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THE MAXIMAL SEMILATTICE DECOMPOSITION OF A SEMIGROUP, RADICALS AND NILPOTENCY

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In paper [7] M. Petrich dealt with the maximal semilattice decomposition of a semigroup and he studied the classes of this decomposition. In the present paper a description of these classes and their products is given in terms of Luh Jiang completely prime radicals and faces of a semigroup $S$. Also the case of the commutative semigroup is discussed.

The last, 5th section is self-contained. Here a characterization of the class of all periodic semigroups with period 1, a characterization of the class of all periodic semigroups with index 1 and some characterizations of the class of all bands are given. We accomplished this using the mappings $M \to N_i(M)$ ($i = 1, 2, 3$), where $N_1(M)$ ($N_2(M)$) $[N_3(M)]$ is the set of all strongly (weakly) [almost] nilpotent elements with respect to the subset $M$ of the semigroup $S$ (see [9]).

1. On an equivalence relation.

Let $S$ be a non-empty set and $\Gamma$ a family of subsets of $S$ with the property
\[(\alpha) : S \in \Gamma, \Box \notin \Gamma.\]

On $S$ the following relation can be introduced (see [5] p. 203): $x \Gamma y$ if and only if for every $A \in \Gamma$, either $x, y \in A$ or $x, y \notin A$. This relation is an equivalence relation on $S$. If $x \Gamma y$ holds, we say that $x$ and $y$ are $\Gamma$-equivalent and the equivalence relation is called $\Gamma$-equivalence relation.

Let $A = \{S\} \cup \{B \mid B = S \setminus A, A \in \Gamma, A \neq S\}$. Evidently the $\Delta$-equivalence relation is equal to the $\Gamma$-equivalence relation.

If $M$ is an arbitrary (non-empty) subset of $S$, then the intersection of all sets of $\Gamma$, which contain the set $M$, will be called a $\Gamma$-hull of the set $M$.

The following Lemmas are evident.

**Lemma 1.** The elements $x$ and $y$ of $S$ are $\Gamma$-equivalent if and only if they have the same $\Gamma$-hull ($\Delta$-hull).

**Lemma 2.** The $\Gamma$-equivalence relation is the intersection of the universal equi-
Corollary. Every class \( M(x \in M_z) \) of the \( I \)-equivalence relation (\( A \)-equivalence relation) is the intersection of all sets of \( I \cup A \) which contain the element \( x \).

Remark. If \( S \) is a semigroup and \( I \) the family of all right (left) [two-sided] ideals of \( S \), then the classes of this \( I \)-equivalence relation are the \( r -(l-\) \( [f-] \) classes, introduced by J. A. Green in his paper [4]. In section 3 we shall show that the foregoing construction is useful also in the case if \( I \) is the family of all completely prime ideals of a semigroup \( S \).

2. Completely prime ideals and faces

A (two-sided) ideal \( P \) of a semigroup \( S \) will be called a completely prime ideal, if \( xy \in P \ (x, y \in S) \) implies that either \( x \in P \) or \( y \in P \).

A subset \( M \subseteq S \) will be called a face of \( S \), if \( S \setminus M \) is a completely prime ideal or \( M = S \) (the empty set will be considered to be a face).

It is known that a subset \( M \) of \( S \) is a face of \( S \) if and only if \( x, y \in M \) is equivalent to \( xy \in M \).

We now establish some Lemmas about completely prime ideals and faces.

Lemma 3. Let \( M_1 \) be a face of \( S \) and \( M_2 \) a face of \( M_1 \). Then \( M_2 \) is a face of \( S \).

Corollary 1. If \( P_1 \) is a completely prime ideal of \( S \) and \( P_2 \) a completely prime ideal of \( S \setminus P_1 \), then \( P_1 \cup P_2 \) is a completely prime ideal of \( S \).

Corollary 2. Let \( M \) be a face of \( S \) and \( P' \) a completely prime ideal of \( M \). Then \( P' \cup (S \setminus M) \) is a completely prime ideal of \( S \).

