Stephen K. McKee

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TRANSFORMS OF VECTOR-VALUED FUNCTIONS AND MEASURES

STEPHEN K. McKEE, Ballarat, Australia

In a previous paper ([6]), we found conditions on a sequence of vectors under which its terms were the coefficients of a vector measure or a vector-valued function with respect to an orthonormal sequence of continuous functions on a finite interval. This paper investigates the continuous analogues of those theorems. We give conditions under which a vector-valued function is the transform of a vector measure or a Bochner integrable function with respect to some kernel.

1. Introduction. Let (A, B) and (C, D) be intervals such that $-\infty \leq A < B \leq +\infty$ and $-\infty \leq C < D \leq +\infty$. For each $N = 1, 2, \ldots$, let λ_N be a real-valued continuous integrable function on (C, D). Let K and H be complex-valued bounded continuous functions on the product $(A, B) \times (C, D)$. For each bounded continuous complex-valued function f on (C, D), define

$$\sigma_{N,f}(t) = \int_C^D \lambda_N(s) f(s) H(t, s)\, \mathrm{d}s, \ t \in (A, B), \ N = 1, 2, \ldots. \tag{1}$$

Then, if f is the K-transform of a complex measure μ, i.e.

$$f(s) = \int_A^B K(t, s) \mu(\mathrm{d}t), \ s \in (C, D), \tag{2}$$

we have

$$\sigma_{N,f}(t) = \int_A^B T_N(t, u) \mu(\mathrm{d}u) \tag{3}$$

where

$$T_N(t, u) = \int_C^D \lambda_N(s)\, K(u, s)\, H(t, s)\, \mathrm{d}s, \quad t, u \in (A, B). \tag{4}$$

Let $C((A, B))$ denote the space of all complex bounded continuous functions on (A, B), $C_0((A, B))$ the subspace of functions vanishing at infinity and $C_{00}((A, B))$ the subspace of functions with compact support. Equip these spaces with the sup-norm.

For each φ in $L_1((C, D))$, define $\hat{\varphi}$ on (A, B) by

$$\hat{\varphi}(t) = \int_C^D K(t, s)\, \varphi(s)\, \mathrm{d}s. \tag{5}$$

For each N, define $\Psi_N : C_{00}((A, B)) \to C((A, B))$ by

$$(\Psi_N\, \psi)(t) = \int_A^B T_N(t, u)\, \psi(u)\, \mathrm{d}u, \quad \psi \in C_{00}((A, B)).$$

We make the following assumptions about K, H and the λ_N:

Assumption I. *For every bounded continuous scalar-valued function f on (C, D) for which $\sigma_{N,f}$ is integrable, $\int_A^B K(t, s)\sigma_{N,f}(t)\, \mathrm{d}t$ converges to $f(s)$ pointwise.*

Assumption II. *The set $\{\hat{\varphi}; \varphi \in L_1((C, D))\}$ is dense in $C_0((A. B))$.*

Assumption III. *There exists a constant M such that*

$$\int_A^B |T_N(t, u)|\, \mathrm{d}t \leq M, \quad u \in (A, B), \quad N = 1, 2, \ldots.$$

Assumption IV. *For every ψ in $C_{00}((A, B))$, $\|\Psi_N\psi - \psi\|_1 \to 0$, $N \to \infty$.*

Remark. It is possible to prove the theorems of this paper under different assumptions. Namely, for each $\varphi \in L_1((A, B))$. define

$$\tilde{\varphi}(s) = \int_A^B H(t, s)\, \varphi(t)\, \mathrm{d}t.$$

Let $\Omega = \{\varphi \in L_1((A, B))$; φ is continuous, $\tilde{\varphi} \in L_1((C, D))$ and $\varphi(t) = \int_C^D K(t, s)\, \tilde{\varphi}(s)\, \mathrm{d}s\}$. Then we may replace Assumptions I and II with the following:

Assumption I′. *Each λ_N has maximum modulus 1 and converges pointwise to 1 on (C, D) and $\{\tilde{\varphi}; \varphi \in \Omega\}$ is dense in $L_1((C, D))$.*

Assumption II′. *$\bar{\Omega}$, the uniform closure of Ω, is equal to $C_0((A, B))$.* The proofs under Assumptions I′, II′, III and IV are basically similar to those given.

