Natalia I. Bezvitnaya
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WEAKLY IRREDUCIBLE SUBGROUPS OF $\text{Sp}(1, n + 1)$

Natalia I. Bezvitnaya

Abstract. Connected weakly irreducible not irreducible subgroups of $\text{Sp}(1, n + 1) \subset \text{SO}(4, 4n + 4)$ that satisfy a certain additional condition are classified. This will be used to classify connected holonomy groups of pseudo-hyper-Kählerian manifolds of index 4.

1. Introduction

The classification of connected holonomy groups of Riemannian manifolds is well known [4, 5, 6, 10]. A classification of holonomy groups of pseudo-Riemannian manifolds is an actual problem of differential geometry. Very recently were obtained classifications of connected holonomy groups of Lorentzian manifolds [3, 11, 9] and of pseudo-Kählerian manifolds of index 2 [7]. These groups are contained in $\text{SO}(1, n + 1)$ and $\text{U}(1, n + 1) \subset \text{SO}(2, 2n + 2)$, respectively. As the next step, we study connected holonomy groups contained in $\text{Sp}(1, n + 1) \subset \text{SO}(4, 4n + 4)$, i.e. holonomy groups of pseudo-hyper-Kählerian manifolds of index 4. By the Wu theorem [12] and the results of Berger for connected irreducible holonomy groups of pseudo-Riemannian manifolds [4], it is enough to consider only weakly irreducible not irreducible groups (each such group does not preserve any proper non-degenerate vector subspace of the tangent space, but preserves a degenerate subspace).

In the present paper we classify connected weakly irreducible not irreducible subgroups of $\text{Sp}(1, n + 1) \subset \text{SO}(4, 4n + 4)$ ($n \geq 1$) that satisfy a natural condition. The case $n = 0$ will be considered separately. We generalize the method of [8, 7]. Let $G \subset \text{Sp}(1, n + 1)$ be a weakly irreducible not irreducible subgroup and $\mathfrak{g} \subset \mathfrak{sp}(1, n + 1)$ the corresponding subalgebra. The results of [7] allow us to expect that if $\mathfrak{g}$ is the holonomy algebra, then $\mathfrak{g}$ contains a certain 3-dimensional ideal $\mathcal{B}$. We will prove this in another paper. Consider the action of $G$ on the space $\mathbb{H}^{1,n+1}$, then $G$ acts on the boundary of the quaternionic hyperbolic space, which is diffeomorphic to the $4n + 3$-dimensional sphere $S^{4n+3}$ and $G$ preserves a point of this space. We define a map $s_1 : S^{4n+3}\{\text{point}\} \to \mathbb{H}^n$ similar to the usual stereographic projection. Then any $f \in G$ defines the map $F(f) = s_1 \circ f \circ s_2 : \mathbb{H}^n \to \mathbb{H}^n$, where $s_2 : \mathbb{H}^n \to S^{4n+3}\{\text{point}\}$ is the inverse of the usual stereographic projection restricted to $\mathbb{H}^n \subset \mathbb{H}^n \oplus \mathbb{R}^3 = \mathbb{R}^{4n+3}$. We get that $F(G)$ is contained in the group

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Sim $\mathbb{H}^n$ of similarity transformations of $\mathbb{H}^n$. We show that $F(G)$ preserves an affine subspace $L \subset \mathbb{R}^{4n} = \mathbb{H}^n$ such that the minimal affine subspace of $\mathbb{H}^n$ containing $L$ is $\mathbb{H}^n$. Moreover, $F(G)$ does not preserve any proper affine subspace of $L$. Then $F(G)$ acts transitively on $L$. We describe subspaces $L$ with this property and using results of [7] we find all connected Lie subgroups $K \subset \text{Sim } \mathbb{H}^n$ preserving $L$ and acting transitively on $L$. Note that the kernel of the Lie algebra homomorphism $dF : g \to \mathcal{L}A(\text{Sim } \mathbb{H}^n)$ coincides with the ideal $\mathcal{B}$. Consequently, $g = (dF)^{-1}(\mathcal{B})$, where $\mathcal{B} \subset \mathcal{L}A(\text{Sim } \mathbb{H}^n)$ is the Lie algebra of one of the obtained Lie subgroups $K \subset \text{Sim } \mathbb{H}^n$.