Lemma 4. Let \( S \) be a semigroup, \( S' \) a subsemigroup and \( M \) a face of \( S \). Then \( M \cap S' \) is a face of \( S' \).

Corollary. Let \( S \) be a semigroup, \( S' \) a subsemigroup and \( P \) a completely prime ideal of \( S \). Then \( P \cap S' \) is a completely prime ideal of \( S' \).

It is evident that the intersection of an arbitrary number of faces of a semigroup \( S \) is a face of \( S \). Moreover the union of an arbitrary number of completely prime ideals of a semigroup \( S \) is a completely prime ideal of \( S \).

Let \( S' \) be a subsemigroup of a semigroup \( S \) and \( M \) a non-empty subset of \( S' \). The intersection of all completely prime ideals of \( S' \), which contain \( M \), will be denoted by \( C(M, S') \). If \( M = \{x\} \) is a one-element set, we shall write \( C(x, S') \) instead of \( C(\{x\}, S') \). If \( M \) is a (two-sided) ideal of \( S' \), then \( C(M, S') \) is called a Luh Jiang radical of the semigroup \( S' \) with respect to \( M \). For \( S' = S \) we shall write \( C(M) \) and \( C(x) \) instead of \( C(M, S) \) and \( C(x, S) \). The principal two-sided (right) [left] ideal of \( S' \), generated by the element \( x \), will
be denoted by $J(x, S') (R(x, S')) [L(x, S')]$. If $S' = S$, we shall write $J(x)$, $R(x)$, $L(x)$ instead of $J(x, S)$, $R(x, S)$ and $L(x, S)$.

It can be shown (see [8]) that $C(M)$ is the set of all such elements $r \in S$ that every face, which contains $r$, has a non-empty intersection with $M$.

**Lemma 5.** For every two elements $x, y \in S$, $C(xy) = C(J(xy)) = C(J(x)J(y)) = C(J(x) \cap J(y)) = C(x) \cap C(y)$ holds.

**Proof.** For every semigroup $S$, $J(xy) \subseteq J(x) \cap J(y)$ holds, therefore $C(J(xy)) \subseteq C(J(x) \cap J(y)) = C(J(x)) \cap C(J(y))$ (see [8]).

Let $r \in C(J(x) \cap J(y))$. Then each face of $S$, which contains $r$, contains an element of $J(x) \cap J(y)$. Hence each face, which contains $r$, contains $x$ and $y$. Thus it contains $xy$ and it has a nonempty intersection with $J(xy)$, which implies $r \in C(J(xy))$. This means that $C(J(x) \cap J(y)) \subseteq C(J(xy))$ and therefore $C(J(xy)) = C(J(x)) \cap C(J(y))$.

Evidently in every semigroup $S$, $J(xy) \subseteq J(x) J(y) \subseteq J(x) \cap J(y)$ holds. Thus $C(J(xy)) = C(J(x) J(y)) = C(J(x) \cap J(y)) = C(J(x)) \cap C(J(y))$.

The rest of Lemma 5 follows from the fact that for every $x \in S$, $C(x) = C(J(x))$ holds.

### 3. The maximal semilattice decomposition of a semigroup $S$.

Let $\mathcal{F}$ be the family of all completely prime ideals of a semigroup $S$. The $\mathcal{F}$-equivalence relation is a congruence. This was shown by M. Petrich (see [7]) and follows immediately from Lemma 2, because every equivalence relation of Lemma 2 is a congruence. Moreover all factor semigroups of $S$ modulo these congruences are semilattices, therefore the factor semigroup modulo $\mathcal{F}$-equivalence relation is a semilattice too. (This holds also for an $\mathcal{A}$-equivalence relation, where $\mathcal{A}$ is a subfamily of the family $\mathcal{F}$.)

