to \mathscr{A}$ is a product functor.

2. Let $\mathscr{A}$ be the category of modules over a commutative ring R with identity and let $\mathscr{I}$ be a category consisting of two objects i and i' (with corresponding identities) and two morphisms $\alpha: i \to i'$ and $\alpha': i \to i'$. Let $M(\alpha)$ consist of

monomorphisms in $\mathscr{A}$ and let $M(\alpha')$ consist of trivial homomorphisms so that $D\alpha'\colon Di \to Di'$ maps Di onto the zero element of Di'. Then a reflector Φ: $\mathscr{D}_I(M) \to \mathscr{A}$ is a quotient functor and $\Phi D = Di'/Di$ for every diagram D in $\mathscr{D}_I(M)$.

Definition. *Let $\Phi\colon \mathscr{D}_I(M) \to \mathscr{A}$ and $\Psi\colon \mathscr{D}_J(M') \to \mathscr{A}$ be reflectors or coreflectors. Then Ψ is invariant under Φ (or Φ-invariant) provided for every morphism τ in $\mathscr{D}_I(M)$ the morphism $\Phi\tau$ is in class $M'(\alpha)$ of Ψ whenever every component of τ is in $M'(\alpha)$.*

Examples. 3. Sum and product functors are invariant under any reflector of coreflector Φ: $\mathscr{D}_I(M) \to \mathscr{A}$ since as shown in example 1 the classes $M'(\alpha)$ of sum and product functors contain only identities.

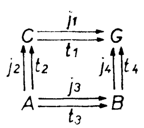
4. A quotient functor as in example 2 is direct sum invariant. For if τ: $D \to D'$ is a morphism in $\mathscr{D}_I$ and τi is a monomorphism for each object i in $\mathscr{I}$ then $\sum_{i \in I} \tau i$ is a monomorphism and if τi is trivial for each object i in $\mathscr{I}$ then $\sum_{i \in I} \tau i$ is trivial.

Definition. *Let $\mathscr{D}_{I \times J}$ $(M \times M')$ be the full subcategory of $\mathscr{D}_{I \times J}$ such that $D(\alpha, e) \in M(\alpha)$, $D(e', \beta) \in M'(\beta)$ whenever α (or β) is a morphism in $\mathscr{I}$ (or $\mathscr{J}$) and e (or e') is an identity in $\mathscr{I}$ ($\mathscr{J}$ respectively). Let D: $\mathscr{I} \times \mathscr{J} \to \mathscr{A}$ be a diagram in $\mathscr{D}_{I \times J}(M \times M')$ and let D_1: $\mathscr{I} \to \mathscr{D}$ and D_2: $\mathscr{J} \to \mathscr{D}$ be the subdiagram functors of D as defined above. Let $\bar{D}_1$: $\mathscr{I} \to \mathscr{D}_J(M')$ and $\bar{D}_2$: $\mathscr{J} \to \mathscr{D}_I(M)$ be restrictions of D_1 and D_2 and suppose Φ: $\mathscr{D}_I(M) \to \mathscr{A}$ and Ψ: $\mathscr{D}_J(M') \to \mathscr{A}$ are functors. Define the composition $\Phi\Psi$: $\mathscr{D}_{I \times J}(M \times M') \to \mathscr{A}$ and the composition $\Psi\Phi$: $\mathscr{D}_{I \times J}(M \times M') \to \mathscr{A}$ by setting $(\Phi\Psi)\ D = \Phi(\Psi\ \bar{D}_1)$ and $(\Psi\Phi)\ D = \Psi(\Phi\ \bar{D}_2)$.*

Theorem 2. *If Φ and Ψ are reflectors then under the composition just defined Φ and Ψ commute provided each is invariant under the other.*

Proof. Since Φ is Ψ-invariant it follows that $\Psi\ \bar{D}_1$: $\mathscr{I} \to \mathscr{A}$ is a diagram in $\mathscr{D}_I(M)$ and since Ψ is Φ-invariant it follows that $\Phi\bar{D}_2$: $\mathscr{J} \to \mathscr{A}$ is a diagram in $\mathscr{D}_J(M')$. Furthermore $\Psi(\Phi\bar{D}_2)$ is isomorphic to $\Phi(\Psi\bar{D}_1)$ since Φ and Ψ are reflectors.

