MATEMATICKÝ ČASOPIS

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HOLONOMY GROUPS OF A FULLY PARALLELIZABLE MANIFOLD

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§1. CURVES IN A METRIC SPACE.

Let (P, ϱ) be a metric space. We shall use the concept of an oriented rectifiable curve in the geometric sense as defined in [2] Chap. 1, § 5. Let us denote by C the set of all such curves in (P, ϱ). For $c \in C$ let $A(c)$, $B(c)$, $\lambda(c)$ denote the starting point, the end point and the length of c, respectively. C can be provided with a natural algebraic structure: $c_1 + c_2$ is defined if and only if $B(c_1) = A(c_2)$. For any $\xi \in P$ we define

$$C_\xi = \{c \in C;\quad A(c) = B(c) = \xi\}$$

The restriction of the algebraic structure of C to C_ξ gives a structure of a semigroup with the neutral element on C_ξ.

Now we shall provide C with the structure of a metric space. For $c \in C$ let $x(\sigma)$, $\sigma \in \langle 0, \lambda(c)\rangle$ be the standard representation of c (σ is the arc length). Let us set $\varphi(t) = \lambda(c) \, . \, t$. The representation $\hat{x}(t) = x(\varphi(t))$, $t \in \langle 0, 1\rangle$ will be called the normal representation of c. For $c_1, c_2 \in C$ let $\hat{x}_1(t), \hat{x}_2(t)$ be their normal representations. We set

$$R(c_1, c_2) = \max_{t \in \langle 0,1\rangle} \varrho(\hat{x}_1(t), \hat{x}_2(t)).$$

Let M be a fully parallelizable manifold and let Γ be a connection on M. The set of all closed curves starting from a fixed point $x \in M$ is provided with such a metric that the mapping assigning to a curve the corresponding element of the holonomy group at x is continuous.

Proposition 1. *R is a metric on C.*

The proof is obvious.

Remark. It can be easily seen that $\lambda(c)$ is not in general a continuous function on (C, R). Neither C_ξ provided with the induced metric is in general a topological semigroup.

§2. CURVES IN A FULLY PARALLELIZABLE MANIFOLD.

Let M be a fully parallelizable paracompact manifold of class C^∞, dim $M = n$, let g be a positive definite metric tensor on M, let ϱ be a metric on M induced by this tensor, and let Γ be a linear connection on M (not necessarily Riemannian). Let $\omega_1, \ldots, \omega_n$ be C^∞ -differentiable 1-forms on M (throughout this paper differentiable $= C^\infty$ -differentiable), linearly independent at every point of M. The existence of such $\omega_1, \ldots, \omega_n$ follows from the parallelizability of M.

Definition 1. *Let $c \in C$. c is said to be piecewise differentiable if there is a piecewise differentiable curve $x(\tau)$, $\tau \in \langle a, b\rangle$ which is a representation of c.*

It is well known that if $c \in C$ is piecewise differentiable then its standard representation is a piecewise differentiable curve. Hence it follows that its normal representation is also a piecewise differentiable curve. Let us denote

$D = \{c \in C;\ c$ is piecewise differentiable$\}$,

$$D_\xi = D \cap C_\xi.$$

Therefore $D \subset C$ and $D_\xi \subset C_\xi$ is a subsemigroup of the semigroup C_ξ. By the restriction of R to D and D_ξ induced metrics we shall also denote by R. Now we introduce one more metric on D. For $d_1, d_2 \in D$ let $x_1(t)$, $x_2(t)$ be their normal representations. Let us denote by $\dot{x}_1(t)$ and $\dot{x}_2(t)$ a tangent vector to the curves x_1(t) and x_2(t) at the point t respectively (at a singular point let us take the lefthand tangent vector). Further let $\alpha > 0$ be a real number, and let m_{d_1} and m_{d_2} be the number of singular points of the curves x_1(t) and x_2(t) respectively. Let us set

$$S(d_1, d_2) = R(d_1, d_2) + \max_i \sup_{t \in \langle 0,1\rangle} | \omega_i(\dot{x}_1(t)) - \omega_i(\dot{x}_2(t)) | + \alpha \, | m_{d_1} - m_{d_2} | .$$

Proposition 2. *S is a metric on D.*

The proof follows easily using Proposition 1. For any $d_1, d_2 \in D$ there is $R(d_1, d_2) \leqq D(d_1, d_2)$.

Definition 2 *.Let (U, φ) be a chart on M. (U, φ) is said to be symmetric with the center at a point $p \in M$ if there is $\eta > 0$ such that $U = \{q \in M;\ \varrho(p, q) < \eta\}$.*

Now we shall define a „function" $\xi(p)$ on M in the following way. Let Ξ_p be the set of all positive real numbers such that for every $\eta \in \Xi_p$ there exists a symmetric chart (U, φ) with the center at p and the radius η. Let us set $\xi(p) = \sup \Xi_p$.