A decomposition of a semigroup $S$ will be called (in agreement with [7]) a semilattice decomposition of $S$ if this decomposition belongs to a congruence of $S$ and the factor semigroup modulo this congruence is a semilattice. M. Petrich has shown (see [7]) that the decomposition belonging to the $\mathcal{F}$-equivalence relation is the maximal semilattice decomposition of the semigroup $S$ in the sense that every homomorphic image of $S$, which is a semilattice, is a homomorphic image of the factor semigroup of $S$ modulo the $\mathcal{F}$-equivalence relation.

M. Petrich [7] has shown some properties of the maximal semilattice decomposition of a semigroup $S$. On the base of the preceding two sections we can establish some other properties of this decomposition.

Let $\mathfrak{M}$ be the set of all faces of a semigroup $S$ without \(\square\). Let $N_x$ be the
class of the maximal semilattice decomposition of $S$ which contains the element $x$. If $S'$ is a subsemigroup of the semigroup $S$, we denote by $N(x, S')$ the intersection of all faces of $S'$ that contain $x$ (it is the minimal face of $S'$ which contains $x$). Instead of $N(x, S)$ we shall write simply $N(x)$.

From the preceding sections the following Theorems follow:

**Theorem 1.** The fulfilment of the following conditions for elements $x, y$ of a semigroup $S$ is equivalent:

a) $x\overline{\mathcal{F}}y$

b) $x\overline{\mathcal{R}}y$

c) $N(x) = N(y)$ (see [7])

d) $C(x) = C(y)$

e) $C(J(x)) = C(J(y))$.

Proof. $N(x)$ is the $\mathcal{R}$-hull of $x$, $C(x)$ is the $\mathcal{F}$-hull of $x$ and $C(x) = C(J(x))$.

Remark. In d) of Theorem 1 we can replace the elements $x$ and $y$ by their principal right (left) ideals, since $C(x) = C(R(x)) = C(L(x))$.

M. Petrich has shown (see [7]) that a class $N_x$ of the maximal semilattice decomposition of a semigroup $S$ contains no proper completely prime ideals. From this follows

**Lemma 6a.** Let $S'$ be a subsemigroup of a semigroup $S$ and let $N_x \subseteq S'$. Then every completely prime ideal of $S'$ is either disjoint with $N_x$ or contains $N_x$.

Proof. Suppose, by way of contradiction, that there exists a completely prime ideal $P'$ of $S'$, which has a non-empty intersection with $N_x$ but $P'$ does not contain $N_x$. Then, as a consequence of Lemma 4, $N_x \cap P'$ is a proper completely prime ideal of $N_x$. But this is a contradiction.

Evidently the following Lemma holds.

**Lemma 6b.** Let $S'$ be a subsemigroup of a semigroup $S$ and let $N_x \subseteq S'$. Then every face of $S'$ is either disjoint with $N_x$ or contains $N_x$.

Now we can easily prove

**Theorem 2.** For each $x \in S$ we have:

a) $N_x$ is the intersection of all completely prime ideals and of all faces of the semigroup $S$ that contain the element $x$.

b) $N_x = N(x) \cap C(x)$,

c) $N_x = C(x, N(x))$,

d) $N_x = N(x, C(x))$.

Proof. a) follows from the Corollary of Lemma 2. b) is equivalent to a). b) and the Corollary of Lemma 4 and Lemma 6a imply c). d) follows from b), Lemma 4 and Lemma 6a.

Remark 1. Evidently $N_x = N(x) \cap C(J(x))$, $N_x = C(J(x, N(x)), N(x))$ and $N_x = N(x, C(J(x)))$ holds.

Remark 2. From Lemma 4 and its Corollary, from Lemma 6a and Lemma
6b it is evident that if $S'$ is a subsemigroup of a semigroup $S$ and $N_x \subseteq S'$, then $N_x$ is also a class of the maximal semilattice decomposition of the semigroup $S'$. Thus if $N_x \subseteq S'$, we have by Theorem 2:

\[
\begin{align*}
N_x &= N(x, S') \cap C(x, S') \\
N_x &= C(x, N(x, S')) \\
N_x &= N(x, C(x, S')).
\end{align*}
\]

We can take for $S'$ an arbitrary face of $S$ which contains $x$ or an arbitrary completely prime ideal of $S$ which contains $x$.

