Bedřich Pondělíček
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A NOTE ON CLASSES OF REGULARITY IN SEMIGROUPS

BEDRICH PONDĚLÍČEK, Poděbrady

Let $S$ be a semigroup. Denote by $\mathcal{R}_S(m, n)$ classes of regularity in $S$ (see R. Croisot [1]), i.e.

$$\mathcal{R}_S(m, n) = \{a \mid a \in a^mSa^n\},$$

where $m, n$ are non-negative integers and $a^0$ means the void symbol.

In [2] I. Fabrici studies sufficient conditions for $\mathcal{R}_S(m, n)$, where $m + n \geq 2$, to be subsemigroups of $S$. In this note we shall study necessary and sufficient conditions for $\mathcal{R}_S(m, n)$ to be subsemigroups, semilattices of groups, right groups and groups, respectively.

It is known [3] that

1. if $0 \leq m_1 \leq m_2$ and $0 \leq n_1 \leq n_2$, then $\mathcal{R}_S(2, 2) \subset \mathcal{R}_S(m_2, n_2) \subset \mathcal{R}_S(m_1, n_1)$;

2. $\mathcal{R}_S(1, 2) = \mathcal{R}_S(1, 1) \cap \mathcal{R}_S(0, 2)$;

3. $\mathcal{R}_S(2, 1) = \mathcal{R}_S(1, 1) \cap \mathcal{R}_S(2, 0)$.

Denote by $E$ the set of all idempotents of a semigroup $S$. Then (see Theorem 3 in [2]).

4. if $1 \leq m$ and $1 \leq n$, then $\mathcal{R}_S(m, n) \neq \emptyset$ if and only if $E \neq \emptyset$.

**Theorem 1.** The class of regularity $\mathcal{R}_S(1, 1)$ is a subsemigroup of a semigroup $S$ if and only if

(5) $E \neq \emptyset$ and $E^2 \subset \mathcal{R}_S(1, 1)$.

**Proof.** Let $\mathcal{R}_S(1, 1)$ be a subsemigroup of $S$. It follows from (4) that $E \neq \emptyset$. Since $E \subset \mathcal{R}_S(1, 1)$, hence $E^2 \subset \mathcal{R}_S(1, 1)$.

Let (5) hold. Then (4) implies that $\mathcal{R}_S(1, 1) \neq \emptyset$. Let $a, b \in \mathcal{R}_S(1, 1)$. Then $a = axa$, $b = byb$ for some $x, y \in S$ and $xa, by \in E$. According to (5) we have $(xa)(by) = (xa)(by)z(xa)(by)$ for some $z \in S$. Therefore, $ab = (axa)(byb) =$
\[ a(xa)(by)b = a(xa)(by)z(xa)(by)b = (axa)b(yzx)a(byb) = (ab)u(ab), \] where \( u = yzx. \) Hence \( ab \in \mathcal{R}_S(1, 1). \)

Remark. From [3] (p. 108) it is known that if \( \mathcal{R}_S(1, 1) \) is a subsemigroup of \( S, \) then \( \mathcal{R}_S(1, 1) \) is a regular semigroup.

**Corollary 1** (cf. [2], Theorem 4(c)). If \( E \) is a subsemigroup of \( S, \) then \( \mathcal{R}_S(1, 1) \) is a subsemigroup of \( S. \)

**Corollary 2** (cf. [2], Theorem 4(d)). \( \mathcal{R}_S(1, 1) \) is an inverse subsemigroup of a semigroup \( S \) if and only if

\[ (6) \quad E \neq \emptyset \text{ and any two idempotents of } S \text{ commute.} \]

Proof. It is known [4] that a semigroup \( S \) is inverse if and only if \( S \) is regular and any two idempotents of \( S \) commute. Evidently (6) implies (5). The rest of the proof follows from Theorem 1 and from the Remark.

Let \( a \) be an element of a semigroup \( S. \) The right (left) principal ideal generated by \( a \) is denoted by \( R(a) = a \cup aS \) (\( L(a) = a \cup Sa \)).

**Lemma 1.** Let \( a, b \in S. \)

1. If \( ab \in \mathcal{R}_S(2, 0), \) then \( ab \in R(aba). \)
2. If \( ab \in R(aba) \) and \( ba \in R(bab), \) then \( ab \in \mathcal{R}_S(2, 0). \)

Proof. 1. If \( ab \in \mathcal{R}_S(2, 0), \) then \( ab = (ab)^2x \) for some \( x \in S. \) This implies that \( ab = aba(bx) \in R(aba). \)

2. If \( ab \in R(aba), \) then \( ab = abax \) for some \( x \in S \) or \( ab = ab \) and in both cases we obtain that \( ab = abu \) for some \( u \in S. \) If \( ba \in R(bab), \) then analogously we can prove that \( ba = babv \) for some \( v \in S. \) Hence we have \( ab = (ab)^2z, \) where \( z = vu. \)