Lemma 1. *There is either $\xi(p) = \infty$ for all $p \in M$ or $\xi(p)$ is a uniformly continuous function on M.*

The proof follows easily from the inequality $|\xi(p) - \xi(q)| \leqslant \varrho(p, q)$.

Lemma 2. *Let $c \in C$ with the normal representation $x(t)$. There exists a partition $0 = t_0 < t_1 \ldots < t_k = 1$ of the interval $\langle 0, 1\rangle$, symmetric charts (U_i, φ_i), $i = 1, \ldots, k$ with the centers $x(t_{i-1})$ and the same radius η, and a number $\delta > 0$ such that the following assertion holds: if $c_1 \in C$ is such that $R(c, c_1) < \delta$ and $x_1(t)$ is its normal representation, then $\{x_1(t); t \in \langle t_{i-1}, t_i\rangle\} \subset U_i$.*

Proof: The assertion is clear in the case $\xi(p) = \infty$. Thus let us consider the case when $\xi(p)$ is a real function. We can restrict ourselves to the case $\lambda(c) > 0$, for in the case $\lambda(c) = 0$ the assertion is also clear. There is

$$0 < \xi_0 = \min_{t \in \langle 0,1\rangle} \xi(x(t)).$$

Let k be a positive integer such that $\frac{1}{k} \leqslant \frac{\xi_0}{4\lambda(c)}$ and let us set $t_i = \frac{i}{k}$ $i = 0, \ldots, k$, $U_i = \{p \in M, \varrho(p, x(t_{i-1})) < \frac{3\xi_0}{4}$, $i = 1, \ldots, k$. Obviously there exist functions φ_i defined on U_i such that (U_i, φ_i) is a symmetric chart. Let us set $\delta = \frac{\xi_0}{4}$. We shall show that just chosen t_i, U_i, δ have the required properties.

Let $c_1 \in C$, $R(c, c_1) < \delta$. For the sake of simplicity let us denote by $c^{(i)}$ and $c_1^{(i)}$ the curves $x(t)$, $t \in \langle t_{i-1}, t_i\rangle$ and $x_1(t)$, $t \in \langle t_{i-1}, t_i\rangle$, respectively. With respect to the fact that ϱ is a Riemannian metric on M we have for any $t \in \langle t_{i-1}, t_i\rangle$ an inequality

$$\varrho(x_1(t), x(t_{i-1})) \leqslant \varrho(x_1(t), x(t)) + \varrho(x(t), x(t_{i-1})) \leqslant$$

$$\leqslant \frac{\xi_0}{4} + \lambda(c^{(i)}) = \frac{\xi_0}{4} + \frac{\lambda(c)}{k} \leqslant \frac{\xi_0}{4} + \frac{\xi_0}{4} < \frac{3\xi_0}{4}.$$

This completes the proof.

Proposition 3. *λ is a continuous function on D.*

Proof: Let us keep the notation from the above lemma. Let $d, d_1 \in D$ and let $S(d, d_1) < \delta$. There is

$$|\lambda(d) - \lambda(d_1)| \leqslant \sum_{i=1}^{k} |\lambda(d^{(i)}) - \lambda(d_1^{(i)})|$$

and both $d^{(i)}$, $d_1^{(i)}$ lie in U_i. Let $\varphi = \{x^1, \ldots, x^n\}$, let $g_{\alpha\beta}$ be the components of the metric tensor with respect to φ and let $x^\alpha(t)$ and $x_1^\alpha(t)$ denote coordinates of points $x(t)$ and $x_1(t)$, respectively. Further let us set $g_{\alpha\beta}(t) = g_{\alpha\beta}(x(t))$, $g^1_{\alpha\beta}(t) = g_{\alpha\beta}(x_1(t))$. Now we set

$$K_i = \max_{\alpha,\beta=1,\ldots,n} \max_{\varrho(p,x(t_{i-1}))\leq\frac{3\xi_0}{4}} |g_{\alpha\beta}(p))|$$

$$L_i = \min_{t\in\langle t_{i-1},t_i\rangle} g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t}.$$

We have

$$|\lambda(d^{(i)}) - \lambda(d_1^{(i)})| = \left|\int_{t_{i-1}}^{t_i}\left(\sqrt{g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t}} - \sqrt{g^1_{\alpha\beta}\frac{\mathrm{d}x_1^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x_1^\beta}{\mathrm{d}t}}\right)\mathrm{d}t\right|$$

$$= \left|\int_{t_{i-1}}^{t_i}\frac{g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t} - g^1_{\sigma\beta}\frac{\mathrm{d}x_1^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x_1^\beta}{\mathrm{d}t}}{\sqrt{g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t}} + \sqrt{g^1_{\alpha\beta}\frac{\mathrm{d}x_1^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x_1^\beta}{\mathrm{d}t}}}\,\mathrm{d}t\right|$$