The set of all faces of a semigroup $S$ is a lattice if for every two faces $M_1$ and $M_2$ of $S$, $M_1 \cap M_2 = M_1 \cap M_2$ and $M_1 \cup M_2$ is the minimal face of $S$, which contains $M_1$ and $M_2$. Then we have

**Theorem 3.** For every two elements $x, y$ of $S$ the following holds:

a) $N_{xy} = (N(x) \cup N(y)) \cap C(x) \cap C(y)$

b) $N_{xy} = (N(x) \cup N(y)) \cap C(J(x) J(y))$.

**Proof.** By Theorem 2, $N_{xy} = N(xy) \cap C(xy)$. Evidently $N(xy) = N(x) \cup \cup N(y)$, but on the other hand $C(xy) = C(J(xy)) = C(J(x) J(y)) = C(J(x)) \cap \cap C(J(y))$ by Lemma 5 and this proves Theorem 3.

4. The case of a commutative semigroup.

There are other possibilities how to express the sets $C(M)$ of a commutative semigroup $S$. This leads to other expressions for the $T$-equivalence relation and for the classes $N_x$.

Let $J$ be a (two-sided) ideal of $S$.

a) Let $x$ be such an element of $S$ that for some positive integers $n$, $x^n \in J$ holds. Then $x$ will be called a nilpotent element of the semigroup $S$ with respect to the ideal $J$. The set of all nilpotent elements of the semigroup $S$ with respect to $J$ will be denoted by $\mathcal{N}(J)$.

b) An ideal $J$ of the semigroup $S$, each element of which is nilpotent with respect to $J$, will be called a nilideal of $S$ with respect to $J$. The union $R^*(J)$ of all nilideals of $S$ with respect to $J$ is called the Clifford radical of $S$ with respect to $J$.

c) An ideal (subsemigroup) $I$ of the semigroup $S$, for which there exists such a positive integer $n$ that $I^n \subseteq J$, is called a nilpotent ideal (subsemigroup) of $S$ with respect to $J$. The union $R(J)$ of all nilpotent ideals of $S$ with respect to $J$ will be called the Schwarz radical of $S$ with respect to $J$.

d) An ideal $I$ of the semigroup $S$, with the property that each subsemigroup of $I$ generated by a finite number of elements of $I$ is nilpotent with respect to $J$, is called a locally nilpotent ideal of $S$ with respect to $J$. The union $L(J)$
of all locally nilpotent ideals of \( S \) with respect to \( J \) will be called the Ševrin radical of \( S \) with respect to \( J \).

e) An ideal \( P \) of the semigroup \( S \) is called a prime ideal of \( S \), if for any two ideals \( A \) and \( B \) of \( S \), \( AB \subseteq P \) implies either \( A \subseteq P \) or \( B \subseteq P \). The intersection \( M(J) \) of all prime ideals of \( S \) that contain the ideal \( J \) is called the McCoy radical of \( S \) with respect to \( J \).

Remark. It is known that in a commutative semigroup an ideal \( P \) is a prime ideal if and only if it is a completely prime ideal.

Then from Theorem 1 we obtain

**Theorem 4.** The following conditions for the elements \( x, y \) of a commutative semigroup \( S \) are equivalent:

a) \( x (\sim) y \)

b) \( x (\sim) y \)

c) \( C(x) = C(y) \)

d) \( M(x) = M(y) \)

e) \( C(J(x)) = C(J(y)) \)

f) \( M(J(x)) = M(J(y)) \)

g) \( \bar{N}(J(x)) = \bar{N}(J(y)) \)

h) \( R^*(J(x)) = R^*(J(y)) \)

i) \( R(J(x)) = R(J(y)) \)

j) \( L(J(x)) = L(J(y)) \).