Note that we classify weakly irreducible not irreducible subgroups of $\text{Sp}(1, n + 1)$ up to conjugacy in $\text{SO}(4, 4n + 4)$. It is also possible to classify these subgroups up to conjugacy in $\text{Sp}(1, n + 1)$, see Remark [1].

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2. Preliminaries

First we summarize some facts about quaternionic vector spaces. Let $\mathbb{H}^m$ be an $m$-dimensional quaternionic vector space and $e_1, \ldots, e_m$ a basis of $\mathbb{H}^m$. We identify an element $X \in \mathbb{H}^m$ with the column $(X_t)$ of the left coordinates of $X$ with respect to this basis, $X = \sum_{t=1}^m X_t e_t$.

Let $f : \mathbb{H}^m \to \mathbb{H}^m$ be an $\mathbb{H}$-linear map. Define the matrix $\text{Mat}_f$ of $f$ by the relation $f e_i = \sum_{t=1}^m (\text{Mat}_f)_{it} e_t$. Now if $X \in \mathbb{H}^m$, then $fX = (X^t \text{Mat}_f^t)^t$ and because of the non-commutativity of the quaternions this is not the same as $\text{Mat}_f X$.

Conversely, to an $m \times m$ matrix $A$ of the quaternions we put in correspondence the linear map $\text{Op}_A : \mathbb{H}^m \to \mathbb{H}^m$ such that $\text{Op}_A \cdot X = (X^t A^t)^t$. If $f, g : \mathbb{H}^m \to \mathbb{H}^m$ are two $\mathbb{H}$-linear maps, then $\text{Mat}_{fg} = (\text{Mat}_g \text{Mat}_f)^t$. Note that the multiplications by the imaginary quaternions are not $\mathbb{H}$-linear maps. Also, for $a, b \in \mathbb{H}$ holds $\overline{ab} = \overline{a} \overline{b}$. Consequently, for two square quaternionic matrices we have $(\overline{AB})^t = B^t \overline{A}^t$.

A pseudo-quaternionic-Hermitian metric $g$ on $\mathbb{H}^m$ is a non-degenerate $\mathbb{R}$-bilinear map $g : \mathbb{H}^m \times \mathbb{H}^m \to \mathbb{H}$ such that $g(aX, Y) = ag(X, Y)$ and $g(Y, X) = g(X, Y)$, where $a \in \mathbb{H}$, $X, Y \in \mathbb{H}^m$. Hence, $g(X, aY) = g(X, Y)a$. There exists a basis $e_1, \ldots, e_m$ of $\mathbb{H}^m$ and integers $(r, s)$ with $r + s = m$ such that $g(e_t, e_l) = 0$ if $t \neq l$, $g(e_t, e_l) = -1$ if $1 \leq t \leq s$ and $g(e_t, e_l) = 1$ if $s + 1 \leq t \leq m$. The pair $(r, s)$ is called the signature of $g$. In this situation we denote $\mathbb{H}^m$ by $\mathbb{H}^{r:s}$. The realification of $\mathbb{H}^m$ gives us the vector space $\mathbb{R}^{4m}$ with the quaternionic structure $(i, j, k)$. Conversely, a quaternionic structure on $\mathbb{R}^{4m}$, i.e. a triple $(I, J, K)$ of endomorphisms of $\mathbb{R}^{4m}$ such that $I^2 = J^2 = K^2 = -\text{id}$ and $K = IJ = -JI$, allows us to consider $\mathbb{R}^{4m}$ as $\mathbb{H}^m$. A pseudo-quaternionic-Hermitian metric $g$ on $\mathbb{H}^m$ of signature $(r, s)$ defines on $\mathbb{R}^{4m}$ the $i, j, k$-invariant pseudo-Euclidean metric $\eta$ of signature $(4r, 4s)$, $\eta(X, Y) = \text{Re} g(X, Y)$, $X, Y \in \mathbb{R}^{4m}$. Conversely, a $I, J, K$-invariant pseudo-Euclidean metric on $\mathbb{R}^{4m}$ defines a pseudo-quaternionic-Hermitian metric $g$ on $\mathbb{H}^m$,

$$g(X, Y) = \eta(X, Y) + i\eta(X, JY) + j\eta(X, JY) + k\eta(X, KY).$$
The converse is not true, see Example 2 below. If weakly irreducible if it does not preserve any non-degenerate proper vector subspace of $\text{Sp}(r,s)$. 