$$\leqslant \frac{1}{L_i}\left|\int_{t_{i-1}}^{t_i}\left(g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t} - g^1_{\alpha\beta}\frac{\mathrm{d}x_1^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x_1^\beta}{\mathrm{d}t}\right)\mathrm{d}t\right|$$

$$\leqslant \frac{1}{L_i}\int_{t_{i-1}}^{t_i}\left|g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t} - g^1_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t} + g^1_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t}\right.$$

$$\left.- g^1_{\alpha\beta}\frac{\mathrm{d}x_1^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x_1^\beta}{\mathrm{d}t}\right|\mathrm{d}t \leqslant \frac{1}{L_i}\int_{t_{i-1}}^{t_i}|g_{\alpha\beta} - g^1_{\alpha\beta}|\cdot\left|\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t}\right|\mathrm{d}t$$

$$+ \frac{1}{L_i}\int_{t_{i-1}}^{t_i}|g^1_{\alpha\beta}|\cdot\left|\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t} - \frac{\mathrm{d}x_1^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x_1^\beta}{\mathrm{d}t}\right|\mathrm{d}t.$$

Components of the metric tensor $g_{\alpha\beta}$ are uniformly continuous functions on $\left\{p \in M;\ \varrho(p\ \ x(t_{i-1})) \leqslant \frac{3\xi_0}{4}\right\}$. Hence it follows that choosing δ sufficiently small, the term $\frac{1}{L_i}\int_{t_{i-1}}^{t_i}\left|g_{\alpha\beta} - g^1_{\alpha\beta}\right|\cdot\left|\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t}\right|\mathrm{d}t$ can be made arbitrarily small. Now deal we shall with the second term of the above expression. First we shall consider the expression $\left|\frac{dx^\alpha}{dt} - \frac{dx_1^\alpha}{dt}\right|$. Let us denote by $\dot{x}(t)$ and $\dot{x}_1(t)$ the tangent vectors to the curves $x(t)$, $t \in \langle t_{i-1}, t_i\rangle$ and $x_1(t)$, $t \in \langle t_{i-1}, t_i\rangle$ at the points $x(t)$and $x_1(t)$, respectively. There is

$$\frac{dx^\alpha}{dt} - \frac{dx_1^\alpha}{dt} = dx^\alpha(\dot{x}(t)) - dx^\alpha(\dot{x}_1(t)).$$

Now let us write $dx^\alpha = a_\gamma^\alpha \omega_\gamma$, where a_γ^α are differentiable functions. Hence we have

$$\left|\frac{dx^\alpha}{dt} - \frac{dx_1^\alpha}{dt}\right| \leqslant |a_l^k(x(t)) - a_l^k(x_1(t))| \,.\, |\omega_l(\dot{x}(t))| + |a_l^k(x_1(t))| \,.\, |\omega_l(\dot{x}(t)) - \\ - \omega_l(\dot{x}_1(t))| .$$

According to the compactness of the set $\{p \in M;\ \varrho(p, x(t_{i-1})) \leqslant \frac{3\xi_0}{4}$ we see again that choosing δ sufficiently small we can make the expression arbitrarily small. We have

$$\int_{t_{i-1}}^{t_i} |g_{\alpha\beta}^1| \cdot \left|\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} - \frac{dx_1^\alpha}{dt}\frac{dx_1^\beta}{dt}\right| \leqslant K_i \int_{t_{i-1}}^{t_i} \left|\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} - \frac{dx_1^\alpha}{dt}\frac{dx_1^\beta}{dt}\right| dt$$

$$= K_i \int_{t_{i-1}}^{t_i} \left|\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} - \frac{dx^\alpha}{dt}\frac{dx_1^\beta}{dt} + \frac{dx^\alpha}{dt}\frac{dx_1^\beta}{dt} - \frac{dx_1^\alpha}{dt}\frac{dx_1^\beta}{dt}\right| dt$$

$$\leqslant K_i \int_{t_{i-1}}^{t_i} \left|\frac{dx^\alpha}{dt}\right| \cdot \left|\frac{dx^\beta}{dt} - \frac{dx_1^\beta}{dt}\right| dt + K_i \int_{t_{i-1}}^{t_i} \left|\frac{dx_1^\alpha}{dt}\right| \cdot \left|\frac{dx^\alpha}{dt} - \frac{dx_1^\alpha}{dt}\right| dt$$

$$\leqslant K_i \int_{t_{i-1}}^{t_i} \left|\frac{dx^\alpha}{dt}\right| \cdot \left|\frac{dx^\beta}{dt} - \frac{dx_1^\beta}{dt}\right| dt + K_i \int_{t_{i-1}}^{t_i} \left|\frac{dx^\beta}{dt}\right| \cdot \left|\frac{dx^\alpha}{dt} - \frac{dx_1^\alpha}{dt}\right| dt$$

$$+ K_i \int_{t_{i-1}}^{t_i} \left|\frac{dx^\beta}{dt} - \frac{dx_1^\beta}{dt}\right| \cdot \left|\frac{dx^\alpha}{dt} - \frac{dx_1^\alpha}{dt}\right| dt.$$

And now the assertion follows easily.