Examples. It is well known that Assumptions I—IV and I′ and II′ hold in the following examples. Let $(A, B) = (C, D) = (-\infty, \infty)$, $K(u, s) = = \frac{1}{\sqrt{2\pi}} e^{-ius}$ and $H(t, s) = \frac{1}{\sqrt{2\pi}} e^{its}$. Then $\lambda_N(s) = 1 - \frac{|s|}{N}$ on $[-N, N]$ and 0 elsewhere gives Cesàro summation of the Fourier transform while $\lambda_N(s) = e^{-s^2/2N^2}$ gives Riesz summation.

2. Scalar-valued Transforms. Lemma 1. *For each N, define the map I_N : : $L_1((A, B)) \to L_1((A, B))$ by*

$$(I_N g)(t) = \int_A^B T_N(t, u)\, g(u)\, du, \quad g \in L_1((A, B)).$$

Then $\|I_N\| \leq M$, $N = 1, 2, \ldots$.

Proof. For each N and each g in $L_1((A, B))$, Assumption III gives

$$\|I_N g\|_1 = \int_A^B \left| \int_A^B T_N(t, u)\, g(u)\, du \right| dt \leq M \|g\|_1.$$

Theorem 1. *Suppose K, H and λ_N satisfy Assumptions* I—IV. *Then if f is a bounded continuous complex-valued function on (C, D), there exists*

(i) *a function g in $L_1((A, B))$ such that f is the K-transform of g,* i.e.

$$f(s) = \int_A^B K(t, s)\, g(t)\, dt; \tag{6}$$

(ii) *a unique finite complex regular Borel measure μ on (A, B) such that f is the K-transform of μ* (i.e. (2));

if and only if, for each N, $\sigma_{N,f}$ is integrable and

(i)′ *they converge in the L_1-norm;*

(ii)′ *they are uniformly bounded inthe L_1-norm.*

Proof. Suppose g is in $L_1((A, B))$ and f is the K-transform of g. If $\varepsilon > 0$, there exists ψ in $C_{00}((A, B))$ with $\|g - \psi\|_1 < \varepsilon$. So, for all N sufficiently large, Assumption IV and Lemma 1 give

$$\|\sigma_{N,f} - g\|_1 = \|I_N g - g\|_1 \leq \|I_N\|\, \|g - \psi\|_1 + \|I_N \psi - \psi\|_1 + \|\psi - g\|_1 \\ \leq M\varepsilon + \varepsilon + \varepsilon.$$

If f is the K-transform of measure μ on (A, B), then

$$\|\sigma_{N,f}\|_1 = \int_A^B \left| \int_A^B T_N(t, u)\, \mu(du) \right| dt \leq M |\mu|((A, B)) < \infty.$$

Conversely, suppose that the $\sigma_{N,f}$ are integrable and converge in the L_1-norm to an integrable function g. Then, for each $s \in (C, D)$,

$$\left| \int_A^B K(t, s) \, (\sigma_{N,f}(t) - g(t)) \, dt \right| \leq \sup_{t \in (A,B)} |K(t, s)| \, \|\sigma_{N,f} - g\|_1 \to 0 .$$

The result now follows from Assumption I.

