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INCIDENCE PRESYSTEMS

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In the present paper the incidence systems are investigated with a distinguished bijection between their point set and line set; these can be characterized with the help of certain halfgroupoids, the binary operation of which represents simultaneously the joining of points and the cutting of lines. Further, the same approach will be used to affine planes with one pencil of parallel lines omitted, which leads to some generalization of Hall's ternary rings. The notions of incidence presystem, geometric halfgroupoid and affine preplane are new. One obtains some results which generalize certain properties of planes with duality.

§ 1 *General part*

An *incidence system* will be defined as a triple (P, L, I), where P and L are sets and I is a binary relation between P and L such that

$$\mathrm{P1}_{\leqq}\ \{a, b\} \subseteqq P \Rightarrow \operatorname{card} \{x \in L \mid a, b \mathrm{I} x\} \leqq 1$$

and

$$\mathrm{P2}_{\leqq}\ \{a, b\} \subseteqq L \Rightarrow \operatorname{card} \{x \in P \mid x \mathrm{I} a, b\} \leqq 1. \quad (^1)$$

An *isomorphism* between incidence systems (P, L, I) and (P', L', I$'$) is defined as a couple (π, λ) of bijections $\pi : P \to P'$ and $\lambda : L \to L'$ such that $a \mathrm{I} b \Leftrightarrow \pi a \mathrm{I}' \lambda b$. An isomorphism between incidence systems (P, L, I) and (L, P, I^{-1}) is called a *duality* of (P, L, I).

An *incidence presystem* is defined as a couple (S, ι) where S is a set, a ι is a binary relation in S such that

$$\mathrm{P1}'_{\leqq}\ \{a, b\} \subseteqq S \Rightarrow \operatorname{card} \{x \in S \mid a, b \,\iota\, x\} \leqq 1$$

and

$$\mathrm{P2}'_{\leqq}\ \{a, b\} \subseteqq S \Rightarrow \operatorname{card} \{x \in S \mid x \,\iota\, a, b\} \leqq 1. \quad (^2)$$

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(1) { . . . } is the symbol for a set of elements.

(2) $\mathrm{P1}_{=}$, $\mathrm{P2}_{=}$, $\mathrm{P1}'_{=}$ and $\mathrm{P2}'_{=}$ mean that the symbol $\leqq 1$ on the right hand is replaced by $=1$.

An *isomorphism* between incidence presystems (S, ι) and (S', ι') is defined as a bijection $\sigma : S \to S'$ such that $a \iota b \langle = \rangle \sigma a \iota' \sigma b$. An incidence system (P, L, I) is said to be a *projective plane* if there are valid P $1_=$, P $2_=$ and
P 0 there is $\{a_1, a_2, a_3, a_4\} \subseteqq P$ such that $\{i, j, k\} \subset \{1, 2, 3, 4\}$

$$\Rightarrow \text{card } \{x \in L \mid a_i, a_j, a_k \text{ I} x\} = 0. \qquad (^3)$$

An incidence presystem (S, ι) is said to be a *projective preplane* if there are valid P $1'_=$, P$2'_=$ and
P 0′there is $\{a_1, a_2, a_3, a_4\} \subseteqq S$ such that $\{i, j, k\} \subset \{1, 2, 3, 4\}$

$$\Rightarrow \text{card } \{x \in S \mid a_i, a_j, a_k \, \iota \, x\} = 0.$$

Remark. Of course, each incidence presystem (S, ι) can be understood as an incidence system (S, S, ι).

Proposition 1. a) *Let* $\boldsymbol{I} = (P, L, \mathrm{I})$ *be an incidence system such that* card $P =$ $=$ card L. *If* S *is a set bijective to* P, $\alpha : P \to S$ *and* $\beta : L \to S$ *two bijections and* ι *a binary relation in* S *defined by* $a \mathrm{I} b \langle = \rangle : \alpha a \, \iota \, \beta b$, *then* $\mathfrak{I}(\boldsymbol{I}, \alpha, \beta) : = (S, \iota)$ *is an incidence presystem.*

b) *Let* $\mathsf{I} = (S, \iota)$ *be an incidence presystem. If* P *and* L *are sets bijective to* S, $\alpha : P \to S$ *and* $\beta : L \to S$ *two bijections and* I *a binary relation between* P *and* L *defined by* $a \mathrm{I} b : \langle = \rangle \alpha a \, \iota \, \beta b$, *then* $\mathcal{J}(\mathsf{I}, \alpha, \beta) : = (P, L, \mathrm{I})$ *is an incidence system.*

The proof is easy.

