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ON THE MAXIMUM VALUE OF A CLASS
OF DETERMINANTS

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1. Result
For \( m = 1, 2, 3, \ldots \) let \( X(m) \) be the set of \( m \times m \)-matrices such that \( x_{i,j} \in \mathbb{R}^1, |x_{i,j}| \leq 1 \) for \( i, j = 1, 2, \ldots m \).
Define
\[
g(m) = \sup_{(x_{i,j}) \in X(m)} |\det(x_{i,j})|
\]
The aim of this note is to prove that
\[
\lim_{m \to \infty} (g(m))^{1/m}/m^{1/2} = 1. \quad (*)
\]

2. Preliminaries
Hadamard inequality reads
\[
|\det (x_{i,j})| \leq \prod_{i=1}^{m} \left( \sum_{j=1}^{m} x_{i,j}^2 \right)^{1/2}
\]

hence
\[
g(m) \leq m^{1/m}
\]
so that it remains to be proved that
\[
\lim \inf_{m \to \infty} (g(m))^{1/m}/m^{1/2} \geq 1.
\]
Equality in (2) holds, iff \( (x_{i,j}) \) is an orthogonal matrix. Hence \( g(m) = m^{1/m} \) iff there exists a matrix \( (y_{i,j}) \in X(m) \) such that \( |y_{i,j}| = 1 \) for \( i, j = 1, 2, \ldots m \) and \( \sum_{j=1}^{m} y_{i,j} y_{k,j} = 0 \) for \( i \neq k \); such a matrix \( (y_{i,j}) \) is called a Hadamard matrix. \( (x_{i,j}) \in X(m) \) is a Hadamard matrix, iff \( |\det(x_{i,j})| = m^{1/m} \).

(*) Formula (1) was needed in an investigation on functional differential equations, cf. [1].
Let $\mathcal{H}$ be the set of such $m = 1, 2, 3, \ldots$ that there exists a Hadamard matrix of order $m$. It is well known (cf. [2], Chapter 14) that

(5) if $m, n \in \mathcal{H}$, then $m n \in \mathcal{H}$,

(6) if $m \in \mathcal{H}$, $m > 2$, then $m \equiv 0 \pmod{4}$,

(7) $2^k \in \mathcal{H}$ for $k = 0, 1, 2, \ldots$

(8) if $m = (p^k + 1) \equiv 0 \pmod{4}$, $p$ being an odd prime, $k = 1, 2, 3, \ldots$, then $m \in \mathcal{H}$,

(9) if $m = h(p^k + 1), h \in \mathcal{H}$, $h > 2$, $p$ being an odd prime, $k = 1, 2, 3, \ldots$, then $m \in \mathcal{H}$

and there are known several other sufficient conditions for $m \in \mathcal{H}$, but the conjecture that $\mathcal{H}$ contains all $m \equiv 0 \pmod{4}$ remains undecided so far.

J. H. E. Cohn in [3] showed that for every $\varepsilon > 0$

(10) $g(m) \geq m^{(1-\varepsilon)m}$ for all sufficiently large $m$.

G. F. Clements and B. Lindström in [4] obtained an estimate of $g(m)$ from below, from which it follows that

(11) $\liminf_{m \to \infty} (g(m))^{1/m}/m^{1/2} \geq \left(\frac{3}{4}\right)^{1/2}$.

3. Lemmas.

**Lemma 1.** Let $\beta$ be irrational $\beta > 0$. Let $S$ be the set of all $u + v\beta$, $u, v$ being nonnegative integers. Let $S = \{s_1, s_2, s_3, \ldots\}$, $s_1 < s_2 < s_3 < \ldots$. For every $\delta > 0$ there exists a $D > 0$ and to every $d \geq D$ there exists a $k$, $K = 1, 2, 3, \ldots$ such that $s_k \leq d < s_k + 1$.

Proof. By the Dirichlet theorem (cf. [5], Chapter 2) for every $\delta > 0$ there exist integers $p, q$ such that $0 < q \leq \delta^{-1} + 1$, $p \geq 0$, $|q\beta - p| < \delta$. Hence $p < \delta^{-1}\beta + \beta + \delta$. Let $D$ be the least integer such that $D > (|q\beta - p|^{-1} + 1)(\delta^{-1}\beta + \beta + \delta)$ and let $r \geq D$ be an integer. Let there be distinguished two cases: (i) $q\beta - p > 0$, (ii) $q\beta - p < 0$. In the case (i) define $w_i = r + i(q\beta - p)$, $i = 0, 1, 2, \ldots, J$, $J$ being the whole part of $|q\beta - p|^{-1} + 1$. It follows that $r - ip \geq 0$, so that $w_i \in S$ for $i = 0, 1, \ldots, J$, $r = w_0 < w_1 < \ldots < w_J$, $w_J \geq r + 1$, $w_{i+1} < w_i + \delta$. Therefore for every $d \in \langle r, r + 1 \rangle$ there exists an $i = 0, 1, \ldots, J - 1$ such that $d \in \langle w_i, w_{i+1} \rangle$, so that $w_i \leq d < w_{i+1} < w_i + \delta$. In the case (ii) it is defined $w_i = r + 1 + i(q\beta - p)$, $i = 0, 1, \ldots, J$ and the argumentation is similar. The proof is complete.
Lemma 2. For every $\varepsilon > 0$ there exists a $C > 1$ and to every $m \geq C$ there exist $a, b \in \mathcal{H}$ such that $a < m < b < a(1 + \varepsilon)$.

Proof: Find $z \in \mathcal{H}$ such that $z \neq 2^k$, $k = 0, 1, 2, \ldots$ (cf. (8), (9)). By (5) and (7) $2^u . z^v = 2^u + \beta v \in \mathcal{H}$ for $u, v = 0, 1, 2, \ldots$ and $\beta = \log z / \log 2$ is irrational. Put $\delta = \log (1 + \varepsilon) / \log 2$, find $D$ according to Lemma 1 and put $C = 2^D$. If $m \geq C$, then $d = \log m / \log 2 > D$ and by Lemma 1 there exist two pairs of nonnegative integers $(u, v), (\bar{u}, \bar{v})$ such that $u + \beta v \leq d < \bar{u} + \beta \bar{v} < u + \beta v + \delta$. Hence $2^u . z^v \leq m < 2^u . z^v < 2^u . z^v . 2^\delta = 2^u . z^v . (1 + \varepsilon)$. The proof is complete.

Lemma 3. $g(m + n) \geq g(m) . g(n)$, $m, n = 1, 2, 3, \ldots$ This follows immediately from the definition of $g(m)$.

4. Proof of (1). By (11) there exists an $\alpha$, $0 < \alpha < 1$ such that

$$g(n) \geq \alpha^n n^{1n}, \quad n = 1, 2, 3, \ldots$$

Choose $\varepsilon$, $0 < \varepsilon < \alpha^{-1}$ and find $C > 1$ according to Lemma 2. Let $m \geq C$. By Lemma 2 there exists an $a \in \mathcal{H}$ such that $a \leq m < a(1 + \varepsilon)$. By Lemma 3 and (12) $g(m) \geq g(a) g(m - a) \geq a^{1n} \alpha^{m-a} (m-a)^{1(m-a)} \geq (a/m) \left(1 - \frac{a}{m}\right)^{1/2} \left(1 - \frac{a}{m}\right) \alpha \geq (1 - \varepsilon)^{1(1-\varepsilon)} \varepsilon \alpha \varepsilon$.

and (4) holds, as $\varepsilon$ is arbitrary. The proof of (1) is complete.

REFERENCES


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