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ON PERIODIC SEMIGROUPS WITH ONE-SIDED IDENTITIES

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In the papers [5] and [6] the structure of semigroups containing invertible elements and totally maximal elements is studied. The purpose of this paper is to study the results of [5] in the case of periodic semigroups, and mutual relation of invertible elements and of totally maximal elements in periodic semigroups.

In the first place we give some known notions and statements we are going to use.

An element a of a semigroup S is called totally maximal if $SaS = S$.

An element a of a semigroup S is called the left (right, twosided) invertible element of S if $Sa = S(aS = S, aSa = S)$. Further, $\mathscr{K}$ will denote the set of all elements of a semigroup S which are neither left nor right invertible, $\mathscr{L}$ the set of all elements of S which are left invertible but not right invertible, $\mathscr{R}$ the set of all elements of S which are right invertible but not left invertible, and finally, $\mathscr{G}$ will denote the set of all elements of S which are both left and right in vertible. In [4] it is proved that each of the sets $\mathscr{K}$, $\mathscr{L}$, $\mathscr{R}$, $\mathscr{G}$, is a subsemigroup of the semigroup S. From [1] it is known that every subsemigroup of a periodic semigroup S is periodic. Consequently each of the sets, $\mathscr{K}$, $\mathscr{L}$, $\mathscr{R}$, $\mathscr{G}$, is a periodic subsemigroup of the periodic semigroup S.

Let us denote by L^* the maximal proper left ideal of a semigroup S, which contains every proper left ideal of the se migroup S. The maximal proper right ideal R^* and the maximal proper two-sided ideal M^* are defined similarly.

Lemma 1 ([5]). *If in a semigroup S there exists at least one left invertible element, then S contains the maximal proper left ideal L^* and a complement of this ideal is the set of all left invertible elements of S, hence*

$$S = L^* \cup \mathscr{L} \cup \mathscr{G}.$$

Remark 1. A similar statement holds in the case if S contains at least one right invertible element.

Lemma 2 ([2]). *Let S be a periodic semigroup. If the ideal $L^*(R^*)$ exists in S, then in S there exists also M^* and $M^* = L^*(M^* = R^*)$.*

Theorem 1. *If a periodic semigroup S contains at least one left (right) invertible element, then S contains at least one right (left) identity.*

Proof. If $\mathscr{G} \neq \emptyset$, then from [4] it follows that $\mathscr{G}$ is a subgroup of S and its identity is the identity of the whole semigroup S. Let $a \in \mathscr{L}$. Then $Sa = S$. Since $\mathscr{L}$ is a periodic subsemigroup of the semigroup S, then $\mathscr{L}$ contains at least one idempotent. Let $e_1 \in \mathscr{L}$. Then evidently $Se_1 = S$. Let $x \in S$ be an arbitrary element. $Se_1 = S$ implies $ye_1 = x$ for some $y \in S$. Hence $xe_1 = (ye_1)e_1 = ye_1 = ye_1 = x$. This means that e_1 is a right identity of S. Analogously we can prove the case where S contains at least one right invertible element.

Theorem 2. *Let S be a periodic semigroup. Then only one subsemigroup from $\mathscr{L}$, $\mathscr{R}$. $\mathscr{G}$ can be non-empty.*

Proof. Let $\mathscr{L} \neq \emptyset$ and $\mathscr{R} \neq \emptyset$. Then from Theorem 1 it follows that S contains at least one right identity e_1, $e_1 \in \mathscr{L}$, and at least one left identity e_2, $e_2 \in \mathscr{R}$. Then $e_2e_1 = e_2$, since e_1 is a right identity of S. But at the same time $e_2e_1 = e_1$, since e_2 is a left unit of S. From this we have $e_1 = e_2$. But it is impossible, since $e_1 \in \mathscr{L}$, $e_2 \in \mathscr{R}$ and $\mathscr{L} \cap \mathscr{R} = \emptyset$. The other possibilities are proved analogously.

Corollary. *If S is periodic semigroup, then only the following cases are possible:* 1. $S = \mathscr{K} \cup \mathscr{L}$, 2. $S = \mathscr{K} \cup \mathscr{R}$, 3. $S = \mathscr{K} \cup \mathscr{G}$, 4. $S = \mathscr{K}$, 5. $S = \mathscr{L}$, 6. $S = \mathscr{R}$, 7. $S = \mathscr{G}$.

We say that a semigroup S is left simple (right simple, simple) if S contains no proper left (right, two-sided) ideal of the semigroup S distinct from a void set and the whole semigroup S. From [7] it is known that a semigroup S is left simple (right simple) if and only if, for every element $a \in S$, $Sa = S(aS = S)$ holds.

An element a of a semigroup S is called the right (left) increasing element of S, if in S there exists such a proper subset $S'(S'')$ that $S'a = S(aS'' = S)$.

Theorem 3. *Let S be a periodic semigroup. Then the subsemigroup $\mathscr{L}$ ($\mathscr{R}$) is the left simple (right simple) subsemigroup of S.*

Proof. We prove our statement only for $\mathscr{L}$. Only the following two cases are possible: 1. $S = \mathscr{L}$, 2. $S = \mathscr{K} \cup \mathscr{L}$. In the first case our statement is evident. Let $S = \mathscr{K} \cup \mathscr{L}$. Then from Lemma 1 it follows that $S = L^* \cup \mathscr{L}$ and for any element $a \in \mathscr{L}$ we have $Sa = S$, hence, $(L^* \cup \mathscr{L})a = L^* \cup \mathscr{L}$. It means that

$$(*) \qquad L^*a \cup \mathscr{L}a = L^* \cup \mathscr{L}.$$

It is easy to show that L^*a is a left ideal. Now, only two cases are possible: either $L^*a \subset L^*$, or $L^*a = S$. If the second case holds, then it means that the element a is the right increasing element of S. But in [4] it is proved that a periodic semigroup contains no increasing elements. Hence the first case must hold, namely, $L^*a \subset L^*$. But from the relation (*) it follows that $\mathscr{L} \subset \mathscr{L}a$, therefore, $\mathscr{L} \subset \mathscr{L}^2$. But $\mathscr{L}$ is a semigroup, therefore $\mathscr{L}^2 \subset \mathscr{L}$. Hence we have $\mathscr{L} \subset \mathscr{L}a \subset \mathscr{L}^2 \subset \mathscr{L}$. The last relation imply that $\mathscr{L}a = \mathscr{L}$ and the proof is complete.

