Mária Polinová

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REPRESENTATION OF LATTICES BY EQUIVALENCE RELATIONS

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Introduction

P. M. Whitman [5] proved that every lattice L can be embedded into the lattice of all equivalence relations on a set M. If L is countable (in particular finite), then P. M. Whitman's construction yields M countable. S. K. Thomason [4] gave a more simple construction for the case of L finite. In this paper we shall show that to any sublattice $\mathscr{L}$ of the lattice of all equivalence relations on a set M with card $\mathscr{L} \leqq m$, where m is an infinite cardinal number, there is a subset $Q \subset M$ with card $Q \leqq m$ such that the lattice of reduced equivalence relations to the set Q is isomorphic to $\mathscr{L}$. An analogous result will be proved for algebraic lattices. By an algebraic lattice (see e.g. [1]) it is meant a complete lattice in which every element is a join of compact elements. Denote by $\mathscr{E}(M)$ and $\mathscr{C}(\mathfrak{M})$ the lattice of all equivalence relations on the set M, or the lattice of all congruence relations on the algebra $\mathfrak{M}$, respectively. Let $\mathscr{L}$ be a sublattice of the lattice $\mathscr{E}(M)$; then, according to B. Jónsson [3], $\mathscr{L}$ is

(1) of type 1 if $\Theta \vee \Phi = \Theta . \Phi$,
(2) of type 2 if $\Theta \vee \Phi = \Theta . \Phi . \Theta$,
(3) of type 3 if $\Theta \vee \Phi = \Theta . \Phi . \Theta . \Phi$

for every $\Theta, \Phi \in \mathscr{L}$ ($\Theta . \Phi$ denotes the product of Θ and Φ). Let Θ be a binary relation on a set M. We denote by Θ_Q the restriction of Θ to the subset $Q \subset M$, i. e. $(x, y) \in \Theta_Q$ if and only if $x, y \in Q$ and $(x, y) \in \Theta$. If Θ is an equivalence relation, then Θ_Q is an equivalence relation, too. If $\mathscr{L}$ is a sublattice of $\mathscr{E}(M)$ and $Q \subset M$, then $\mathscr{L}_Q = \{\Theta_Q \mid \Theta \in \mathscr{L}\}$.

Results

Theorem 1. *Let $\mathscr{L}$ be a sublattice of $\mathscr{E}(M)$ with* card $\mathscr{L} \leqq m$, *where m is an infinite cardinal number. Then there exists a subset $Q \subset M$ with* card $Q \leqq m$

such that $\mathscr{L}_Q$ is a sublattice of $\mathscr{E}(Q)$ isomorphic to $\mathscr{L}$. Moreover if $\mathscr{L}$ is of type p ($p \in \{1, 2, 3\}$), then $\mathscr{L}_Q$ is of type p, too.

Corollary 1. *Let m be an infinite cardinal number and let L be a lattice with* $\operatorname{card} L \leqq m$. *Then L is isomorphic to a sublattice of $\mathscr{E}(Q)$ with* $\operatorname{card} Q \leqq m$. *In particular any countable (or finite) lattice is isomorphic to a sublattice of $\mathscr{E}(N)$ with* $\operatorname{card} N \leqq \aleph_0$.

Theorem 2. *If the lattice $\mathscr{L}$ of Theorem 1 is a complete sublattice* [1] *of $\mathscr{E}(M)$, then the lattice $\mathscr{L}_Q$ of Theorem 1 is a complete sublattice of $\mathscr{E}(Q)$, too.*

Corollary 2. *Any algebraic lattice L with* $\operatorname{card} L \leqq m$, *where m is an infinite cardinal number, is isomorphic to a complete sublattice of $\mathscr{E}(Q)$ with* $\operatorname{card} Q \leqq m$.

Corollary 3. *Let $\mathfrak{A} = (A, F)$ be an algebra having only finitary operations and let C be a sublattice of the lattice $\mathscr{C}(\mathfrak{A})$ with* $\operatorname{card} C \leqq m$, *where m is an infinite cardinal number. Then there exists a subalgebra $\mathfrak{A}' = (A', F)$ of the algebra $\mathfrak{A}$ with* $\operatorname{card} A' \leqq (m + \operatorname{card} F)\,\aleph_0$ *such that the lattice C is isomorphic to a sublattice C' of the lattice $\mathscr{C}(\mathfrak{A}')$. In particular if* $\operatorname{card} F \leqq \aleph_0$, *then* $\operatorname{card} A' \leqq m$. *If C is of type p ($p \in \{1, 2, 3\}$) then C' is of the type p, too. If C is a complete sublattice of $\mathscr{C}(\mathfrak{A})$, then C' is a complete sublattice of $\mathscr{C}(\mathfrak{A}')$, too.*