Examples. 5. In the category of R-modules direct sums commute with quotients. However, quotients are not self-invariant: otherwise we should be able to prove that if $A \subset B \subset G$ and if $A \subset C \subset G$ then $(G/C)/(B/A)$ is isomorphic to $(G/B)/(C/A)$ using the diagram



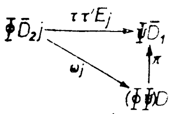
in which the j's are monomorphisms and the t's are trivial. In case $C = A$ *the monomorphisms are preserved under quotients and it follows that $(G/A)/(B/A)$ is isomorphic to G/B.*

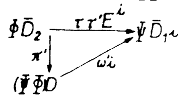
By a trivial modification of the proof of Theorem 2 it follows that *coreflectors Φ: $\mathscr{D}_I(M) \to \mathscr{A}$ and Ψ: $\mathscr{D}_J(M') \to \mathscr{A}$ commute provided each is invariant under the other.*

Definition. *Morphisms between diagrams may be generalized as follows: let $\mathscr{K}$ be a small category and let D: $\mathscr{I} \to \mathscr{A}$ and D': $\mathscr{J} \to \mathscr{A}$ be diagrams such that there are projectors P: $\mathscr{K} \to \mathscr{I}$ and P': $\mathscr{K} \to \mathscr{J}$ onto $\mathscr{I}$ and onto $\mathscr{J}$: then a $\mathscr{K}$-morphism $D \to D'$ is a natural transformation $DP \to D'P'$. A mapping $D \to D'$ is an $\mathscr{I}$-morphism and a comapping is a $\mathscr{J}$-morphism.*

Let Φ: $\mathscr{D}_I(M) \to \mathscr{A}$ be a coreflector and let Ψ: $\mathscr{D}_J(M') \to \mathscr{A}$ be a reflector and suppose that each is invariant under the other. Let D: $\mathscr{I} \times \mathscr{J} \to \mathscr{A}$ be a diagram in $\mathscr{D}_{I \times J}$ $(M \times M')$. As in the proof of Theorem 2 $\Psi\bar{D}_1$: $\mathscr{I} \to \mathscr{A}$ is a diagram in $\mathscr{D}_I(M)$ and $\Phi\bar{D}_2$: $\mathscr{J} \to \mathscr{A}$ is a diagram in $\mathscr{D}_J(M')$.

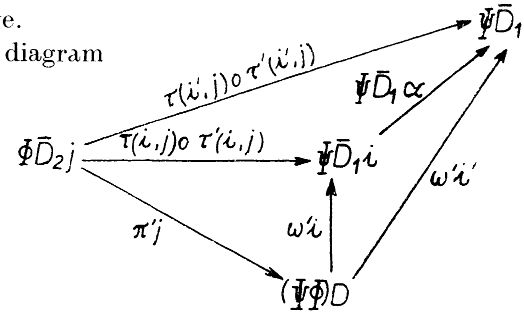
There is a natural transformation τ: $D \to \Psi\bar{D}_1P_1$ that defines a relative reflection $D \to \Psi\bar{D}_1$ and there is a natural transformation τ': $\Phi\bar{D}_2P_2 \to D$ that defines a relative coreflection $\Phi\bar{D}_2 \to D$. Consequently the natural transformation $\tau\tau'$: $\Phi\bar{D}_2P_2 \to \Psi\bar{D}_1P_1$ is an $\mathscr{I} \times \mathscr{J}$-morphism $\Phi\bar{D}_2 \to \Psi\bar{D}_1$. Hence for every object i in $\mathscr{I}$ there is a mapping $(\tau\tau')E^i$: $\Phi\bar{D}_2 \to \Psi\bar{D}_1 i$ and for every object j in $\mathscr{J}$ there is a comapping $(\tau\tau')E_j$: $\Phi\bar{D}_2 j \to \Psi\bar{D}_1$. Let π': $\Phi\bar{D}_2 \to (\Psi\Phi)D$ be a reflection and let π: $(\Phi\Psi)D \to \Psi\bar{D}_1$ be a coreflection. Then obviously there exist unique morphisms $\omega'i$ and ωj for each object i in $\mathscr{I}$ and j in $\mathscr{J}$ that makes the diagrams





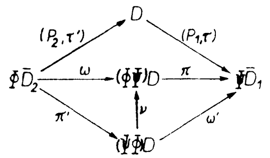
commutative.

Since the diagram



commutes it follows that ω': $(\Psi\Phi)D \to \Psi\bar{D}_1$ is a comapping. Similarly ω: $\Phi\bar{D}_2 \to (\Phi\Psi)D$ is a mapping.

We now show that there exists a unique morphism ν such that the diagram



commutes. The existence of such a morphism follows from the existence of a unique morphism ν such that $\nu\pi' = \omega$. This yields $\pi\nu\pi' = \pi\omega = \tau\tau'$ and therefore $\pi\nu = \omega'$, which establishes commutativity.

If the natural transformation $\Psi\Phi \to \Phi\Psi$ were an equivalence then reflection functors would commute naturally with coreflection functors provided that each is invariant under the other. However, there are counterexamples such as (disjoint) sums and products in the category of sets. Nonetheless, we obtain a generalization of the celebrated minimax inequality: namely that there is a natural transformation $\sum_{j \in J} \prod_{i \in I} \to \prod_{i \in I} \sum_{j \in J}$ from which we have as a special case the one-sided distributive law $(x \,.\, y) + (x \,.\, z) \to x \,.\, (y + z)$. For the cases in which sum and product are lattice operations these reduce to the usual laws. [3]

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