Finally, suppose that $\|\sigma_{N,f}\|_1 \leq \alpha$ for all N. For every φ in $L_1((C, D))$, put

$$\Phi(\hat{\varphi}) = \int_C^D \varphi(s) f(s) \, ds .$$

For each N, define the scalar-valued map Φ_N on $C_0((A, B))$ by

$$\Phi_N(\psi) = \int_A^B \psi(t) \, \sigma_{N,f}(t) \, dt , \qquad \psi \in C_0((A, B)) .$$

Using Fubini's theorem and Assumption I, the Lebesgue dominated convergence theorem gives, for all φ in L_1 $((C, D))$,

$$\Phi_N(\hat{\varphi}) = \int_A^B \sigma_{N,f}(t) (\int_C^D K(t, s) \, \varphi(s) \, ds) \, dt$$

$$= \int_C^D \varphi(s) \, (\int_A^B K(t, s) \, \sigma_{N,f}(t) \, dt) \, ds \to \Phi(\hat{\varphi}), \ N \to \infty .$$

Now $\|\sigma_{N,f}\|_1 \leq \alpha$ implies that $|\Phi_N(\psi)| \leq \alpha \|\psi\|_\infty$ for all ψ in $C_0((A, B))$. Therefore, since the functions $\hat{\varphi}$ lie densely in $C_0((A, B))$. (Assumption II), $\lim \Phi_N(\psi)$ exists for all ψ in $C_0((A, B))$. Denote this limit by $\Phi(\psi)$. Then Φ is a bounded linear functional on $C_0((A, B))$ and so the Riesz representation theorem gives the existence of a unique regular complex measure μ such that

$$\lim_N \Phi_N(\psi) = \int_A^B \psi(t) \, \mu(dt) , \qquad \psi \in C_0((A, B)) . \tag{7}$$

So, for all φ in $L_1((C, D))$,

$$\int_C^D \varphi(s) f(s) \, ds = \int_A^B \hat{\varphi}(t) \, \mu(dt) = \int_C^D \varphi(s) \, (\int_A^B K(t, s) \, \mu(dt)) \, ds .$$

Hence f is the K-transform of μ.

3. Vector-valued Transforms. Let X be a quasi-complete, locally convex topological vector space. For each N, let $\Phi_N : C_0((A, B)) \to X$ be a linear map. The set of Φ_N is said to be weakly equi-compact if there exists a weakly compact subset W of X such that

$$\{\Phi_N(\psi) ; \ \psi \in C_0((A, B)), \ \|\psi\|_\infty \leq 1, \ N = 1, 2, \ldots \} \subset W .$$

Suppose that K, H and the λ_N again satisfy Assumptions I—IV. Let $\mathscr{B}((A, B))$ denote the σ-algebra of all Borel sets in (A, B).

Theorem 2. *Given a bounded, weakly continuous function $f:(C, D) \to X$, there exists a regular measure $\mu : \mathscr{B}((A, B)) \to X$ such that f is the K-transform of μ if and only if, for each N, $\sigma_{N,f}$ is integrable and the set of maps Φ_N : $C_0((A, B)) \to X$, defined by*

$$\Phi_N(\psi) = \int_A^B \psi(t)\, \sigma_{N,f}(t)\, \mathrm{d}t, \qquad \psi \in C_0((A, B)),$$

is weakly equi-compact.

In the proof of this theorem, we use the following lemma (see [5]).

Lemma 2. *Let $F : (A, B) \to X$ be a function such that, for every $\psi \in C_{00}((A, B))$ there exists an element $x_\psi \in X$ with*

$$\langle x_\psi, x' \rangle = \int_A^B \psi(t)\ \langle F(t), x' \rangle\, \mathrm{d}t, \quad x' \in X'.$$

Suppose that $\{x_\psi ; \psi \in C_{00}((A, B)), \|\psi\|_\infty \leq 1\}$ is a relatively weakly compact subset of X. Then F is integrable and

$$x_\psi = \int_A^B \psi(t)\, F(t)\, \mathrm{d}t, \qquad \psi \in C_{00}((A, B)).$$

Proof. For $\psi \in C_{00}((A, B))$, define $\Lambda(\psi) = x_\psi$. Then there exists a regular measure $\mu : \mathscr{B}((A, B)) \to X$ such that

$$\Lambda(\psi) = \int_A^B \psi(t)\, \mu(\mathrm{d}t), \qquad \psi \in C_{00}((A, B)),$$

(see [5]; Proposition 1). For every $E \in \mathscr{B}((A, B))$,

$$\langle \mu(E), x' \rangle = \int_E \langle F(t), x' \rangle\, \mathrm{d}t, \qquad x' \in X'.$$

Hence F is integrable.