Proofs of Results

Lemma. *Let $\mathscr{L}$ be a sublattice of the lattice $\mathscr{E}(M)$ and let $Q \subset M$. Then for the elements of $\mathscr{L}_Q$ the following conditions hold ($\Theta, \Phi, \Theta_\gamma \in \mathscr{L}$).*

(1) *If $\Theta \leqq \Phi$, then $\Theta_Q \leqq \Phi_Q$.*

(2) $$\Big(\bigwedge_{\gamma\in\Gamma} \Theta_\gamma\Big)_Q = \bigwedge_{\gamma\in\Gamma} (\Theta_\gamma)_Q .$$

(3) $$\Big(\bigvee_{\gamma\in\Gamma} \Theta_\gamma\Big)_Q \geqq \bigvee_{\gamma\in\Gamma} (\Theta_\gamma)_Q .$$

Proof of Lemma.

(1) If $(x, y) \in \Theta_Q$ then $x, y \in Q \subset M$ and $(x, y) \in \Theta$.

This implies $x, y \in Q$ and $(x, y) \in \Phi$, hence $(x, y) \in \Phi_Q$.

(2) $(x, y) \in (\bigwedge_{\gamma\in\Gamma} \Theta_\gamma)_Q$ if and only if $x, y \in Q$ and $(x, y) \in \bigwedge_{\gamma\in\Gamma} \Theta_\gamma$. This is true if and only if $x, y \in Q$ and $(x, y) \in \Theta_\gamma$ for each $\gamma \in \Gamma$. This is equivalent to $(x, y) \in \bigwedge_{\gamma\in\Gamma} (\Theta_\gamma)_Q$.

(3) follows from (1).

Proof of Theorem 1. According to the Lemma it is sufficient to show that there exists $Q \subset M$ such that the following three conditions are fulfilled:

(4) $\operatorname{card} Q \leqq m$.

(5) The correspondence $\Theta \mapsto \Theta_Q$ is one-one.

(6) $(\Theta \vee \Phi)_Q \leqq \Theta_Q \vee \Phi_Q$ for any $\Theta, \Phi \in \mathscr{L}$.

We shall construct a sequence of sets Q_n by induction. For every $\Theta, \Phi \in \mathscr{L}$ with $\Theta < \Phi$ choose elements $a, b \in M$ with $(a, b) \in \Phi$ but $(a, b) \notin \Theta$; denote Q_0 the set of all these elements a, b. Obviously, $\operatorname{card} Q_0 \leqq m$. Now we construct the sets $O_i, i \in \{1, 2, \ldots\}$, as follows. Let us suppose that we have already constructed Q_i $(i \in \{0, 1, \ldots,\})$. For every pair $\Theta, \Phi \in \mathscr{L}$ and for every pair $(a, b) \in Q_i \times Q_i$ with $(a, b) \in (\Theta \vee \Phi)_{Q_i}$ but $(a, b) \notin \Theta_{Q_i} \vee \Phi_{Q_i}$ choose a finite sequence $t_0, t_1, \ldots, t_n \in M$ such that $a = t_0 \Theta t_1 \Phi t_2 \ldots t_{n-1} \Phi t_n = b$ and all elements of these sequences add to the set Q_i. Thus we obtain the set Q_{i+1}. It is easy to prove that $\operatorname{card} Q_{i+1} \leqq m$. Obviously, $Q_i \subset Q_{i+1}$ for each $i \in \{0, 1, \ldots\}$. Let $Q = \bigcup_{i=0}^{\infty} Q_i$. Obviously, $\operatorname{card} Q \leqq m$, which proves (4). Now we prove (5). If $\Theta \neq \Phi$, then either $\Theta \wedge \Phi < \Theta$ or $\Theta \wedge \Phi < \Phi$. If $\Theta \wedge \Phi < \Theta$, then there exist elements $a, b \in Q_0 \subset Q \subset M$ with $(a, b) \in \Theta$ but $(a, b) \notin \Theta \wedge \Phi$, i. e. $(a, b) \notin \Phi$. This means $\Theta_Q \neq \Phi_Q$. The proof for $\Theta \wedge \Phi < \Phi$ is analogous. It remains to prove (6). If $a, b \in Q$ and $(a, b) \in (\Theta \vee \Phi)_Q$, then there exists an $i \in N$ such that $(a, b) \in (\Theta \vee \Phi)_{Q_i}$. If $(a, b) \in \Theta_{Q_i} \vee \Phi_{Q_i}$, then obviously $(a, b) \in \Theta_Q \vee \Phi_Q$. If $(a, b) \notin \Theta_{Q_i} \vee \Phi_{Q_i}$, then there exists a finite sequence $t_0, t_1, \ldots, t_n \in Q_{i+1}$ such that $a = t_0 \Theta t_1 \Phi t_2 \ldots t_{n-1} \Phi t_n = b$; this means $(a, b) \in \Theta_{Q_{i+1}} \vee \Phi_{Q_{i+1}}$ and also $(a, b) \in \Theta_Q \vee \Phi_Q$. It can easily be seen that if $\mathscr{L}$ is of type p $(p = 1, 2, 3)$ the construction of Q can be realised in such a way that $\mathscr{L}_Q$ is of the type p, too.