Remark. If (α, β) runs over all pairs of bijections of the prescribed type then in case a) the corresponding $\mathfrak{I}(\boldsymbol{I}, \alpha, \beta)$ are in general not mutually isomorphic, whereas in case b), $\mathcal{J}(\mathsf{I}, \alpha, \beta)$ are mutually isomorphic.

Proposition 2. *Let* $\mathsf{I} = (S, \iota)$ *be an incidence presystem and* $\boldsymbol{I} : = \mathcal{J}(\mathsf{I}, \alpha, \beta)$ *for some choice of bijections* $\alpha : P \to S$ *and* $\beta : L \to S$. *Then* ι *is symmetric iff* $\beta^{-1} \circ \alpha$ *is a duality of* $\boldsymbol{I}$. *If* ι *is symetric and if there exist* $a, b \in S$ *such that* $a \iota b \neq a$, *then* ι *is not reflexive.*

Proof. By the definition of I in $\boldsymbol{I} = (P, L, \mathrm{I})$ we can rewrite $x \iota y \Rightarrow y \iota x$ as $\alpha^{-1} x \, \mathrm{I} \, \beta^{-1} y \Rightarrow \alpha^{-1} y \, \mathrm{I} \, \beta^{-1} x$. The last relation can be written as $(\beta^{-1} \circ \alpha)^{-1}$ $\beta^{-1} y \, \mathrm{I} (\beta^{-1} \circ \alpha) \, \alpha^{-1} x$ and from this if follows that $\beta^{-1} \circ \alpha$ is a duality of $\boldsymbol{I}$, and conversely. Now, let ι be symmetric and reflexive. Use elements $a, b \in S$ with $a \iota b \neq a$ so that $a \iota a, b$ and simultaneously $b \iota a, b$, which contradicts P $2'_{\leqq}$.Q.E.D.

A *pairing system*(3) is defined as a quadruple $(P, L, \cdot, \cap)$, where

(3) The denotation pairing system is new, whereas the concept of this notion is contained in [1], p. 5.

P and L are sets and $\cdot$ and $\cap$ are mappings of some set dom $(\cdot) \subseteqq (P \times P) \setminus \text{diag}$ $(P \times P)$ into L, respectively, of some set dom $(\cap) \subseteqq (L \times L) \setminus \text{diag}\,(L \times L)$ into P and the following conditions are fulfilled:

$$\text{PS 1 } (a, b), (c, d) \in \text{dom}\,(\cdot) \;\&\; a \cdot b = c \cdot d \;\&\; b \neq c \Rightarrow (b, c) \in \text{dom}\,(.) \;\&\; a \cdot b = b \cdot c,$$

$$\text{PS 2 } (a, b), (a, c) \in \text{dom}\,(\cdot) \;\&\; a \cdot b \neq a \cdot c \Rightarrow (a \cdot b, a \cdot c) \in \text{dom}\,(\cap) \;\&\; (a \cdot b) \cap (a \cdot c) = a.$$

If $P = L$ and $\cdot = \cap$ then the pairing system will be called a *geometric half-groupoid*. The condition PS 2 has then the form

$$\text{PS 2* } (a, b), (a, c) \in \text{dom}\,(\cdot) \;\&\; a \cdot b \neq a \cdot c \Rightarrow (a \cdot b, a \cdot c) \in \text{dom}\,(\cdot) \;\&\; (a \cdot b) \cdot (a \cdot c) = a.$$

Remark. If $(P, L, \cdot, \cap)$ is a pairing system, then the operation $\cdot$ is commutative in the following sense:

$$(a, b) \in \text{dom}\,(\cdot) \Rightarrow (b, a) \in \text{dom}\,(\cdot) \;\&\; a \cdot b = b \cdot a.$$

This follows from PS 1 for $a = c$, $b = d$.