Remark: It can be easily seen that D_ξ, even with the metric S, is not a topological semigroup.

§ 3. MAPPING OF THE SPACE (D_ξ, S) INTO THE HOLONOMY GROUP OF A LINEAR CONNECTION Γ ON M.

First of all we shall prove

Lemma 3. *Let a_{ij}, $i, j = 1, \ldots, n$ be continuous functions on an interval $\langle x_0, x_1\rangle$. Let y_i be a solution of the system*

$$\frac{dy_i}{dx} + \sum_{j=1}^{n} a_{ij} y_j = 0, \quad i = 1, \ldots, n$$

in the interval $\langle x_0, x_1\rangle$ with the initial conditions $y_i(x_0) = y_i^{(0)}$. Then there exist $N > 0$, $\delta_0 > 0$ such that if $0 < \delta < \delta_0$ and if b_{ij}, $i, j = 1, \ldots, n$ are continuous functions on $\langle x_0, x_1\rangle$ such that $\max_{i,j} \max_{x \in \langle x_0, x_1\rangle} | a_{ij}(x) - b_{ij}(x) | < \delta$ and if z_i is a solution of the system

$$\frac{dz_i}{dx} + \sum_{j=1}^{n} b_{ij} z_j = 0, \quad i = 1, \ldots, n$$

such that $\max_i | y_i^{(0)} - z_i^{(0)} | < \delta$ *then*

$$\max_i \max_{x \in \langle x_0, x_1\rangle} | y_i(x) - z_i(x) | < N\delta.$$

Proof: Let us define the following sequences of functions on $\langle x_0, x_1\rangle$:

$$y_i^{(0)} = y_i(x_0), \; z_i^{(0)} = z_i(x_0)$$

$$y_i^{(k+1)} = y_i^{(0)} - \int_{x_0}^{x} \sum_{j=1}^{n} a_{ij}\, y_j^{(k)}\, dx$$

$$z_i^{(k+1)} = z_i^{(0)} - \int_{n_0}^{x} \sum_{j=1}^{n} a_{ij}\, y_j^{(k)}\, dx$$

There is $y_i = \lim_{k\to\infty} y_i^{(k)}$, resp. $z_i = \lim_{k\to\infty} z_i^{(k)}$ uniformly on $\langle x_0, x_1\rangle$ (see for instance [3] Chap. VII, § 2). Let $K > 0$, $L > 0$ be such that $\max_{i,j} \max_{x \in \langle x_0, x_1\rangle} | a_{ij} | < \frac{1}{2} K$, $\max_i \max_{x \in \langle x_0, x_1\rangle} | y_i^{(k)} | < \frac{1}{2} L$ for all k, $\delta_0 = \frac{1}{2} \min (K, L)$ and let $0 < \delta < \delta_0$, $\max_{i,j} \max_{x \in \langle x_0, x_1\rangle} | a_{ij}(x) - b_{ij}(x) | < \delta$, $\max_i | y_i^{(0)} - z_i^{(0)} | < \delta$. For $i = 1, \ldots, n$ we have

$$| y_i^{(0)} - z_i^{(0)} | < \delta$$

$$y_i^{(1)} - z_i^{(1)} = y_i^{(0)} - z_i^{(0)} + \int_{x_0}^{x} \sum_{j=1}^{n} (b_{ij} - a_{ij})\, y_j^{(0)}\, \mathrm{d}x + \int_{x_0}^{x} \sum_{j=1}^{n} b_{ij}(z_j^{(0)} - y_j^{(0)})\, \mathrm{d}x .$$

From this we have the estimation

$$| y_i^{(1)} - z_i^{(1)} | \leqslant \delta \left[1 + \frac{K + L}{K} (nK)\,(x - x_0) \right] .$$

By induction we can easily prove that for every k there is

$$| y_i^{(k)} - z_i^{(k)} | \leqslant \delta \left[\sum_{i=0}^{k-1} \frac{(nK)^i\,(x - x_0)^i}{i!} + \frac{K + L}{K} \sum_{i=1}^{k} \frac{(nK)^i\,(x - x_0)^i}{i!} \right] .$$

Now it is sufficient to set

$$N = \left(1 + \frac{K + L}{K} \right) \exp\,[nK(x_1 - x_0)] .$$

Definition 3. *A function $f(x)$ defined on $\langle x_0, x_1 \rangle$ is said to be piecewise continuous on $\langle x_0, x_1 \rangle$ with the index of discontinuity m if there is a partition*

$$x_0 = t_0 < t_1 < \ldots < t_m = x_1$$

of the interval $\langle x_0, x_1 \rangle$ such that

(i) *$f(x)$ is continuous on $\langle t_{i-1}, t_i)$, $i = 1, \ldots, m - 1$ and on $\langle t_{m-1}, t_m \rangle$,*
(ii) *there exists the finite limit*
$\lim_{x \to t_i -} f(x) \quad i = 1, \ldots, m,$
(iii) *the points t_i, $i = 1, \ldots, m - 1$ are the points of discontinuity of the function $f(x)$.*

