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ONE-SIDED BASES OF SEMIGROUPS

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T. Tamura in [5] introduced the notion of a right (left) base of a semigroup and by means of this notion some properties of semigroups are investigated. In the present paper we shall describe the structure of semigroups containing one-sided bases. We shall consider only right bases, since for left bases analogous statements hold.

Definition 1. ([5]). We say that a subset \( A \) of a semigroup \( S \) is a right base of \( S \) if \( SA \cup A = S \), but there exists no proper subset \( B \subset A \), for which \( SB \cup \cup B = S \).

Now we introduce a quasi-ordering into \( S \), namely
\[ a \preceq b \text{ means } a \cup Sa \subseteq b \cup Sb. \]

Lemma 1. ([5]). Let \( A \) be a right base of \( S \). If \( a, b \in A \) and \( a \in Sb \), then \( a = b \).

Lemma 2. ([5]). A non-empty subset \( A \) of \( S \) is a right base of \( S \) if and only if \( A \) satisfies the following conditions:
(1) for any \( x \in S \) there exists \( a \in A \) such that \( x \preceq a \).
(2) for any two distinct elements \( a, b \in A \) neither \( a \preceq b \), nor \( b \preceq a \).

The set of all elements of \( S \) generating the same principal left ideal as a fixed element of \( S \) is called an \( L \)-class of \( S \) (see [2]). The principal left ideal \( a \cup Sa \) will be denoted by \( (a)_L \).

Simple examples of semigroups show that a right base \( A \) of \( S \) need not be a subsemigroup and therefore not a left ideal, either.

Further we show some conditions when a right base of \( S \) is a left ideal, and also a subsemigroup of \( S \).

Remark 1. We can show easily that a right base \( A \) of a semigroup \( S \) is a left ideal of \( S \) if and only if \( A = S \).

A semigroup \( S \) is called right singular if for every two elements \( x, y \in S \) we have \( xy = y \).

Theorem 1. A right base \( A \) of a semigroup \( S \) is a subsemigroup of \( S \) if and only if \( A \) is a right singular semigroup.
Proof. (a) Let a right base $A$ of $S$ be a subsemigroup of $S$. It is necessary to show that for any $a, b \in A$, $ab = b$. According to the assumption $ab \in A$. Therefore, $ab = c$ for some $c \in A$, whence it follows that $c \in Sb$. Lemma 1 implies that $c = b$, therefore for arbitrary $a, b \in A$, $ab = b$.

(b) The converse statement is evident.

**Corollary.** If a right base $A$ of a semigroup $S$ is a subsemigroup of $S$, then $S$ contains at least one idempotent.

By the following example of a semigroup we can ascertain that if a right base of $S$ is a subsemigroup and therefore a right singular subsemigroup, the whole semigroup need not be such one.

**Example 1.** Let $S = \{a, b, c, d\}$ be a semigroup with the multiplication table

$$
\begin{array}{c|cccc}
\text{a} & \text{b} & \text{c} & \text{d} \\
\hline
\text{a} & a & a & a & a \\
\text{b} & a & b & a & d \\
\text{c} & c & c & c & c \\
\text{d} & a & b & a & d \\
\end{array}
$$

$A = \{b, d\}$ is a right base and a subsemigroup, but the whole semigroup is not right singular.

The notion of a maximal proper left ideal is used in the same sense as in [3]. We say that a semigroup $S$ contains a left ideal $L^*$, if $L^*$ is such a maximal proper left ideal, in which every proper left ideal of $S$ is included (see [4]).

**Lemma 3.** Let $A$ be a right base of a semigroup $S$. Let $a_0 \in A$ be any element of $A$. If $(a_0)_L = (b)_L$ for some $b \in S, b \neq a_0$, then $b$ is an element of a right base of $S$, distinct from $A$.