Proposition 3.(4) a) *Each incidence system* $\boldsymbol{I} = (P, L, \mathrm{I})$ *canonically determines a pairing system* $\mathscr{P}(\boldsymbol{I}) := (P, L, \cdot(\boldsymbol{I}), \cap(\boldsymbol{I}))$, *where* $a \cdot (\boldsymbol{I})\, b$ *is defined iff* card $\{x \in L \mid a, b\, \mathrm{I}\, x\} = 1$ *as the* $x \in L$, *which satisfies* a, b, $\mathrm{I}\, x$ *and* $a \cap (\boldsymbol{I}) b$ *is defined iff* card $\{x \in P \mid x\, \mathrm{I}\, a, b\} = 1$ *as the* $x \in P$ *with* $x\, \mathrm{I}\, a, b$.

b) *Each pairing system* $\boldsymbol{P} = (P, L, \cdot, \cap)$ *canonically determines an incidence system* $\mathscr{I}(\boldsymbol{P}) := (P, L, \mathrm{I})$, *where* I *is defined by* $a\, \mathrm{I}\, b :\Leftrightarrow$ *there is a* $c \in P$ *such that* $(a, c) \in \text{dom}\,(\cdot) \;\&\; a \cdot c = b$. *We have then* $\cdot(\mathscr{I}(\boldsymbol{P})) = \cdot$ *and* $\cap(\mathscr{I}(\boldsymbol{P})) \subseteqq \cap$.

Proof. a) From the definition of $\cdot(\boldsymbol{I})$ and from $\mathrm{P}\, 1_{\leqq}$, it follows PS 1. Similarly it follows from the definitions of $\cdot(\boldsymbol{I})$ and $\cap(\boldsymbol{I})$ and from $\mathrm{P}2_{\leqq}$ that PS 2 is valid. If some $b \in L$ satisfied card $\{x \in P \mid x\, \mathrm{I}\, b\} > 1$ then $a\, \mathrm{I}\, b$ holds iff there is a $c \in P$ with $(a, c) \in \text{dom}(\cdot(\boldsymbol{I})) \;\&\; a \cdot(\boldsymbol{I})\, c = b$, as it follows at once from the definition of $\cdot(\boldsymbol{I})$.

b) Let $a, b \in P \;\&\; c, d \in L \;\&\; a, b\, \mathrm{I}\, c \;\&\; a, b\, \mathrm{I}\, d$. Then there exist elements $a_c, b_c, a_d, b_d \in P$ with $(a, a_c), (b, b_c), (a, a_d), (b, d_d) \in \text{dom}\,(\cdot) \;\&\; a \cdot a_c = b \cdot b_c = c \;\&\; a \cdot a_d = b \cdot b_d = d$. Thus for $a \neq b$ we obtain by PS 1 $(a, b) \in \text{dom}\,(\cdot) \;\&\; c = a \cdot b = d$. This may be expressed also as $c \neq d \Rightarrow a = b$ so that $\mathrm{P}\, 1_{\leqq}$ and $\mathrm{P}\, 2_{\leqq}$ hold. Further verify that $\cdot(\mathscr{I}(\boldsymbol{P})) = \cdot$ and $\cap(\mathscr{I}(\boldsymbol{P})) \subseteqq \cap$. In fact, if $(a, b) \in \text{dom}\, \cdot(\mathscr{I}(\boldsymbol{P}))$ then there exist $a', b' \in P$ such that $(a, a'), (b, b') \in \text{dom}\,(\cdot) \;\&\; a \cdot a' = b \cdot b'$. Thus for $a \neq b$, PS 1 gives $(a, b) \in \text{dom}\,(\cdot) \;\&\; a \cdot (\mathscr{I}(\boldsymbol{P}))\, b = a \cdot b$. Conversely, if $(a, b) \in \text{dom}\,(\cdot)$, then by PS 1, $a, b\, \mathrm{I}\, a \cdot b$ implies $(a, b) \in$

(4) This theorem is taken from [1], pp. 5—6. We reconstruct the proof here because without it the background of further investigations would not be clear.

dom $\cdot(\mathscr{I}(\boldsymbol{P}))$ & $a \cdot b = a \cdot(\mathscr{I}(\boldsymbol{P}))\, b$. Finally, if $(a, b) \in$ dom $\cap(\mathscr{I}(\boldsymbol{P}))$, then there exist $a_1, b_1 \in P$ such that $(a \cap(\mathscr{I}(\boldsymbol{P})\, b), a_1)$, $(a \cap(\mathscr{I}(\boldsymbol{P})\, b,)\, b_1) \in$ dom $(\cdot)$ & $(a \cap(\mathscr{I}(\boldsymbol{P}))\, b) \cdot a_1 = a$ & $(a \cap(\mathscr{I}(\boldsymbol{P}))\, b) \cdot b_1 = b$ so that PS 2 gives $(a, b) \in$ $\in$ dom $(\cap)$ & $a \cap(\mathscr{I}(\boldsymbol{P}))\, b = a \cap b$. Q.E.D.