Definition 4. *Let a_{ij}; $i, j = 1, \ldots, n$ be piecewise continuous functions on $\langle x_0, x_1 \rangle$. Let*

$$x_0 = u_0 < u_1 < \ldots < u_r = x_1$$

be a partition of the interval $\langle x_0, x_1 \rangle$ such that
(i) *any interval (u_{k-1}, u_k) does not contain a point of discontinuity of any function a_{ij},*
(ii) *every point u_k, $k = 1, \ldots, r - 1$ is a poin of discontinuity of at least one of the functions a_{ij}.*

Let us define the functions ${}^{+}a_{ij}^{(k)}$ on $\langle u_{k-1}, u_k \rangle$, $k = 1, \ldots, r$ by

$$+a_{ij}^{(k)} \begin{cases} a_{ij} \text{ for } x \in \langle u_{k-1}, u_k) \\ \lim\limits_{x \to u_k -} a_{ij} \text{ for } x = u_k. \end{cases}$$

We say that functions the y_i defined on $\langle x_0, x_1 \rangle$ are a solution of the generalized system

$$\frac{dy_i}{dx} + \sum_{j=1}^{n} a_{ij} y_j = 0$$

if (i) y_i are continuous on $\langle x_0, x_1 \rangle$,

(ii) y_i are on $\langle u_{k-1}, u_k \rangle$ a solution of the system

$$\frac{dy_i}{dx} + \sum_{j=1}^{n} + a_{ij}^{(k)} y_j = 0$$

for all $k = 1, \ldots, r$

The generalization of lemma 3 is

Lemma 4. *Let a_{ij}; $i, j = 1, \ldots, n$ be piecewise continuous functions on the interval $\langle x_0, x_1 \rangle$. Let y_i be the solution of the generalized system*

$$\frac{dy_i}{dx} + \sum_{j=1}^{n} a_{ij} y_i = 0, \quad i = 1, \ldots, n$$

on the interval $\langle x_0, x_1 \rangle$ with the initial conditions $y_i(x_0) = y_i^{(0)}$. Let P be a non-negative integer. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if b_{ij}; $i, j =$ $= 1, \ldots, n$ are piecewise continuous functions on $\langle x_0, x_1 \rangle$ such that the index of discontinuity of each of them is $\leqslant P$ and $\max\limits_{i,j} \max\limits_{x \in \langle x_0, x_1 \rangle} | a_{ij}(x) - b_{ij}(x) | < \delta$ and if z_i is a solution of the generalized system

$$\frac{dz_i}{dx} + \sum_{j=1}^{n} b_{ij} z_j = 0, \quad i = 1, \ldots, n$$

such that $\max\limits_{i} | y_i^{(0)} - z_i(x_0) | < \delta$, then there is

$$\max_{i} \max_{x \in \langle x_0, x_1 \rangle} | y_i(x) - z_i(x) | < \varepsilon$$

The proof follows easily from Lemma 3.

Now let us denote respectively by $T(M)$ and $T_p(M)$ the tangent bundle and the tangent space at the point $p \in M$ of a fully parallelizable Riemannian manifold M. Let $X_1, \ldots, X_n (n = \dim M)$ be differentiable vector fields, linearly independent at every point of M. Now we shall define on $T(M)$ a pseudometric σ in the following way. Let Y_p, $Y_q \in T(M)$, $Y_p =$

$= \sum_{i=1}^{n} \xi^i (X_i)_p$, $Y_q = \sum_{i=1}^{n} \eta^i (X_i)_q$. We set

$$\sigma(Y_p, Y_q) = \max_i | \xi^i - \eta^i | .$$

It can be easily seen that the restriction of σ to $T_p(M)$ is a metric.

Proposition 4. *Let (U, φ), $\varphi = \{x^1, \ldots, x^n\}$ be a chart on M. Let $d \in D$, $x(t)$ be its normal representation. Let us suppose $\{x(t);\ t \in \langle 0, 1 \rangle\} \subset U$. Finally let $W(0) \in T_{x(0)}(M)$ and $W(t) \in T_{x(t)}(M)$ be the vector obtained by the parallel displacement of $W(0)$ along the curve $x(t)$ with respect to Γ. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $d_1 \in D$ with the normal representation $x_1(t)$ such that $S(d, d_1) < \delta$ and if $V(0) \in T_{x_1(0)}(M)$ such that $\sigma(W(0), V(0)) < \delta$, then*

(i) $\{x_1(t);\ t \in \langle 0, 1 \rangle\} \subset U$;