**Proof.** Let $B = [A - \{a_0\}] \cup \{b\}$. It is clear that $A \neq B$. We show that $B$ is also a right base of $S$. It is sufficient to show that $B$ satisfies conditions (1), (2) of Lemma 2. Let $x$ be an arbitrary element of $S$. Then, since $A$ is a right base of $S$, there exists $a \in A$ such that $x \leq a$. Now, there are only two possibilities. 1. $a \neq a_0$, 2. $a = a_0$. If $a = a_0$, then $a \in B$. If $a = a_0$, then $a \in B$, but $(a_0)_L = (b)_L$, therefore if $x \leq a$ then $x \cup Sx \subseteq a \cup Sa = b \cup Sb$, whence it follows that $x \leq b$ and $b \in B$. It means that $B$ satisfies condition (1) of Lemma 2. Now let $b_1, b_2 \in B$ be arbitrary elements, but distinct. If both elements are distinct from $b$, then $b_1 \in A, b_2 \in A$ and since $A$ is a right base of $S$, then neither $b_1 \leq b_2$, nor $b_2 \leq b_1$. Let for instance $b_1 = b$. If $b_1 \leq b_2$ then $a_0 \leq b_2$, where $a_0 \in A, b_2 \in A$. But $A$ is a right base of $S$, therefore this is not possible. Similarly we can show that the relation $b_2 \leq b_1$ cannot be fulfilled. But it means that $B$ also satisfies condition (2) of Lemma 2, therefore, $B$ is a right base of $S$, distinct from $A$. 

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**Theorem 2.** Let $\mathcal{A}$ be the union of all right bases of a semigroup $S$. If $L = S - \mathcal{A}$ is non-empty, then $L$ is a left ideal of $S$.

**Proof.** To prove our statement, we must show that if $x \in S$, $a \in L = S - \mathcal{A}$, then $xa \in L$. Let us assume that $xa \in L$. Then $b = xa \in \mathcal{A}$, and thus $b$ belongs at least to one right base $A$ of $S$, and we have that $b \in Sa$, therefore $Sb \subseteq Sa$, $b \cup Sb \subseteq a \cup Sa$. We show that $(b)_L \neq (a)_L$. If $(b)_L = (a)_L$, then, since $b \in \mathcal{A}$, according to Lemma 3 $a \in \mathcal{A}$, and it is a contradiction to the assumption, because $a \in S - \mathcal{A}$. It means that $(b)_L \neq (a)_L$. And since $A$ is a right base, then to the element $a$ there exists an element $b_1 \in A$ such that $a \leq b_1$. Thus, we have $b \leq a \leq b_1$, therefore $b \leq b_1$, but it is a contradiction to condition (2) of Lemma 2, because $b, b_1 \in A$. Hence, $xa \in S - \mathcal{A}$.

The following example of a semigroup shows that $L = S - \mathcal{A}$ need not be a maximal left ideal of $S$.

**Example 2.** Let $S = \{a, b, c, d\}$ be a semigroup with the multiplication table:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>b</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>b</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>c</td>
</tr>
<tr>
<td>d</td>
<td>a</td>
<td>b</td>
<td>d</td>
<td>d</td>
</tr>
</tbody>
</table>

All right bases of $S$ are: $A_1 = \{b, c\}$ and $A_2 = \{b, d\}$. $S - \mathcal{A} = \{a\}$ is a left ideal of $S$, but it is not a maximal proper one.

In the following we shall find conditions when $L = S - \mathcal{A}$ is a maximal proper left ideal of $S$.

**Theorem 3.** Let $\mathcal{A} \neq \emptyset$. $S - \mathcal{A}$ is a maximal proper left ideal of a semigroup $S$ if and only if $\mathcal{A} \neq S$ and $\mathcal{A} \subseteq a \cup Sa$ for all $a \in \mathcal{A}$.

**Proof.** (a) Let $L = S - \mathcal{A}$ be a maximal proper left ideal of a semigroup $S$. Therefore $\mathcal{A} \neq S$. Let $a \in \mathcal{A}$. If $\mathcal{A} \subseteq a \cup Sa$ does not hold, then $(S - \mathcal{A}) \cup \cup (a)_L$ as a union of two left ideals is a left ideal of $S$, but a proper one. Then $S - \mathcal{A}$ is not a maximal left proper ideal, which is a contradiction to the assumption.

(b) Let $\mathcal{A} \subseteq a \cup Sa$ for all $a \in \mathcal{A}$, and $\mathcal{A} \neq S$. We have to prove that $S - \mathcal{A}$ is a maximal proper left ideal of $S$. According to Theorem 2, $S - \mathcal{A}$ is a (proper) left ideal of $S$. Let $S - \mathcal{A} \subseteq L'$, where $L'$ is a left ideal of $S$ and $S - \mathcal{A} \neq L'$. Then $L' \cap \mathcal{A} \neq \emptyset$. Let $a \in L' \cap \mathcal{A}$, so $a \in L'$. It follows that $Sa \subseteq SL' \subseteq L'$, $a \cup Sa \subseteq L'$. Whence, and according to the assumption, we obtain $\mathcal{A} \subseteq a \cup Sa \subseteq L'$. Consequently, $\mathcal{A} \subseteq L'$, $S - \mathcal{A} \subseteq L'$, therefore $S = L'$.