Proof of Theorem 2. Suppose that such a measure exists. Then $\sigma_{N,f}(t) = \int_A^B T_N(t, u)\, \mu(\mathrm{d}u)$ and so $\sigma_{N,f}$ is continuous for each N. Let $\psi \in C_{00}((A, B))$ and N be arbitrary. For each $\varphi \in C_{00}((A, B))$, there exists $x_\varphi \in X$ such that

$$\langle x_\varphi, x' \rangle = \int_A^B \varphi(t)\, \psi(t)\, \langle \sigma_{N,f}(t), x' \rangle\, \mathrm{d}t, \qquad x' \in X'$$

(see [2]; III. 3.3 Proposition 7); moreover, if $\|\varphi\|_\infty \leq 1$, then x_φ belongs to a scalar multiple of the closed convex hull of the range of $\psi\, \sigma_{N,f}$ which is compact. Lemma 2 implies that $\psi\, \sigma_{N,f}$ is integrable. Since

$$\int_A^B \psi(t)\,\sigma_{N,f}(t)\,\mathrm{d}t = \int_A^B \psi(t)\,\Big(\int_A^B T_N(t,u)\,\mu(\mathrm{d}u)\Big)\,\mathrm{d}t = \int_A^B \Big(\int_A^B \psi(t)\,T_N(t,u)\,\mathrm{d}t\Big)\,\mu(\mathrm{d}u)$$

and

$$\Big|\int_A^B \psi(t)\,T_N(t,u)\,\mathrm{d}t\Big| \leq M\,\|\psi\|_\infty,$$

we have

$$\int_A^B \psi(t)\,\sigma_{N,f}(t)\,\mathrm{d}t \in M\,\|\psi\|_\infty\,\bar{C}R(\mu)$$

(where $\bar{C}R(\mu)$ denotes the symmetric closed convex hull of the range of μ). Since the range of μ is weakly compact (see [8]), the Krein theorem ([8]) implies the weak compactness of the restrictions of Φ_N to $C_{00}((A, B))$. Lemma 2 implies the integrability of $\sigma_{N,f}$. Since $C_0((A, B))$ is the uniform closure of $C_{00}((A, B))$, the weak equicompactness of the Φ_N on $C_0((A, B))$ follows.

Suppose conversely that the $\sigma_{N,f}$ are integrable and there exists a weakly compact subset W of X such that

$$\{\Phi_N(\psi);\ \psi \in C_0((A, B)),\ \|\psi\|_\infty \leq 1,\ N = 1, 2, \ldots\} \subset W.$$

Fix x' in X'. Then there exists a constant $\alpha_{x'}$ such that

$$|\langle \Phi_N(\psi), x'\rangle| \leq \alpha_{x'}\,\|\psi\|_\infty$$

for all N and all $\psi \in C_0((A, B))$. Therefore, proceeding as in Theorem 1, there exists a unique complex regular Borel measure $\mu_{x'}$, such that

$$\langle f(s), x'\rangle = \int_A^B K(t, s)\mu_{x'}, (\mathrm{d}t), \qquad s \in (C, D), \tag{8}$$

and (as in (7)),

$$\lim_N \langle \Phi_N(\psi), x'\rangle = \int_A^B \psi(t)\mu_{x'}, (\mathrm{d}t), \qquad \psi \in C_0((A, B)). \tag{9}$$