(ii) if *$V(t) \in T_{x_1(t)}(M)$ denotes the vector obtained by the parallel displacement of $V(0)$ along $x_1(t)$, then*

$$\sigma(W(1), V(1)) < \varepsilon.$$

Proof: (i) is obvious. As to (ii) we shall first prove the following lemma: Let $\delta_1 > 0$, $p \in U$, $Y_p \in T_p(M)$. Then there exists $\delta > 0$ such that if $q \in U$, $Y_q \in T_q(M)$ are such that $\varrho(p, q) < \delta$, $\sigma(Y_p, Y_q) < \delta$, then writing $Y_p = \sum_{i=1}^{n} \xi^i \left(\frac{\partial}{\partial x^i} \right)_p$, $Y_q = \sum_{i=1}^{n} \eta^i \left(\frac{\partial}{\partial x^i} \right)_q$ we have $\max_i | \xi^i - \eta^i | < \delta_1$. Let us write therefore

$$Y_p = \sum_{i=1}^{n} \bar{\xi}^i (X_i)_p,\quad Y_q = \sum_{i=1}^{n} \bar{\eta}^i (X_i)_q \quad (X_i)_r = \sum_{j=1}^{n} A_i^j(r) \left(\frac{\partial}{\partial x^j} \right)_r;\ A_i^j(r)\ i, j = 1, \ldots, n$$

are differentiable functions onu U. From these relations we obtain

$$\xi^i = \sum_{j=1}^{n} A_j^i(p)\, \bar{\xi}^j,\quad \eta^i = \sum_{j=1}^{n} A_j^i(q) \bar{\eta}^j$$

and therefore

$$\xi^i - \eta^i = \sum_{j=1}^{n} (A_j^i(p)\, \bar{\xi}^j - A_j^i(q)\, \bar{\eta}^j) =$$

$$= \sum_{j=1}^{n} [(A_j^i(p) - A_j^i(q))\, \bar{\xi}^j + A_j^i(q)\, (\bar{\xi}^j - \bar{\eta}^j)]$$

$$|\,\xi^i - \eta^i\,| \leqslant \sum_{j=1}^{n} [\,|\,A^i_j(p) - A^i_j(q)\,|\,.\,|\,\bar{\xi}^j\,| + |\,A^i_j(q)|\,.\,|\bar{\xi}^j - \bar{\eta}^j|\,]\,.$$

From the last inequality our lemma follows easily.

Now let us write $W(t) = \sum_{i=1}^{n} w^i(t)\left(\frac{\partial}{\partial x^i}\right)_{x(t)}$, $V(t) = \sum_{i=1}^{n} v^i(t)\left(\frac{\partial}{\partial x^i}\right)_{x_1(t)}$ and let us denote by Γ^i_{jk} the components of Γ with respect to the coordinate system $\{x^1, \ldots, x_n\}$. The functions $w^i(t)$ and $v^i(t)$ are on $\langle 0, 1\rangle$ solutions of the generalized systems

$$\frac{\mathrm{d}w^i}{\mathrm{d}t} + \sum_{j,k=1}^{n} \Gamma^i_{jk}(x(t))\frac{\mathrm{d}x^k}{\mathrm{d}t}w^j = 0$$

and

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} + \sum_{j,k=1}^{n} \Gamma^i_{jk}\,(x_1(t))\frac{\mathrm{d}x^k_1}{\mathrm{d}t}v^j = 0,$$

respectively (see [1], chap. III, § 7).

Hence let us have $\varepsilon > 0$. According to Lemma 4 there exists $\delta_1 > 0$ such that if

$$\max_{i,j}\max_{t\in\langle 0,1\rangle}\left|\sum_{k=1}^{n}\left(\Gamma^i_{jk}(x(t))\,\frac{\mathrm{d}x^k}{\mathrm{d}t} - \Gamma^i_{jk}(x_1(t))\,\frac{\mathrm{d}x^k_1}{\mathrm{d}t}\right)\right| < \delta_1$$

$$\max_i |w^i(0) - v^i(0)\,| < \delta_1,$$

then $\max_i |w^i(1) - v^i(1)| < \varepsilon$. From the equality

$$\sum_{k=1}^{n}\left(\Gamma^i_{jk}(x(t))\,\frac{\mathrm{d}x^k}{\mathrm{d}t} - \Gamma^j_{ik}\,(x_1(t))\,\frac{\mathrm{d}x^k_1}{\mathrm{d}t}\right) =$$

$$= \sum_{k=1}^{n}\left[(\Gamma^i_{jk}(x(t)) - \Gamma^i_{jk}(x_1(t)))\,\frac{\mathrm{d}x^k}{\mathrm{d}t} + \Gamma^i_{jk}(x_1(t))\left(\frac{\mathrm{d}x^k}{\mathrm{d}t} - \frac{\mathrm{d}x^k_1}{\mathrm{d}t}\right)\right]$$