**Theorem 2.** Let \( S \) be a semigroup and \( \mathcal{R}_S(2, 0) \neq \emptyset. \) Then \( \mathcal{R}_S(2, 0) \) is a subsemigroup of \( S \) if and only if

\[ (7) \quad ab \in R(aba) \text{ for any } a, b \in \mathcal{R}_S(2, 0). \]

Proof. Let \( \mathcal{R}_S(2, 0) \) be a subsemigroup of \( S. \) If \( a, b \in \mathcal{R}_S(2, 0), \) then \( ab \in \mathcal{R}_S(2, 0). \) It follows from Lemma 1 that (7) holds.

Let (7) hold. If \( a, b \in \mathcal{R}_S(2, 0), \) then from Lemma 1 it follows that \( ab \in \mathcal{R}_S(2, 0). \) This means that \( \mathcal{R}_S(2, 0) \) is a subsemigroup of \( S. \)

**Corollary 1** (cf. [2], Theorem 5(b)). If the product of local right identities of the elements \( a, b \in \mathcal{R}_S(2, 0) \) is a right identity of the element \( ab, \) then \( \mathcal{R}_S(2, 0) \) is a subsemigroup of a semigroup \( S. \)

Proof. If \( a, b \in \mathcal{R}_S(2, 0), \) then \( a = a^2x \) and \( b = b^2y \) for some \( x, y \in S. \)
The element \( ax (by) \) is a local right identity of \( a \) (of \( b \)). According to the assumption we have \( ab = ab(ax)(by) \in R(aba) \). Hence Theorem 2 implies that \( R_S(2, 0) \) is a subsemigroup of \( S \).

**Corollary 2** (cf. [2], Theorem 5 (e)). If every local right identity of any element of \( R_S(2, 0) \) belongs to the centre of a semigroup \( S \), then \( R_S(2, 0) \) is a subsemigroup of \( S \).

**Proof.** If \( a, b \in R_S(2, 0) \), then \( a = a^2x \) for some \( x \in S \). Therefore \( ab = (a^2x)b = a(ax)b = ab(ax) \in R(aba) \). It follows from Theorem 2 that \( R_S(2, 0) \) is a subsemigroup of \( S \).

**Theorem 3.** The class of regularity \( R_S(2, 1) \) is a subsemigroup of a semigroup \( S \) if and only if (5) and

\[(8) \quad ab \in R(aba) \text{ for any } a, b \in R_S(2, 1)\]

hold.

**Proof.** Let \( R_S(2, 1) \) be a subsemigroup of \( S \). It follows from (4) that \( E = \emptyset \). Since \( E \subseteq R_S(2, 1) \), hence, by (1), we have \( E^2 \subseteq R_S(2, 1) \subseteq R_S(1, 1) \). This means that (5) holds. If \( a, b \in R_S(2, 1) \), then \( ab \in R_S(2, 1) \). According to (1) we have \( ab \in R_S(2, 0) \). It follows from Lemma 1 that \( ab \in R(aba) \) and thus (8) holds.

Let (5) and (8) hold. Then (4) implies that \( R_S(2, 1) \neq \emptyset \). Let \( a, b \in R_S(2, 1) \). Then by (1) we have \( a, b \in R_S(1, 1) \). Theorem 1 and (5) imply that \( R_S(1, 1) \) is a subsemigroup of \( S \) and thus \( ab \in R_S(1, 1) \). According to (8) we have \( ab \in R(aba) \) and \( ba \in R(bab) \). Lemma 1 implies that \( ab \in R_S(2, 0) \). It follows from (3) that \( ab \in R_S(2, 1) = R_S(1, 1) \cap R_S(2, 0) \). The class of regularity \( R_S(2, 1) \) is a subsemigroup of \( S \).

**Corollary.** \( R_S(2, 1) \) is a subsemigroup of a semigroup \( S \) if and only if \( R_S(1, 1) \) is a subsemigroup of \( S \) and \( R_S^2(2, 1) \subseteq R_S(2, 0) \).

**Lemma 2.** The class of regularity \( R_S(2, 2) \) is a union of all subgroups of a semigroup \( S \).

**Proof.** From [3] (pp. 139, 424) it is known that an element \( a \in S \) belongs to some subgroup of \( S \) if and only if \( a \) is totally regular, i.e. \( a = axa \) for some \( x \in S \) and \( xa = ax \). We shall prove that \( R_S(2, 2) \) is the set of all totally regular elements of \( S \).

Let \( a \) be a totally regular element of \( S \). Then \( a = axa \) for some \( x \in S \) and \( ax = xa \). This implies that \( a = (axa)x(axa) = a^2x^3a^2 \in R_S(2, 2) \).