Definition 1. A subgroup $G \subset \text{SO}(r,s)$ (or a subalgebra $g \subset \text{so}(r,s)$) is called weakly irreducible if it does not preserve any non-degenerate proper vector subspace of $\mathbb{R}^{r,s}$.

Let $\mathbb{R}^{4,4n+4}$ be a $(4n+8)$-dimensional real vector space endowed with a quaternionic structure $I, J, K \in \text{End}(\mathbb{R}^{4,4n+4})$ and an $I, J, K$-invariant metric $\eta$ of signature $(4, 4n+4)$. We identify this space with the $(n+2)$-dimensional quaternionic space $\mathbb{H}^{1,n+1}$ endowed with the pseudo-quaternionic-Hermitian metric $g$ of signature $(1, n+1)$ as above.

Obviously, if a Lie subgroup $G \subset \text{Sp}(1, n+1)$ acts weakly irreducibly not irreducibly on $\mathbb{R}^{4,4n+4}$, then $G$ acts weakly irreducibly not irreducibly on $\mathbb{H}^{1,n+1}$. The converse is not true, see Example 2 below. If $G$ acts weakly irreducibly not irreducibly on $\mathbb{H}^{1,n+1}$, then $G$ preserves a proper degenerate subspace $W \subset \mathbb{H}^{1,n+1}$. Consequently, $G$ preserves the intersection $W \cap W^\perp \subset \mathbb{H}^{1,n+1}$, which is an isotropic quaternionic line.

Fix a Wit basis $p, e_1, \ldots, e_n, q$ of $\mathbb{H}^{1,n+1}$, i.e. the Gram matrix of the metric $g$ with respect to this basis has the form $\begin{pmatrix} 0 & 0 & 1 \\ 0 & E_n & 0 \\ 1 & 0 & 0 \end{pmatrix}$, where $E_n$ is the $n$-dimensional identity matrix. Denote by $\text{Sp}(1, n+1)_{\mathbb{H}p}$ the Lie subgroup of $\text{Sp}(1, n+1)$ acting on $\mathbb{H}^{1,n+1}$ and preserving the quaternionic isotropic line $\mathbb{H}p$. Note that any weakly irreducible and not irreducible subgroup of $\text{Sp}(1, n+1)$ is conjugated to a weakly irreducible subgroup of $\text{Sp}(1, n+1)_{\mathbb{H}p}$. The Lie subalgebra $\text{sp}(1, n+1)_{\mathbb{H}p} \subset \text{sp}(1, n+1)$ corresponding to the Lie subgroup $\text{Sp}(1, n+1)_{\mathbb{H}p} \subset \text{Sp}(1, n+1)$ has the following form

$$\text{sp}(1, n+1)_{\mathbb{H}p} = \left\{ \text{Op} \begin{pmatrix} \bar{a} & -X^t & b \\ 0 & \text{Mat}_h & X \\ 0 & 0 & -a \end{pmatrix} \mid a \in \mathbb{H}, \quad X \in \mathbb{H}^n, \quad h \in \text{sp}(n), \quad b \in \text{Im} \mathbb{H} \right\}.$$ 

Let $(a, A, X, b)$ denote the above element of $\text{sp}(1, n+1)_{\mathbb{H}p}$. Define the following vector subspaces of $\text{sp}(1, n+1)_{\mathbb{H}p}$:

$$\mathcal{A}_1 = \{(a, 0, 0, 0) \mid a \in \mathbb{R}\}, \quad \mathcal{A}_2 = \{(a, 0, 0, 0) \mid a \in \text{Im} \mathbb{H}\},$$

$$\mathcal{N} = \{(0, 0, X, 0) \mid X \in \mathbb{H}^n\}, \quad \mathcal{B} = \{(0, 0, 0, b) \mid b \in \text{Im} \mathbb{H}\}.$$ 