we have the estimation

$$\left|\sum_{k=1}^{n}\left(\Gamma^i_{jk}(x(t))\frac{\mathrm{d}x^k}{\mathrm{d}t} - \Gamma^i_{jk}(x_1(t))\frac{\mathrm{d}x_1^k}{\mathrm{d}t}\right)\right| \leqslant$$

$$\leqslant \sum_{k=1}^{n}\left[|\Gamma^i_{jk}(x(t)) - \Gamma^i_{jk}(x_1(t))| \cdot \left|\frac{\mathrm{d}x^k}{\mathrm{d}t}\right| + |\Gamma^i_{jk}(x_1(t))| \cdot \left|\frac{\mathrm{d}x^k}{\mathrm{d}t} - \frac{\mathrm{d}x_1^k}{\mathrm{d}t}\right|\right].$$

Writing similarly as in the proof of proposition 3

$$\frac{\mathrm{d}x^k}{\mathrm{d}t} = \mathrm{d}x^k(\dot{x}(t)),\ \frac{\mathrm{d}x_1^k}{\mathrm{d}t} = \mathrm{d}x^k(\dot{x}_1(t)),\ \mathrm{d}x^k = \sum_{l=1}^{n} a_l^k\,\omega_l,$$

we get

$$\frac{\mathrm{d}x^k}{\mathrm{d}t} - \frac{\mathrm{d}x_1^k}{\mathrm{d}t} = \sum_{l=1}^{n}\left[a_l^k(x(t))\,\omega_l(\dot{x}(t)) - a_l^k(x_1(t))\,\omega_l(\dot{x}_1(t))\right] =$$

$$= \sum_{l=1}^{n}\left[(a_l^k(x(t)) - a_l^k(x_1(t)))\,\omega_l(\dot{x}(t)) + a_l^k(x_1(t))\,(\omega_l(\dot{x}(t)) - \omega_l(\dot{x}_1(t)))\right]$$

and from this equality we have the estimation

$$\left|\frac{\mathrm{d}x^k}{\mathrm{d}t} - \frac{\mathrm{d}x_1^k}{\mathrm{d}t}\right| \leqslant \sum_{l=1}^{n}\left[\,|\,a_l^k(x(t)) - a_l^k(x_1(t))\,|\,.\,|\,\omega_l(\dot{x}(t))\,| + \right.$$

$$\left. + \,|\,a_l^k(x_1(t))\,|\,.\,|\,\omega_l(\dot{x}(t)) - \omega_l(\dot{x}_1(t))\,|\,\right].$$

Now it can be easily seen that there exists $\delta_2 > 0$ such that if $S(d_1, d) < \delta_2$, then

$$\max_{i,j}\max_{t\in\langle 0,1\rangle}\left|\sum_{k=1}^{n}\left(\Gamma^i_{jk}(x(t))\frac{\mathrm{d}x^k}{\mathrm{d}t} - \Gamma^i_{jk}(x_1(t))\frac{\mathrm{d}x_1^k}{\mathrm{d}t}\right)\right| < \delta_1$$

Choosing δ_2 sufficiently small then according to our lemma at the beginning of the proof $\sigma(W(0),\ V(0)) < \delta_2$ implies $\max_i |\,w^i(0) - v^i(0)\,| < \delta_1$. Setting $\delta = \delta_2$, then according to Lemma 4 $S(d_1, d) < \delta$, $\sigma(W(0),\ V(0)) < \delta$ imply

$$\max_i |\,w^i(1) - v^i(1)\ \ < \varepsilon'.$$

Now it is easy to show that for sufficiently small δ, ε' there is

$$\sigma(W(1),\ V(1)) < \varepsilon$$

and this completes the proof of the proposition.

Proposition 5. *Let $d \in D$, $x(t)$ be its normal representation. Let $W(0) \in T_{x(0)}(M)$ and let $W(t) \in T_{x(t)}(M)$ be the vector obtained by the parallel displacement of $W(0)$ along the curve $x(t)$ with respect to Γ. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $d_1 \in D$ with the normal representation $x_1(t)$ such that $S(d, d_1) < \delta$ and $V(0) \in T_{x1(0)}(M)$ such that $\sigma(W(0),\ V(0)) < \delta$, then*

$$\sigma(W(1),\ V(1)) < \varepsilon,$$

where $V(t) \in T_{x1(t)}(M)$ denotes again the vector obtained by the parallel displacement of $V(0)$ along $x_1(t)$.