The proof follows from the Remark preceding Theorem 4 and from the fact that in every commutative semigroup \( S \), \( C(J) = M(J) = \bar{N}(J) = R^*(J) = R(J) = L(J) \) holds for each ideal \( J \) of \( S \) (see [8] and [1]).

From Theorem 2 we obtain

**Theorem 5.** For every \( x \in S \) we have:

a) \( N_x = N(x) \cap C(x) = N(x) \cap M(x) \)

b) \( N_x = N(x) \cap C(J(x)) = N(x) \cap M(J(x)) = N(x) \cap \bar{N}(J(x)) = N(x) \cap R^*(J(x)) = N(x) \cap R(J(x)) = N(x) \cap L(J(x)) \)

c) \( N_x = C(x, N(x)) = M(x, N(x)) \)

d) \( N_x = C(J(x, N(x)), N(x)) = M(J(x, N(x)), N(x)) = \bar{N}(J(x, N(x)), N(x)) = N(x, C(J(x))) = N(x, M(J(x))) = N(x, \bar{N}(J(x))) = N(x, R^*(J(x))) = N(x, R(J(x))) = N(x, L(J(x))) \).

A similar adaptation of Theorem 3 for commutative semigroups is obvious.

5. On nilpotency.

In paper [9] the notions of strong nilpotency, weak nilpotency and almost
nilpotency of an element of a semigroup $S$ with respect to an arbitrary subset of $S$ were introduced. We shall characterize three classes of periodic semigroups using these notions.

Let $S$ be a semigroup and $M$ a subset of $S$.

a) An element $x \in S$ will be called strongly nilpotent with respect to $M$ if there exists such a positive integer $N$ that for every integer $n \geq N$, $x^n \in M$ holds. The set of all strongly nilpotent elements of $S$ with respect to $M$ will be denoted by $N_1(M)$.

b) An element $x \in S$ will be called weakly nilpotent with respect to $M$ if for infinitely many positive integers $n$, $x^n \in M$ holds. The set of all weakly nilpotent elements of $S$ with respect to $M$ will be denoted by $N_2(M)$.

c) An element $x \in S$ will be called almost nilpotent with respect to $M$, if for some positive integers $n$, $x^n \in M$ holds. The set of all almost nilpotent elements of $S$ with respect to $M$ will be denoted by $N_3(M)$.

In paper [9] the mappings $M \rightarrow N_1(M), i = 1, 2, 3$ were studied. We shall show some other properties of these mappings.

**Theorem 6.** The class of all periodic semigroups with the period 1 is equal to the class of all semigroups in which the mappings $M \rightarrow N_1(M)$ and $M \rightarrow N_2(M)$ are equal.

**Proof.** Let $a \in S$. Let $A = \langle a \rangle$ (the cyclic semigroup generated by $a$) and let $A'$ be the set of the elements of the sequence $\{a^{2k}\}^{\infty}_{k=1}$. Clearly $a \in N_2(A')$. If the mappings $M \rightarrow N_1(M)$ and $M \rightarrow N_2(M)$ are equal, then $a \in N_1(A')$. Hence there exists such a positive integer $N_1$ that for all integers $n > N_1$, $a^n \in A'$ holds. Therefore $\langle a \rangle$ is a cyclic semigroup of finite order (and $S$ is a periodic semigroup). Let $r$ be the index and $m$ the period of $\langle a \rangle$. Then $a \in N_2(a^r) = N_1(a^r)$. Thus there exists such a positive integer $N_2$ that for all integers $n' > N_2$, $a^{n'} = a^r$, i. e. the period $m = 1$. This means that the semigroup $S$ is a periodic semigroup with the period 1.