It is proved in [5] that if $a \in S$ is left invertible or right invertible, then a is totally maximal. The following example shows that in some semigroups there exist such elements which are totally maximal, but neither left invertible nor right invertible.

Example 1. Let us consider the semigroup generated by a set $\{e, a, b, c\}$ subject to the generating relations $ex = xe = x$ for any $x \in S$, $ab = c$, $ca = b$, $a^3 = e$. $Sa \subset S$, since the set of elements of Sa contains no words finishing by c. $aS \subset S$, since the set of elements of aS contains no words finishing with b. But $SaS = S$. Therefore, the element a is a totally maximal, but neither left invertible nor right invertible. However the element e is two-sided invertible.

Remark 2. Let us denote by $\mathscr{M}$ the set of all totally maximal elements of a semigroup S which are not invertible. Then, evidently, $\mathscr{M} \subset \mathscr{K}$.

The following theorem makes us acquainted with the situation in periodic semigroups, containing invertible elements.

Theorem 4. *Let S be a periodic semigroup, containing invertible elements. Then every totally maximal element is either left invertible or right invertible.*

Proof. Let $\mathscr{M} \neq \emptyset$. Corollary of Theorem 2 implies that only one of the following cases is possible: 1. $S = \mathscr{K} \cup \mathscr{L}$, 2. $S = \mathscr{K} \cup \mathscr{R}$, 3. $S = \mathscr{K} \cup \mathscr{G}$. We give the proof for the case $S = \mathscr{K} \cup \mathscr{M}$. Since, according to the assumption $\mathscr{M} \neq \emptyset$, then there exists at least one totally maximal element which is not invertible. In [6] it is proved that if S contains at least one totally maximal element, then S contains the maximal proper two-sided ideal M^* and the complement of this ideal is the set of all totally maximal elements. Hence, $S = M^* \cup \mathscr{M} \cup \mathscr{L}$. But from Lemma 1 it follows that S contains the maximal proper left ideal L^* and we have $S = L^* \cup \mathscr{L}$. On the other hand we have from Lemma 2 $L^* = M_1^*$ and this is a contradiction with the assumption that M^* is a maximal proper ideal of S, therefore, $\mathscr{M} = \emptyset$ and all totally maximal elements are left invertible.

We say that an element a of a periodic semigroup S belongs to the idempotent e if there exists such an integer $n \geqslant 1$ that $a^n = e$. Let us denote by K_α the set of all elements of S, belonging to the idempotent e_α. From [1] it is known that a periodic semigroup can be written as a union of disjoint K-classes, therefore $S = \bigcup_\alpha K_\alpha$.

We say that a group G_α is a maximal group belonging to the idempotent e_α if G_α contains e_α and if there exists no group $G' \neq G_\alpha$ such that $G_\alpha \subset G' \subset K_\alpha$.

Finally, we wish to remark on relation between K-classes of the periodic semigroup S and the subsemigroups $\mathscr{K}$, $\mathscr{L}$, $\mathscr{R}$, $\mathscr{G}$.

Theorem 5. *Each of the subsemigroups $\mathscr{K}$, $\mathscr{L}$, $\mathscr{R}$, $\mathscr{G}$ of a periodic semigroup S is a union of some K-classes of the semigroup S.*

Proof. In order to prove our statement it is sufficient to show that no K-class has a non-empty intersection with two subsemigroups from $\mathscr{K}$, $\mathscr{L}$, $\mathscr{R}$, $\mathscr{G}$. Let us assume that $K_\alpha \cap \mathscr{K} \neq \emptyset$ and at the same time $K_\alpha \cap \mathscr{L} \neq \emptyset$. Let $a \in K_\alpha \cap \mathscr{K}$. Since $a \in K_\alpha$, then the element a belongs to the idempotent e_α. But $a \in \mathscr{K}$. It means that $\{a, a^2, a^3, \dots\} \subset \mathscr{K}$, therefore $e_\alpha \in \mathscr{K}$. Let $b \in K_\alpha \cap \mathscr{L}$. The element b belongs to the idempotent e_α, too. And since $\{b, b^2, b^3, \dots\} \subset \mathscr{L}$, then $e_\alpha \in \mathscr{L}$. Finally we obtain that $e_\alpha \in \mathscr{L}$ and $e_\alpha \in \mathscr{K}$. But it is impossible, because $\mathscr{K} \cap \mathscr{L} = \emptyset$.

Remark 3. In [3] it is proved that a left (right) simple semigroup, having at least one idempotent is a union of disjoint isomorphic groups G_α. But $\mathscr{L}$ ($\mathscr{R}$) is a left (right) simple subsemigroup of a periodic semigroup S, therefore it is itself periodic and it contains at least one idempotent. From this we have:

Theorem 6. *Each of the subsemigroups $\mathscr{L}$, $\mathscr{R}$ of a periodic semigroup S is a union of disjoint isomorphic maximal groups G-.*

The proof follows from Theorem 5 and Remark 3.

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