Proof: Let us have $\varepsilon > 0$. Let $\bar{\delta} > 0$ and let

$$0 = t_0 < t_1 < \ldots < t_k = 1$$

be the partition of the interval $\langle 0, 1\rangle$ with the properties described in Lemma 2. We shall keep the notation of Lemma 2, only instead of $c^{(i)}$ we shall write $d^{(i)}$. According to the inclusion $\{x(t),\ t \in \langle t_{i-1}, t_i\rangle\} \subset U_i$ and according to the fact that $\hat{x}(t') = x((t_i - t_{i-1})\, t' + t_{i-1}),\ t' \in \langle 0, 1\rangle$ is the normal representation of $d^{(i)}$, it follows from proposition 4 that there exists $\delta_k > 0$ such that if $\bar{d}^{(k)} \in D$ is such that $S(d^{(k)}, \bar{d}^{(k)}) < \delta_k$ and if $V^{(k)} \in T_{A(\bar{d}^{(k)})}(M)$ is such that $\sigma(W(t_{k-1}),\ V^{(k)}) < \delta_k$ and if we denote by $\bar{V}^{(k)} \in T_{B(\bar{d}^{(k)})}(M)$ the vector obtained by the parallel displacement of $V^{(k)}$ along $\bar{d}^{(k)}$, then $\sigma(W(t_k),\ \bar{V}^{(k)}) < \varepsilon$. Successively we can find $\delta_i > 0,\ i = 1, \ldots, k$ such that if $\bar{d}^{(i)} \in D$ is such that $S(d^{(i)}, \bar{d}^{(i)}) < \delta_i$ and if $V^{(i)} \in T_{A(d^{(i)})}(M)$ is such that $\sigma(W(t_{i-1}),\ V^{(i)}) < \delta_i$ and if $\bar{V}^{(i)} \in T_{B(\bar{d}^{(i)})}(M)$ is the vector obtained by the parallel displacement of $V^{(i)}$ along $\bar{d}^{(i)}$, then $\sigma(W(t_i),\ \bar{V}^{(i)}) < \delta_{i+1}$.

Now let us choose $\delta < \min(\bar{\delta}, \delta_1, \ldots, \delta_k)$ so small that $S(d, d_1) < \delta$ will imply $S(d^{(i)}, d_1^{(i)}) < \delta_i$. Thus if $W(0) \in T_{x(0)}(M)$, $V(0) \in T_{x_1(0)}(M)$ are two vectors such that $\sigma(W(0),\ V(0)) < \delta$, we can easily see that $\sigma(W(1),\ V(1) < \varepsilon$ and this proves the proposition.

Now let $\xi \in M$ and let $E_1, \ldots, E_n \in T_\xi(M)$ be an orthonormal frame. Let $\Phi(\xi) \subseteq GL(n, \mathbf{R})$ be the holonomy group of Γ with the reference point ξ. We define the mapping $H\colon (D_\xi, S) \to \Phi(\xi)$ in the following way: let $d \in D_\xi$ and let $E'_1, \ldots, E'_n \in T_\xi(M)$ be the vectors obtained by the parallel displacement of the vectors $E_1, \ldots, E_n$ along the curve d. Let $a_{ij};\ i, j = 1, \ldots, n$ be such that $E_i = \sum_{j=1}^{n} a_{ij} E_j$ and let $A = (a_{ij})$. We set $H(d) = A$. If we take $\Phi(\xi)$ with the topology induced from $GL(n, \mathbf{R})$, we have

Proposition 6. *The mapping $H\colon (D_\xi, S) \to \Phi(\xi)$ is continuous.*

Proof: Let us have $\varepsilon > 0$, $d \in D_\xi$. We denote by $\| \dots \|$ the norm on $T_\xi(M)$ arising from the metric tensor g. It is clear that there are k_1, $k_2 > 0$ such that for any $X \in T_\xi(M)$ we have $k_1\sigma(X, 0) \leqslant \|X\| \leqslant k_2\sigma(X, 0)$. According to this fact and proposition 5 it is obvious that there exists $\delta > 0$ such that if $d_1 \in D_\xi$ is such that $S(d, d_1) < \delta$, then

$$\|F_i - F_i^{(1)}\| < \varepsilon, \quad i = 1, \dots, n,$$

where F_i and $F_i^{(1)}$ are the vectors from $T_\xi(M)$ obtained by the parallel displacement of the vector E_i along the curves d and d_1, respectively. Let us write

$$F_i = \sum_{j=1}^{n} a_{ij} E_j, \quad F_i^{(1)} = \sum_{j=1}^{n} a_{ij}^{(1)} E_j.$$

From this we have

$$F_i - F_i^{(1)} = \sum_{j=1}^{n} (a_{ij} - a_{ij}^{(1)})\, E_j$$

and after the scalar multiplication

$$\sum_{j=1}^{n} (a_{ij} - a_{ij}^{(1)})^2 = \|F_i - F_i^{(1)}\|^2 < \varepsilon^2.$$

This implies the inquality $\max\limits_{i,j} | a_{ij} - a_{ij}^{(1)} | < \varepsilon$. The proposition is therefore proved.

REFERENCES

[1] Kobayashi S., Nomizu K., *Foundations of differential geometry*, New York—London 1963.

[2] Busemann H., *The geometry of geodesics*, New York 1955.

[3] Stěpanov V. V., *Kurs diferenciálních rovnic*, Praha 1950.

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