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REGULAR GRAPHS, EACH EDGE OF WHICH BELONGS TO EXACTLY ONE s-GON

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At the Colloquium on Graph Theory at Tihany (1966) the first author expressed the following conjecture:

To every pair of positive integers r, s there exists a regular graph G of the degree $2r$, in which each edge belongs to exactly one s-gon.

We are going to prove this conjecture.

The conjecture is evidently true for $s = 1$ and $s = 2$.

We define the (r, s)-graph as a graph whose vertex set is the union of two disjoint sets U and V, while each vertex of U has the degree r, each vertex of V has the degree s and each edge of this graph joins a vertex of U with a vertex of V.

We shall prove a theorem, which has the character of a lemma for us.

Theorem 1. *Let g, r, s be given positive integers. There exists such an (r, s)-graph which does not contain any circuit of the length less than g.*

Proof. P. Erdös and H. Sachs [1, 3] have proved that for every two positive integers $d \neq 1$, t there exists a regular graph of the degree d, whose girth is equal to t (the girth of the graph is the minimal length of a circuit in this graph). Let p be a common multiple of the numbers r and s and construct such a graph G_1 for $d = 2p$ and $t = g$. The graph G_1 is a regular graph of an even degree, therefore it can be decomposed into quadratic factors $F_1, \ldots, F_p$. Now direct this graph so that each circuit which is a component of any of the factors $F_1, \ldots, F_p$ might become a cycle (directed circuit). In this way we obtain from the non-directed graph G_1 a directed graph G_2, in which from each vertex exactly p edges go out and also exactly p edges come into it. The vertices of the graph G_2 will be denoted by $u_1, \ldots, u_n$. Now construct the non-directed graph G_3. Its vertices will be $v_1, \ldots, v_n$, $w_1, \ldots, w_n$. The vertex v_i is joined with the vertex $w_j (1 \leqq i \leqq n, 1 \leqq j \leqq n)$ by a non-directed edge if and only if in G_2 a directed edge goes from the vertex u_i to the vertex u_j. No two of the vertices $v_1, \ldots, v_n$ and no two of the vertices $w_1, \ldots, w_n$ are joined by an edge. Thus the graph G_3 is a bipartite regular graph of the degree p; its girth evidently cannot be less than the girth of the graph G_1 and therefore it is greater than or equal to g. Finally construct the graph G. The vertices of the graph G will be $v_i^{(\alpha)}$, $w_i^{(\beta)}$ for $i = 1, \ldots, n$; $\alpha = 1, \ldots, p/r$; $\beta = 1, \ldots, p/s$. In the graph G_3 we decompose the set of the edges incident with the vertex v_i (for each $i = 1, \ldots, n$) in an arbitrary way into p/r subsets $H_i^{(1)}, \ldots, H_i^{(p/r)}$, which are pairwise disjoint and each of which contains exactly r elements. Similarly we decompose the set of the edges incident with the vertex w_i into p/s subsets $K_i^{(1)}, \ldots, K_i^{(p/s)}$, which are pairwaise disjoint and each of them contains exactly s elements. In the graph G the vertex $v_i^{(\alpha)}$ is joined with the vertex $w_j^{(\beta)}$ if and only if in the graph G_3 the vertex v_i is joined with the vertex w_j by an edge belonging to $H_i^{(\alpha)} \cap K_j^{(\beta)}$. No two of the vertices $v_i^{(\alpha)}$ and no two of the vertices $w_j^{(\beta)}$ are joined by an edge. Therefore G is a bipartite graph, each of the vertices $v_i^{(\alpha)}$ has the degree r, each of the vertices $w_j^{(\beta)}$ has the degree s. The girth of the graph G cannot be less than the girth of the graph G_3 and therefore it is greater than or equal to g.

With the help of Theorem 1 we shall prove our conjecture.

Theorem 2. *To each pair of positive integers r and s there exists a regular graph of the degree $2r$, in which each edge belongs to exactly one s-gon in the graph.*

Proof. Construct the graph G from Theorem 1 for $g = 2s + 1$. It is not useful here to use the complicated symbols from the proof of Theorem 1, hence denote the vertices of the degree r by $v_1, \ldots, v_k$ and the vertices of the degree s by $w_1, \ldots, w_l$. Now construct the graph H_0 consisting of l components, each of which is an s-gon. Denote these s-gons by $C_1, \ldots, C_l$. For $i = 1, \ldots, l$ denote the vertices of the s-gon C_i by the symbols u_{ij}, where j runs through all such positive integers for which there exists an edge joining v_j and w_i in G. It does not matter in what order we number those vertices. When we identify the vertices u_{ij} for all i (always at the fixed j), we obtain from the graph H_0 the graph H; the vertex created by identifying the vertices u_{ij} for all i will be denoted by u_j. We can easily show that the thus constructed graph H is the wanted graph.

For some special cases we can construct the graph satisfying our conditions in a more simple way.

Construction A (for odd s and arbitrary r).