That is, for each fixed ψ, $\{\langle \Phi_N(\psi), x'\rangle\}$ is convergent for all x' in X'. Thus $\{\Phi_N(\psi)\}$ is weakly Cauchy and therefore weakly convergent since $\{\Phi_N(\psi);\ N = 1, 2, \ldots\}$ is in the weakly compact set $\|\psi\|_\infty W$. Denote this limit by $\Phi(\psi)$. Then, for all ψ with $\|\psi\|_\infty \leq 1$, $\Phi(\psi)$ is in W; that is Φ is weakly compact. So (see [5] Proposition 1), there exists a regular measure $\mu : \mathscr{B}((A. B)) \to X$ such that

$$\Phi(\psi) = \int_A^B \psi(t)\,\mu(dt), \qquad \psi \in C_0((A, B)).$$

It follows from (9) and the uniqueness of $\mu_{x'}$, that, for all $E \in \mathscr{B}((A, B))$ and all $x' \in X'$, $\langle \mu(E), x' \rangle = \mu_{x'}(E)$. The result is now immediate from (8).

Now let X be a Banach space. Suppose that K, H and the λ_N again satisfy Assumptions I—IV.

Theorem 3. *Given a bounded continuous function f on (C, D) with values in X, there exists a regular measure $\mu : \mathscr{B}((A, B)) \to X$ of finite total variation ν such that f is the K-transform of μ if and only if there exists a constant J such that*

$$\int_A^B \|\sigma_{N,f}(t)\| \, dt \leq J, \qquad N = 1, 2, \ldots . \tag{10}$$

Proof. Suppose that such a measure exists. Then, by Assumption III,

$$\int_A^B \|\sigma_{N,f}(t)\| \, dt = \int_A^B \| \int_A^B T_N(t, u) \, \mu(du) \| \, dt \leq M\nu((A, B)) < \infty .$$

Conversely, suppose that (10) holds. Define $\Phi_N : C_0((A, B)) \to X$ by

$$\Phi_N(\psi) = \int_A^B \psi(t) \, \sigma_{N,f}(t) \, dt, \qquad \psi \in C_0((A, B)).$$

Then for $\varphi \in L_1((C, D))$, Assumption I and the Lebesgue dominated convergence theorem give that, for each x' in X',

$$\langle \Phi_N(\hat{\varphi}), x' \rangle = \int_A^B \langle \sigma_{N,f}(t), x' \rangle \left(\int_C^D K(t, s) \, \varphi(s) \, ds \right) dt$$

$$= \int_C^D \varphi(s) \left(\int_A^B K(t, s) \, \sigma_{N,f_{x'}}(t) \, dt \right) ds \to \int_C^D \varphi(s) f_{x'}(s) \, ds, \; N \to \infty$$

where $f_{x'}(s) = \langle f(s), x' \rangle$. Since φ is integrable and f is bounded and continuous, we conclude that $\lim \langle \Phi_N(\hat{\varphi}), x' \rangle = \langle \int_C^D \varphi(s) f(s) \, ds, x' \rangle$ for all x' in X'. It follows that weak-limit $\Phi_N(\psi)$ exists for all ψ in a dense subset of $C_0((A, B))$ (Assumption II) and therefore, by (10), this weak limit exists for all ψ in $C_0((A, B))$. Denote this limit by $\Phi(\psi)$. We obtain the required measure from the following lemma (see [4]; III 19, 3, Theorems 2 and 3).

Lemma 3. *If $F : C_0((A, B)) \to X$ is a linear map, then there exists a regular measure $\mu : \mathscr{B}((A, B)) \to X$ with finite variation such that*

$$F(\psi) = \int_A^B \psi(t) \, \mu(dt), \qquad \psi \in C_0((A, B)),$$

if and only if there exists a constant Q such that, for any finite family of functions $\psi_1, \ldots, \psi_n$ *in* $C_0((A, B))$ *with* $\sum_{i=1}^{n} |\psi_i(t)| \leq 1$ *for all t in* (A, B), $\sum_{i=1}^{n} \|F(\psi_i)\| \leq Q$.