Remark. $\mathscr{I}(\mathscr{P}(\boldsymbol{I}))$ can differ from $\boldsymbol{I}$ and also $\mathscr{P}(\mathscr{I}(\boldsymbol{P}))$ can differ from $\boldsymbol{P}$.

Corollary. *If $\boldsymbol{I} = (S, \iota)$ is an incidence presystem with symmetric ι, then the pairing system $\mathscr{G}(\boldsymbol{I}) := \mathscr{P}(\mathscr{I}(\boldsymbol{I}, \mathrm{id}_S, \mathrm{id}_S))$*[5] *is a geometric halfgroupoid. Conversely, if $\boldsymbol{G} = (S, \cdot)$ is a geometric halfgroupoid, then $\mathfrak{I}(\boldsymbol{G}) := \mathfrak{I}(\mathscr{I}(\boldsymbol{G}), \mathrm{id}_S, \mathrm{id}_S) =: (S, \iota)$ is an incidence presystem with symmetrical ι.*

The proof is easy.

Proposition 4. *Let $\boldsymbol{G} = (S, \cdot)$ be a geometric halfgroupoid and $\mathfrak{I}(\boldsymbol{G}) =: (S, \iota)$., Then ι is irreflexive iff $a \cdot b \neq a$ for all $(a, b) \in$ dom $(\cdot)$.*

Proof. Because of the mutual relation between ι and $\cdot$, we have $a \,\iota\, a \Leftrightarrow$ there exists $b \in S$ such that $(a, b) \in$ dom $(\cdot)$ & $a \cdot b = a$. From this we obtain the required result. Q.E.D.

Proposition 5a. *If $\boldsymbol{G} = (S, \cdot)$ is a geometric halfgroupoid with $a \cdot b \neq a$ for all $(a, b) \in$ dom $(\cdot)$ then the following conditions hold:*

(i) $(a, b), (a, a \cdot b) \in$ dom $(\cdot) \Rightarrow (a, a \cdot (a \cdot b)) \in$ dom $(\cdot)$ & $a \cdot (a \cdot (a \cdot b)) = a \cdot b$,

(ii) $(a, b), (a \cdot b, a) \in$ dom $(\cdot) \Rightarrow (a \cdot b, (a \cdot b) \cdot a) \in$ dom $(\cdot)$ & $(a \cdot b) \cdot ((a . b) \cdot a) = a$,

(iii) $(a, c), (a, a \cdot c), (b, c), (b, b \cdot c) \in$ dom $(\cdot)$ & $a \cdot (a \cdot c) = b \cdot (b \cdot c) = c$ & $a \neq b \Rightarrow (a, b) \in$ dom $(\cdot)$ & $a \cdot b = c$.

Proof. Put $(S, \iota) := \mathfrak{I}(\boldsymbol{G})$. Then for $(a, b), (a, a \cdot b) \in$ dom $(\cdot)$; it follows $a, a \cdot (a \cdot b)\, \iota\, a \cdot b$ & $a \neq a \cdot (a \cdot b)$ so that PS 1 gives $(a, a \cdot (a \cdot b)) \in$ $\in$ dom $(\cdot)$ & $a \cdot (a \cdot (a \cdot b) = a \cdot b$. Further for $(a, b), (a \cdot b, a) \in$ dom $(\cdot)$ we have $a \cdot b \neq (a \cdot b) \cdot a$ so that PS 2* implies $(a \cdot b, (a \cdot b) \cdot a) \in$ dom $(\cdot)$ & $(a \cdot b) \cdot ((a \cdot b) \cdot a) = a$. Finally for $(a, c), (a, a \cdot c), (b, c), (b, b \cdot c) \in$ dom $(\cdot)$ & $a \cdot (a \cdot c) = c$ & $b \cdot (b \cdot c) = c$ we have $a, b\, \iota\, c$. If also $a \neq b$, then PS 1 gives $(a, b) \in$ dom $(\cdot)$ & $a \cdot b = c$. Q.E.D.

Proposition 5b. *Let $\boldsymbol{G} = (S, \cdot)$ be a halfgroupoid*[6] *satisfying the following conditions*

$$(1) \qquad a \in S \Rightarrow (a, a) \notin \text{dom}\,(\cdot)$$
$$(2) \qquad (a, b) \in \text{dom}\,(\cdot) \Rightarrow (b, a) \in \text{dom}\,(\cdot) \;\&\; a \cdot b = b \cdot a,$$

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[5] id_S denotes identity mapping on the set S.