Let now \( a \in R_S(2, 2) \). Then \( a = a^2ya^2 \) for some \( y \in S \). Put \( x = aya \). Then we have \( a = axa \) and \( xa = aya^2 = a^2ya^2ya^2 = a^2ya = ax \).
Lemma 3. $R_S(2, 2) = R_S(2, 0) \cap R_S(0, 2)$. 

(See Lemma 1 in [2].)

Proof. It follows from (1) that $R_S(2, 2) \subseteq R_S(2, 0) \cap R_S(0, 2)$. Let $x \in R_S(2, 0) \cap R_S(0, 2)$. Then $x \in xS \subseteq R(x^2)$ and $x^2 \in xS \subseteq R(x)$. It follows that $R(x) = R(x^2)$. Analogously we can prove that $L(x) = L(x^2)$. From [5] it is known that $x$ belongs to some subgroup of $S$. Lemma 2 implies that $x \in R_S(2, 2)$. Therefore $R_S(2, 2) = R_S(2, 0) \cap R_S(0, 2)$.

Theorem 4. The class of regularity $R_S(2, 2)$ is a subsemigroup of a semigroup $S$ if and only if $E \neq \emptyset$ and 

(9) $ab \in R(aba) \cap L(bab)$ for any $a, b \in R_S(2, 2)$

holds.

Proof. Let $R_S(2, 2)$ be a subsemigroup of $S$. It follows from (5) that $E \neq \emptyset$. If $a, b \in R_S(2, 2)$, then $ab \in R_S(2, 2)$. By Lemma 3 we have $ab \in R_S(2, 0) \cap R_S(0, 2)$. Lemma 1 and its dual imply that $ab \in R(aba) \cap L(bab)$. Hence (9) holds.

Let $E \neq \emptyset$ and let (9) hold. Then (5) implies that $R_S(2, 2) \neq \emptyset$. Let $a, b \in R_S(2, 2)$, then by (9) we have $ab \in R(aba) \cap L(bab)$ and $ba \in R(bab) \cap L(aba)$. Lemma 1 and its dual imply that $ab \in R_S(2, 0) \cap R_S(0, 2)$. It follows from Lemma 3 that $ab \in R_S(2, 2)$. The class of regularity $R_S(2, 2)$ is a subsemigroup of $S$.

Corollary 1. If $R_S(2, 2)$ is a subsemigroup of a semigroup $S$, then $R_S(1, 1)$ is a subsemigroup of $S$.

Proof. If $R_S(2, 2)$ is a subsemigroup of $S$, then $E \neq \emptyset$. Since $E \subseteq R_S(2, 2)$, hence, by (1), we have $E^2 \subseteq R_S(2, 2) \subseteq R_S(1, 1)$. It follows from Theorem 1 that $R_S(1, 1)$ is a subsemigroup of $S$.

Corollary 2. $R_S(2, 2)$ is an inverse subsemigroup of a semigroup $S$ if and only if (6) and (9) hold.

The proof follows from Theorem 4 and from Lemma 2.

Corollary 3. $R_S(2, 2)$ is an inverse subsemigroup of a semigroup $S$ if and only if $R_S(2, 2)$ is a subsemigroup of $S$ and $R_S(1, 1)$ is an inverse subsemigroup of $S$.

Lemma 4. A semigroup $S$ is a semilattice of groups if and only if $S$ is regular and $E \subseteq Z$, where $Z$ is the centre of a semigroup $S$.

Proof. Let $S$ be a regular semigroup and $E \subseteq Z$. If $a \in S$, then $a = a^2$ for some $x \in S$. Evidently $ax \in E$ and thus we have $a = (ax)a = a^2x$. From this it follows that $S = R_S(2, 0)$. Analogously we can prove that $S = R_S(0, 2)$. It follows from Lemma 3 that $S = R_S(2, 2)$. Lemma 2 implies that $S$ is
a union of groups. Hence, by Corollary 2 of Theorem 2 in [6] we obtain that $S$ is a semilattice of groups.

Let $S$ be a semilattice of groups. If $a \in S$, then according to Lemma 6 in [7] we have $R(a) = L(a)$. Since $S$ is a regular semigroup, then $aS = a \cup aS = R(a) = L(a) = Sa \cup a = Sa$. This means that $S$ is a normal semigroup. It follows from Lemma 1 in [8] that $E \subset Z$.