Proof of Corollary 1. By Whitman's theorem [5] L is isomorphic to a sublattice $\mathscr{L}$ of the lattice $\mathscr{E}(M)$ on a set M. By Theorem 1, there exists a set $Q \subset M$ with $\operatorname{card} Q \leqq m$ such that $\mathscr{L}$ is isomorphic to $\mathscr{L}_Q$. Hence L is isomorphic to $\mathscr{L}_Q$.

Proof of Theorem 2. Using the isomorphism $\Theta \to \Theta_Q$ of Theorem 1, we get $(\Theta_1 \vee \Theta_2 \ldots \vee \Theta_n)_Q = (\Theta_1)_Q \vee (\Theta_2)_Q \vee \ldots \vee (\Theta_n)_Q$ for an arbitrary natural number n. This implies immediately the following inequality

(7) $(\bigvee_{\gamma \in \Gamma} \Theta_\gamma)_Q \leqq \bigvee_{\gamma \in \Gamma} (\Theta_\gamma)_Q$ for $\Theta_\gamma \in \mathscr{L}$.

Proof of Corollary 2. By [2], L is isomorphic to the lattice $\mathscr{L} = \mathscr{C}(\mathfrak{M})$ on a finitary algebra $\mathfrak{M} = (M, F)$. By Theorem 2, there exists $Q \subset M$ with $\operatorname{card} Q \leqq m$ such that $\mathscr{L}$ is isomorphic to $\mathscr{L}_Q$. Hence L is isomorphic to $\mathscr{L}_Q$.

Proof of Corollary 3. Let us construct a sequence of sets A_n by induction. Let A_0 have the same meaning as Q_0 in the proof of Theorem 1. Now we construct sets A_i, $i \in \{1, 2, \ldots\}$ as follows. Let us suppose that we have already constructed A_i, $i \in \{0, 1, \ldots\}$. In the case of i being even we construct A_{i+1} from A_i in the same way as in the proof of Theorem 1 we constructed Q_{i+1} from Q_i. If i is odd, we set $A_{i+1} = [A_i]$, where $([A_i], F)$ is the algebra generated by A_i. It is easy to prove that $\operatorname{card} A_{i+1} \leqq (m + \operatorname{card} F)\aleph_0$ (see e. g. [1]) in any case. Obviously, $A_i \subset A_{i+1}$. Let $A' = \bigcup_{i=0}^{\infty} A_i$. Obviously, $\operatorname{card} A' \leqq (m + \operatorname{card} F)\aleph_0$ and every equivalence relation $\Theta_{A'}$ is a congruence relation of $\mathfrak{A}'$. It suffices to show that the following statements are true:

(8) $\mathfrak{A}' = (A', F)$ is a subalgebra of the algebra $\mathfrak{A}$.

(9) The correspondence $\Theta \mapsto \Theta_{A'}$ is one-one.

(10) $(\Theta \vee \Phi)_{A'} = \Theta_{A'} \vee \Phi_{A'}$.

If $a_0, a_1, \ldots, a_{n-1} \in A'$, then for every $i \in \{0, 1, \ldots, n-1\}$ $a_i \in A_{j(i)}$ for some $j(i) \in N$. Let $k = \max\{j(i),\ i = 0,1, \ldots, n-1\}$, then $a_i \in A_k$ for every $i \in \{0, 1, \ldots, n-1\}$. Hence for every $f_\gamma \in F$, $f_\gamma(a_0, a_1, \ldots, a_{n-1}) \in A_{k+2} \subset A'$, which proves (8). The proof of (9) is analogous to that of (5). It remains to prove (10). If $a, b \in A'$ and $(a, b) \in (\Theta \vee \Phi)_{A'}$, then there exists $i \in N$ such that $(a, b) \in (\Theta \vee \Phi)_{A_i}$. If $(a, b) \in \Theta_{A_i} \vee \Phi_{A_i}$, then obviously $(a, b) \in \Theta_{A'} \vee \Phi_{A'}$. If $(a, b) \notin \Theta_{A_i} \vee \Phi_{A_i}$, then there exists a finite sequence $t_0, t_1, \ldots, t_n \in A_{i+2}$ such that $a = t_0\, \Theta t_1 \Phi t_2 \ldots t_{n-1}\, \Phi t_n = b$. This means $(a, b) \in \Theta_{A_{i+2}} \vee \Phi_{A_{i+2}}$ and $(a, b) \in \Theta_{A'} \vee \Phi_{A'}$, too. The last assertion of Corollary 3 can be obtained using Theorem 2.

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Katedra algebry a teórie čísel
Prírodovedecká fakulta UK
816 31 Bratislava