[6] A *halfgroupoid* is a couple $(S,.)$ where S is a set and . is a binary halfoperation in S , i. e. a map of some subset of $S \times S$ into S.

(3) $(a, b) \in \mathrm{dom}\,(\cdot) \Rightarrow (a, a \cdot b) \in \mathrm{dom}\,(\cdot)\ (\underset{\text{by}(1)}{\Rightarrow} a \neq a \cdot b),$

(4) $(a, b) \in \mathrm{dom}\,(\cdot) \Rightarrow a \cdot (a \cdot (a \cdot b)) = a \cdot b,$

(5) $(a, c), (b, c) \in \mathrm{dom}\,(\cdot)\ \&\ a \neq b\ \&\ a \cdot (a \cdot c) = b \cdot (b \cdot c) = c$
$\Rightarrow (a, b) \in \mathrm{dom}\,(\cdot)\ \&\ a \cdot b = c.$

*Then **G** satisfies PS* 1.
Proof. Let $(a, b), (c, d) \in \mathrm{dom}\,(\cdot)$, $a \cdot b = c \cdot d = e$, $a \neq c$. Then, by (3) and (4), $a \cdot (a \cdot (a \cdot b) = a \cdot b$, $c \cdot (c \cdot (c \cdot d)) = c \cdot d$, so that, by (5), $a \cdot c = c \cdot d$. According to (2) this expresses the same as PS 1. Q.E.D.

Proposition 5c. *Let* $\boldsymbol{G} = (S, \cdot)$ *be a halfgroupoid satisfying* (1), (2), (3), (5) *and*

(6) $(a, b) \in \mathrm{dom}\,(\cdot) \Rightarrow (a \cdot b) \cdot ((a \cdot b) \cdot a) = a.$

*Then **G** satisfies* PS 2.
Proof. For $(a, b), (a, c) \in \mathrm{dom}\,(\cdot)$ we have by (6) $(a \cdot b) \cdot ((a \cdot b) \cdot a) = (a \cdot c) \cdot ((a \cdot c) \cdot a) = a$ so that the assumption $a \cdot b \neq a \cdot c$ gives by (5) the required equality $(a \cdot b) \cdot (a \cdot c) = a$. *Q.E.D.*

Remark. In [2] it is proved in another formulation that a halfgroupoid $\boldsymbol{G} = (S, \cdot)$ satisfying (1) to (6) and $\mathrm{dom}\,(\cdot) = (S \times S) \setminus \mathrm{diag}\,(S \times S)$ is geometric and that $\mathfrak{J}(\boldsymbol{G})$ satisfies $P\,1'_{=}$, $P\,2'_{=}$.

§2 *Affine preplanes.*

Define an *affine plane* as an incidence system (P, L, I) such that

AP 1 there exists $\{a, b, c\} \subseteqq P$ such that $x \in L \Rightarrow$ non $(a, b, c\ \mathrm{I}\ x)$,

AP 2 $\{a, b\} \subseteqq P \Rightarrow$ there is exactly one $c \in L$ such that $a, b\ \mathrm{I}\ c$,

AP 3 $(a, b) \in P \times L\ \&\ a$ non $\mathrm{I}\ b \Rightarrow$ there is exactly one $c \in L$ such that

$$a\ \mathrm{I}\ c\ \&\ \{x \in P \mid x\ \mathrm{I}\ b, c\} = \emptyset.$$

Further define an *affine preplane* as an incidence presystem $(S \times S, \iota)$ such that

AP 1′ there is $\{a, b, c\} \subseteqq S \times S$ such that $x \in S \times S \Rightarrow$ non $(a, b, c\ \iota\ x)$,

AP 2′ $(a_1, b_1), (a_2, b_2) \in S \times S\ \&\ a_1 \neq a_2 \Rightarrow$ there is exactly one $(u, v) \in S \times S$ such that $(a_1, b_1), (a_2, b_2)\ \iota\ (u, v)$,

AP 3′ $(a, b), (c, d) \in S \times S\ \&\ (a, b)$ non $\iota\,(c, d) \Rightarrow$ there is exactly one $(u, v) \in S \times S$ such that $(a, b)\ \iota (u, v)\ \&\ \{x \in S \times S \mid x\ \iota\,(c, d)\ (u, v)\} = \emptyset$,

AP 4′ $a, c, d \in S \Rightarrow$ there is exactly one $y \in S$ such that $(a, y)\ \boldsymbol{I}\ (c, d)$.