**Theorem 5** (cf. [2], Theorem 6). Let $S$ be a semigroup and let $1 \leq m, 1 \leq n$. Then the class of regularity $\mathcal{R}_S(m, n)$ is a semilattice of groups if and only if

(10) \[ E \neq \emptyset \text{ and } ae = ea \text{ for any } a \in \mathcal{R}_S(m, n) \text{ and any } e \in E. \]

**Proof.** If $\mathcal{R}_S(m, n)$ is a semilattice of groups, then from Lemma 4 and (4) it follows that (10) holds.

Let (10) hold. If $a \in \mathcal{R}_S(m, n)$, then from (1) it follows that $a \in \mathcal{R}_S(1, 1)$. This means that $a = axa$ for some $x \in S$. Since $ax \in E$, hence, by (10), we have $a = (ax)a = ax^2 \in \mathcal{R}_S(2, 0)$. Similarly we obtain that $a \in \mathcal{R}_S(0, 2)$. From Lemma 3 we have $a \in \mathcal{R}_S(2, 2)$ and thus $\mathcal{R}_S(m, n) \subset \mathcal{R}_S(2, 2)$. By (1) $\mathcal{R}_S(m, n) = \mathcal{R}_S(2, 2)$.

We shall prove that (9) holds. If $a, b \in \mathcal{R}_S(2, 2)$, then $a = a^2xa$ for some $x \in S$. Since $axa^2 \in E$, hence, by (10), $ab = a(axa^2)b = ab(axa^2) \in R(aba)$. Similarly we obtain that $ab \in L(bab)$. Theorem 4 implies that $\mathcal{R}_S(2, 2)$ is a subsemigroup of $S$. It follows from Lemma 2 that $\mathcal{R}_S(2, 2)$ is a regular semigroup. According to Lemma 4 and (10) we obtain that $\mathcal{R}_S(m, n) = \mathcal{R}_S(2, 2)$ is a semilattice of groups.

**Corollary.** Let $S$ be a semigroup and let $1 \leq m, 1 \leq n$. If (10) holds, then $\mathcal{R}_S(m, n) = \mathcal{R}_S(m + k, n + l)$ for any non-negative integers $k, l$.

A semigroup $S$ is called right simple if $S$ is the only right ideal of $S$. A semigroup $S$ is said to be left cancellative if in $S$ the left cancellation law holds, that is $ax = ay$ implies $x = y$ for all $a, x, y$ in $S$. A semigroup $S$ is called a right group if it is right simple and left cancellative.

**Lemma 5.** A semigroup $S$ is a right group if and only if $S$ is regular and $fe = e$ for any $e, f \in E$.

**Proof.** Let $S$ be a regular semigroup and $fe = e$ for any $e, f \in E$. Let $a, b \in S$. Then $a = auu, b = bvb$ for some $u, v \in S$. Put $x = ub$. Since $au, bv \in E$, hence $ax = aub = (au)(bv)b = (bv)b = b$. Therefore, $S$ is right simple. Let $ax = ay$ for $a, x, y \in S$. Since $S$ is regular, hence $a = axa, x = xux, y = yvy$ for some $z, u, v \in S$. Thus we have $axux = ayvy$. Postmultiplying by $z$, we have $zaxux = zayvy$. Since $zu, xu, yv \in E$, then $x = (zu)x = (za)(xu)x = (za)(yv)y = (yv)y = y$. Therefore, $S$ is left cancellative. Thus $S$ is a right group.
Let $S$ be a right group. From Theorem 1.27 in [4] it follows that $S$ is regular and $E$ is a right zero semigroup.

**Theorem 6.** Let $S$ be a semigroup and let $1 \leq m, 1 \leq n$. Then the class of regularity $R_S(m, n)$ is a right group if and only if

(11) $E \neq \emptyset$ and $fe = e$ for any $e, f \in E$.

**Proof.** If $R_S(m, n)$ is a right group, then from Lemma 5 and (4) it follows that (11) holds.

Let (11) hold. This and (4) imply that $R_S(1, 1) \neq \emptyset$. It follows from the Remark and from Lemma 5 that $R_S(1, 1)$ is a right group. Since $R_S(1, 1)$ is a union of groups, then, by Lemma 2, we have $R_S(1, 1) \subseteq R_S(2, 2)$. According to (1) we obtain that $R_S(m, n) \subseteq R_S(1, 1) \subseteq R_S(2, 2) \subseteq R_S(m, n)$. Therefore, $R_S(m, n) = R_S(1, 1)$ is a right group.

**Corollary.** Let $S$ be a semigroup and let $1 \leq m, 1 \leq n$. If (11) holds, then $R_S(1, 1) = R_S(2, 1) = R_S(1, 2) = R_S(2, 2)$.

**Theorem 7** (cf. [2], Corollary of Theorem 4). Let $S$ be a semigroup and let $1 \leq m, 1 \leq n$. Then the class of regularity $R_S(m, n)$ is a group if and only if $\text{card } E = 1$.

The proof follows from Theorem 6 and its dual.

**Corollary.** Let $S$ be a semigroup. If $\text{card } E = 1$, then $R_S(1, 1) = R_S(2, 1) = R_S(1, 2) = R_S(2, 2)$.

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Katedra matematiky
Elektrotechnické fakulty
Českého vysokého učení technického
Poděbrady