Let $n = (s - 1)/2$, $R = \{1, 2, \ldots, r\}$, $S = \{1, 2, \ldots, s\}$. By the symbol V we denote the Cartesian product

$$V = \underbrace{S \times S \times \ldots \times S}_{r \text{ times}}$$

Now construct the graph G_s^r in the following way. Its vertex set is the set V. The vertex $x = (x_1, x_2, \ldots, x_r) \in V$ is joined by an edge with the vertex $y = (y_1, y_2, \ldots, y_r) \in V$ if and only if there exists a number $i \in R$ such that $| x_i - y_i | \equiv 1 \pmod s$ and for each $j \neq i$, $j \in R$ we have $x_j = y_j$. The numbers of an r-tuple of V will be called co-ordinates of a vertex and the i-th of them will be called the i-th co-ordinate of a vertex. Now we shall prove that we have obtained the wanted graph.

From the construction it is evident that G_s^r is a regular graph of the degree $2r$. Let e be an arbitrary edge of G_s^r and let e join the vertex $u = (u_1, u_2, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_r)$ with the vertex $v = (u_1, u_2, \ldots, u_{i-1}, 1 + u_i, u_{i+1}, \ldots, u_r)$. Evidently the edge e belongs to the s-gon K_0 of the graph G_s^r with the vertices $(u_1, u_2, \ldots, u_{i-1}, z, u_{i+1}, \ldots, u_r)$, where z runs through all values of S. Thus it suffices to prove that e does not belong to any other s-gon of the graph G_s^r. Let K_1 be an arbitrary s-gon in G_s^r containing the edge e. Go around the circuit K_1 issuing for example from the vertex u through the edge e and consider this: to the vertex u we shall return after going through an odd number of edges (because $s = 2n + 1$ is odd) and after going through an arbitrary edge exactly one of the co-ordinates of the vertex will be changed and it will be in such a way that it will be increased or decreased by one. As the total number of such changes at the mentioned „travel" is odd, at least for one number $k \in R$ the following must be true: the k-th co-ordinate will be changed at this „travel" an odd number of times. After the mentioned changes we must obtain from the number u_k again the number u_k (because the first vertex and the last are u). From the construction of the fraph G_s^r it follows that an odd number of the mentioned changes can lead to the initial value u_k only if we go through the whole set S (i. e. if we increase s times or decrease s times the k-th co-ordinate by one — it is understood modulo s). Thus the k-th co-ordinate will be changed at least s times. As the total number of changes is s and at each change only one co-ordinate is changed, it means that no other than the k-th co-ordinate will be changed during that „travel". The vertices u and v belong to K_1 and have the different i-th co-ordinates. From that it follows that $i = k$ and also $K_1 = K_0$. Therefore an arbitrary edge e belongs to exactly one s-gon of the graph G_s^r, which was to be proved.

In the following the complete graph with q vertices will be denoted by $\langle q \rangle$.

Construction B (for $r = 2$ and arbitrary s).

We shall construct a regular graph E of the degree s whose girth is equal to $s + 1$ (see [1], [3]). Let F be the interchange graph (see [2]) of the graph E. The following is evident: the set of edges incident with some vertex $x \in E$ corresponds to the set of s vertices of some subgraph of F isomorphic with

$\langle s \rangle$ and each vertex $v \in F$ belongs to exactly two such complete subgraphs (because each edge in E is incident with two vertices). Also this is true: if e_1, e_2 are two edges of F incident with the vertex v in F, but not belonging to the same subgraph isomorphic with $\langle s \rangle$, then in F there does not exist any circuit of the length less than $s + 1$ which would contain both e_1 and e_2. Therefore each edge of F belongs to exactly one subgraph of F isomorphic with $\langle s \rangle$. Thus from each subgraph of F isomorphic with $\langle s \rangle$ omit all edges except the edges of some of its s-gons. We obtain the graph Q which has the following two properties: (1) its degree is 4; (2) each of its edges belongs to exactly one s-gon. Thus the construction is finished.

At the end of the article we shall prove the non-existence of such a graph of an odd degree.

Theorem 3. *There does not exist any regular graph of an odd degree such that each edge of it would be contained in exactly one s-gon where s is a positive integer.*

Proof. Assume that such a graph G exists; let its degree be p. Let u be a vertex of G and h an edge incident with u. The edge h belongs to exactly one s-gon in G; denote this s-gon by C_0. The vertex u is incident with the edge h of C_0, hence it belongs to C_0 and must be incident with exactly two edges of C_0. We see that if C is an s-gon of G, then the vertex u is incident either with exactly two edges of C, or with no edge of C; in both the cases the number $n(u, C)$ of the edges of C incident with u is even. As each edge of G belongs to exactly one s-gon in G, the number of edges incident with u, i. e. the degree of u, is the sum of $n(u, C)$ taken over all s-gons C in G. As all $n(u, C)$ are even, their sum must be even, too. But the degree of the vertex u is p and p is odd, because G is a regular graph of an odd degree. Thus we have obtained a contradiction.

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