If conversely $S$ is a periodic semigroup with the period 1 and $a \in N_2(M)$, then there exist infinitely many positive integers $n$ such that $a^n \in M$. But then there exists such a positive integer $N$ that for all integers $n > N$, $a^n \in M$. Hence $a \in N_1(M)$. Therefore we have $N_2(M) = N_1(M)$ for every subset $M \subseteq S$ and the mappings $M \rightarrow N_1(M)$ and $M \rightarrow N_2(M)$ are equal.

**Theorem 7.** The class $\mathcal{S}$ of all bands is equal to the class of all semigroups in which the mappings $M \rightarrow N_1(M)$ and $M \rightarrow N_3(M)$ are equal. The class $\mathcal{S}$ is also the class of all semigroups in which the mappings $M \rightarrow N_1(M)$, $M \rightarrow N_2(M)$ and $M \rightarrow N_3(M)$ are equal.

**Proof.** Let $a \in S$. Then $a \in N_3(a)$. If $M \rightarrow N_1(M)$ and $M \rightarrow N_3(M)$ are equal, then $a \in N_1(a) = N_3(a)$. Hence $a$ is strongly nilpotent with respect to $\{a\}$ and there exists such a positive integer $N$ that for all integers $n > N$,
This holds for every $a \in S$ i. e. $S$ is a periodic semigroup with the period 1 and the index 1. Thus $S$ is a band.

If conversely $S$ is a band, then $a \in N_2(M)$ implies $a \in N_1(M)$. Therefore $N_3(M) = N_1(M)$ for every subset $M$ of $S$, i. e. the mappings $M \to N_1(M)$ and $M \to N_3(M)$ are equal.

The last statement of Theorem 7 follows immediately.

If $S$ is a band, then the mappings $M \to N_i(M)$, $i = 1, 2, 3$ are clearly identity mappings. Moreover, we have

**Theorem 8.** The class of all bands is equal to the class of

a) all semigroups in which the mapping $M \to N_1(M)$ is the identity mapping,

b) all semigroups in which the mapping $M \to N_2(M)$ is the identity mapping,

c) all semigroups in which the mapping $M \to N_3(M)$ is the identity mapping.

**Proof.** a) If $M \to N_1(M)$ is the identity mapping, then $N_1(a) = a$, i. e. there exists such a positive integer $N$ that for all integers $n > N$, $a^n = a$ holds. Hence $\langle a \rangle$ is a cyclic semigroup with the period 1 and the index 1, therefore $a$ is an idempotent.

b) Let $M \to N_2(M)$ be the identity mapping. Then $N_2(a) = a$ and for some positive integers $n > 1$, $a^n = a$ holds. Therefore $\langle a \rangle$ is a cyclic group with the identity $e$. Hence there exists such a positive integer $m$, that $a^m = e$, which implies $a \in N_2(e) = e$. Thus $a$ is an idempotent.

c) If $M \to N_3(M)$ is the identity mapping, then $a \in N_3(a^2) = a^2$ implies $a = a^2$ for every $a \in S$.

The converse statement is evident.

**Theorem 9.** The class of all periodic semigroups with the index 1 is equal to the class of all semigroups in which the mappings $M \to N_2(M)$ and $M \to N_3(M)$ are equal.

**Proof.** Let the mappings $M \to N_2(M)$ and $M \to N_3(M)$ be equal. Let $a \in S$. Then $a \in N_3(a) = N_2(a)$ and for infinitely many positive integers $n$, $a^n = a$ holds. Thus $\langle a \rangle$ is a finite cyclic group. Hence $S$ is a periodic semigroup with the index 1.

Let $a \in N_3(M)$ and let $S$ be a periodic semigroup with the index 1. Then for infinitely many positive integers $n$, $a^n \in M$ holds. Hence $a \in N_2(M)$ q. e. d.

**Remark.** Theorem 6 follows also from Theorem 7 and 9, because $N_1(M) \subseteq N_2(M) \subseteq N_3(M)$ for every subset $M$ of $S$.

**REFERENCES**


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