To show that the linear map Φ satisfies Lemma 3, let $\psi_1, \ldots, \psi_n$ be any family in $C_0((A, B))$ with $\sum_{i=1}^{n} |\psi_i| \leq 1$. Then, for each N,

$$\sum_{i=1}^{n} \|\Phi_N(\psi_i)\| \leq \sum_{i=1}^{n} \int_A^B |\psi_i(t)| \, \|\sigma_{N,f}(t)\| \, \mathrm{d}t \leq J .$$

Therefore, since $\|\Phi(\psi_i)\| \leq \limsup \|\Phi_N(\psi_i)\|$, $\sum_{i=1}^{n} \|\Phi(\psi_i)\| \leq J$ and so there exists a regular measure $\mu : \mathscr{B}((A, B)) \to X$ with finite variation such that

$$\Phi(\psi) = \int_A^B \psi(t) \, \mu(\mathrm{d}t), \qquad \psi \in C_0((A, B)).$$

So, for all φ in $L_1((C, D))$,

$$\int_C^D \varphi(s) f(s) \, \mathrm{d}s = \Phi(\hat{\varphi}) = \int_A^B \hat{\varphi}(t) \, \mu(\mathrm{d}t) = \int_C^D \varphi(s) \left(\int_A^B K(t, s) \, \mu(\mathrm{d}t) \right) \mathrm{d}s$$

and hence f is the K-transform of μ.

Theorem 4. *Given a bounded continuous function* $f : (C, D) \to X$, *there exists an X-valued Bochner integrable function g on* (A, B) *such that f is the K-transform of g if and only if*

$$\lim_{N,P \to \infty} \int_A^B \|\sigma_{N,f}(t) - \sigma_{P,f}(t)\| \, \mathrm{d}t = 0 .$$

Proof. Suppose that g is Bochner integrable and f is its K-transform. Let $\{E_i\}_1^n$ be a finite family of Borel subsets of (A, B) with finite Lebesgue measure and $\{\beta_i\}_1^n$ a finite family of vectors in X. Define $h : (A, B) \to X$ by $h(t) = \sum_1^n \beta_i \chi_{E_i}(t)$. Then

$$\int_A^B \| \int_A^B T_N(t, u) \, h(u) \, \mathrm{d}u - h(t)\| \, \mathrm{d}t = \int_A^B \| \sum_1^n \beta_i (\int_A^B T_N(t, u) \, \chi_{E_i}(u) \, \mathrm{d}u - \chi_{E_i}(t))\| \, \mathrm{d}t \leq$$

$$\leq \sum_1^n \left(\|\beta_i\| \int_A^B | \int_A^B T_N(t, u) \, \chi_{E_i}(u) \, \mathrm{d}u - \chi_{E_i}(t)| \, \mathrm{d}t \right)$$

which tends to 0 as $N \to \infty$ (by the first part of Theorem 1). Therefore, since the set of all such functions h is dense in the space of all Bochner integrable functions and as $\sigma_{N,f}(t) = \int_A^B T_N(t, u)\, g(u)\, \mathrm{d}u$, we have

$$\lim_N \int_A^B \|\sigma_{P,f}(t) - g(t)\|\, \mathrm{d}t = 0 .$$

Conversely, suppose that the sequence $\{\sigma_{N,f}\}$ is Cauchy in the Bochner space norm. Since this space is complete, $\sigma_{N,f}$ converges in the Bochner norm to a Bochner integrable function g. So, for each s in (C, D),

$$\|\int_A^B (g(t) - \sigma_{N,f}(t))\, K(t, s)\, \mathrm{d}t\| \leq \int_A^B \|g(t) - \sigma_{N,f}(t)\|\, |K(t, s)|\, \mathrm{d}t \leq$$

$$\leq \sup_t |K(t, s)|\, \|\sigma_{N,f} - g\|_B \to 0, \quad N \to \infty .$$

The result now follows from Assumption I.

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Mathematics Department
Ballarat Institute of Advanced Education
Ballarat