Proposition 6. a) *Let* $\boldsymbol{A} = (P, L, \mathrm{I})$ *be an affine plane, Y a „pencil of parallel lines“*(7) *in it, S a set bijective to Y and* $\mu\colon Y \to S$, $\pi\colon P \to S \times S$, $\lambda\colon L \setminus Y \to S \times S$ *bijections such that* $\pi^{-1}(x, y)\ \mathrm{I}\ \mu^{-1}x$ *for all* $x, y \in S$. *Define a binary*

(7) i.e., a set consisting of all lines parallel to a given line in $\boldsymbol{A}$.

relation ι *in* $S \times S$ *by* $\pi p \, \iota \, \lambda \, l : \Leftrightarrow p \, \mathrm{I} \, l$. *Then* $\mathbf{A} = (S \times S, \iota)$ *is an affine preplane.*

b) *Let* $\mathbf{A} = (S \times S, \iota)$ *be an affine preplane,* Y *a set bijective to* S, P *and* L^* *sets bijective to* $S \times S$, *disjoint to Y *and* $\eta \colon Y \to S$, $\pi \colon P \to S \times S$, $\lambda \colon L^* \to S \times S$ *bijections. Define a binary relation* I *between* P *and* $L^* \cup Y$ *by* $p \, \mathrm{I} \, l \colon - \pi p \, \iota \, \lambda l - \mathrm{pr}_1 \, \pi p = \eta l$.(8) *Then* $\boldsymbol{A} = (P, L^* \cup Y, \mathrm{I})$ *is an affine plane.* The proof is easy.

Remark. If $\mathbf{A} = (S \times S, \iota)$ is an affine preplane, then define a ternary operation τ on S by $\tau(a, c, d) = b : \Leftrightarrow (a, b)\iota(c, d)$. Conditions AP1′—3′ for ι can be rewritten as the following conditions for τ:

AP 1″ there is $\{(a_1, a_2), (b_1, b_2), (c_1, c_2)\} \subseteqq S \times S$ such that $(u, v) \in S \times S \Rightarrow \Rightarrow \text{non}\, (\tau(a_1, u, v) = a_2 \;\&\; \tau(b_1, u, v) = b_2 \;\&\; \tau(c_1, u, v) = c_2)$,

AP 2″ $(a_1, b_1), (a_2, b_2) \in S \times S \;\&\; a_1 \neq a_2 \Rightarrow$ there is exactly one $(u, v) \in S \times S$ such that $\tau(a_i, u, v) = b_i$ for $i = 1, 2$,

AP 3″ $a, b, c, d \in S \;\&\; \tau(a, c, d) \neq b \Rightarrow$ there is exactly one $(u, v) \in S$ such that $\tau(a, u, v) = b \;\&\; \{(x, y) \mid \tau(x, c, d) = y \;\&\; \tau(x, u, v) = y\} = \emptyset$.

The ternary groupoid (S, τ) satisfying AP 1″—3″ is a natural generalization of the well-known Hall's ternary ring of an affine plane.

Proposition 7. *Let* $\mathbf{A} = (S \times S, \iota)$ *be an affine preplane where* $\boldsymbol{T} = (S, +, \cdot)$ *is a skew-field and* $(a, b) \, \iota \, (c, d) : \Leftrightarrow a \cdot c + d = b$. *If* $\boldsymbol{T}$ *has a characteristic* $\neq 2$, *then* ι *is not symmetric. If* $\boldsymbol{T}$ *is a field of a characteristic* 2, *then* ι *is symmetric.*

Proof. $(a, b) \, \iota \, (c, d) \Leftrightarrow (c, d) \, \iota \, (a, b)$ can be written as $a \cdot c + d = b \Leftrightarrow \Leftrightarrow c \cdot a + b = d$. The last equivalence is valid iff $a \cdot c = - c \cdot a$, which implies in case of a characteristic $\neq 2$ that zero divisors exist in T. This contradiction confirms that ι is not symmetric. If $\boldsymbol{T}$ is a field with a characteristic 2 then $x \cdot y = y \cdot x = -y \cdot x$ for all $x, y \in S$ so that ι must be symmetric. Q. E. D.

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(8) $\mathrm{pr}_1(x, y)$: x