It is clear that $S$ may contain maximal proper left ideals distinct from
The question arises: when will \( S - \mathcal{A} \) be such a maximal proper left ideal of \( S \) that every proper left ideal of \( S \) will be included in it?

We say that an element \( a \in S \) is left invertible if \( Sa = S \).

**Lemma 4.** ([1]). Let a semigroup \( S \) contain at least one left invertible element. Then \( S \) contains the ideal \( L^* \) and the complement of this ideal is the set of all left invertible elements.

**Theorem 4.** Let \( \emptyset \neq \mathcal{A} \subseteq S \). \( S - \mathcal{A} = L^* \) if and only if every right base of \( S \) is one-element and one from the following conditions holds:

1. Every right base of \( S \) is formed by a left invertible element.
2. A semigroup \( S \) contains only one right base \( A = \{a\} \) and we have: \( a \cup Sa = S \), but \( a \in Sa \).

**Proof.** (a) Let be \( S - \mathcal{A} = L^* \). Thus \( S - \mathcal{A} \) is a maximal proper left ideal. Theorem 3 implies that for any \( a \in \mathcal{A} \) the following holds: \( \mathcal{A} \subseteq a \cup Sa \). But, moreover, every proper left ideal of \( S \) is included in \( S - \mathcal{A} \). Now we shall show that \( S - \mathcal{A} \subseteq a \cup Sa \) for any \( a \in \mathcal{A} \) as well. Till now we have: \( \mathcal{A} \subseteq a \cup Sa \) and \( S - \mathcal{A} = L^* \). There are only two possibilities: either \( a \cup Sa \) is a proper left ideal of \( S \) and then \( a \cup Sa \subseteq S - \mathcal{A} \), or \( a \cup Sa = S \). The first possibility cannot hold, because at least \( a \in S - \mathcal{A} \). Therefore the other possibility must hold, so \( a \cup Sa = S \), for any \( a \in \mathcal{A} \). Thus \( \{a\} \) is a right base of \( S \). And as \( a \) is an arbitrary element of \( \mathcal{A} \), then all right bases are one-element. Therefore, only the following three cases are possible.

1. \( a \cup Sa = S \), \( a \in Sa \) for any element \( a \in \mathcal{A} \). (It means that every element \( a \in \mathcal{A} \) is left invertible.)
2. \( a \cup Sa = S \), \( a \in Sa \) for any element \( a \in \mathcal{A} \).
3. \( a \cup Sa = S \), \( a \in Sa \) for some element \( a \in \mathcal{A} \), but \( b \cup Sb = S \), \( b \in Sb \) for another element \( b \in \mathcal{A} \), \( b \neq a \).

We shall show that if \( S - \mathcal{A} \) is a maximal proper left ideal of \( S \), then case (3) cannot occur and in case (2) the semigroup \( S \) contains only one such base.

Let us assume that in case (2) a semigroup \( S \) contains at least two right bases, \( A_1 = \{a_1\} \), \( A_2 = \{a_2\} \) such that \( a_1 \cup Sa_1 = S \), \( a_1 \in Sa_1 \), \( a_2 \cup Sa_2 = S \), \( a_2 \in Sa_2 \). Then \( S - \mathcal{A} \subseteq S \mathcal{A}_1 \subseteq S \), where \( S - \mathcal{A} \neq S \mathcal{A}_1 \), because \( a_2 \in Sa_1 \). But it means that \( S - \mathcal{A} \) is not a maximal proper left ideal, and this is a contradiction. If case (3) occurs, then again \( S - \mathcal{A} \subseteq Sb \subseteq S \), where \( S - \mathcal{A} \neq Sb \), because \( a \in Sb \). It means that \( S - \mathcal{A} \) is not a maximal proper left ideal, which is again a contradiction to the assumption.

(b) Let us assume that all right bases of \( S \) are one-element and that one condition from (1), (2) is satisfied. If (1) holds, then the statement follows from Lemma 4. If (2) holds, then \( S - \{a\} = S - \mathcal{A} = L \) is a left ideal. It is evident that it is a maximal proper left one. We show that every proper left ideal of \( S \) is included in \( L \). Let \( L_1 \) be a left ideal of \( S \) which is not included
in $L$. Then, evidently $a \in L_1$, therefore $Sa \subseteq SL_1 \subseteq L_1$. But, since $a \in L_1$, then $S = a \cup Sa \subseteq L_1$, therefore $L_1 = S$. It means that $L = L^*$.  

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