Obviously, $\text{sp}(n)$ is a subalgebra of $\text{sp}(1, n+1)_{\mathbb{H}p}$ with the inclusion

$$h \in \text{sp}(n) \mapsto \text{Op} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \text{Mat}_h & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \text{sp}(1, n+1)_{\mathbb{H}p}.$$
We obtain that $A_1$ is a one-dimensional commutative subalgebra that commutes with $A_2$ and $sp(n)$, $A_2$ is a subalgebra isomorphic to $sp(1)$ and commuting with $sp(n)$, $B$ is a commutative ideal, which commutes with $sp(n)$ and $N$. Also,

\[
[(a, 0, 0, 0), (0, 0, X, b)] = (0, 0, aX, 2 \text{Im } ab),
\]

\[
[(0, 0, X, 0), (0, 0, Y, 0)] = (0, 0, 0, 2 \text{Im } g(X, Y)),
\]

\[
[(0, A, 0, 0), (0, 0, X, 0)] = (0, 0, (X^t A^t)^t, 0),
\]

where $a \in \mathbb{H}, X, Y \in \mathbb{H}^n$, $A = \text{Mat}_h$, $h \in sp(n)$, $b \in \text{Im H}$. Thus we have the decomposition

\[
sp(1, n+1)_{\mathbb{H}p} = (A_1 \oplus A_2 \oplus sp(n)) \ltimes (N + B) \cong (\mathbb{R} \oplus sp(1) \oplus sp(n)) \ltimes (\mathbb{H}^n + \mathbb{R}^3).
\]

Now consider two examples.

**Example 1.** The subalgebra $g = \{(0, 0, X, b) \mid X \in \mathbb{R}^n, b \in \text{Im H}\} \subset sp(1, n+1)_{\mathbb{H}p}$ acts weakly irreducibly on $\mathbb{R}^{4,4n+4}$.

**Proof.** Assume the converse. Let $g$ preserve a non-degenerate proper vector subspace $L \subset \mathbb{R}^{4,4n+4}$. Suppose the projection of $L$ to $\mathbb{H}q \subset \mathbb{H}^{1,n+1} = \mathbb{R}^{4,4n+4}$ is non-zero, then there is a vector $v \in L$ such that $v = v_0 p + v_1 + v_2 q$, where $v_0, v_2 \in \mathbb{H}$, $v_2 \neq 0$ and $v_1 \in \mathbb{H}^n$. Consider elements $\xi_1 = (0, 0, X, 0) \in g$ with $g(X, X) = 1$ and $\xi_2 = (0, 0, 0, b) \in g$. Then, $\xi_1(\xi_1 v) = -v_2 p \in L$ and $\xi_2 v = v_2 bp \in L$. Since $v_2 \neq 0$, we have $\mathbb{H}p \subset L$. It follows that $L^\perp \subset \mathbb{H}p \oplus \mathbb{H}^n$ and $L^\perp$ is a $g$-invariant non-degenerate proper subspace. Now we can assume that $g$ preserves a non trivial non-degenerate vector subspace $L \subset \mathbb{H}p \oplus \mathbb{H}^n$. Let $v = v_0 p + v_1 \in L$, $v \neq 0$. If $v_1 = 0$, then $L$ is degenerate. If $v_1 \neq 0$, then there is $X \in \mathbb{R}^n$ with $g(v_1, X) \neq 0$. We get $(0, 0, X, 0)v = -g(v_1, X)p \in L$. Hence $L$ is degenerate. Thus we have a contradiction.

**Example 2.** The subalgebra $g = \{(0, 0, X, 0) \mid X \in \mathbb{R}^n\} \subset sp(1, n+1)_{\mathbb{H}p}$ acts weakly irreducibly on $\mathbb{H}^{1,n+1}$ and not weakly irreducibly on $\mathbb{R}^{4,4n+4}$.

**Proof.** The proof of the first statement is similar to the proof of Example 1. Clearly, the subalgebra $g$ preserves the non-degenerate vector subspace $\text{span}_{\mathbb{R}} \{p, e_1, \ldots, e_n, q\} \subset \mathbb{R}^{4,4n+4}$. The classification of the holonomy algebras contained in $u(1, n+1)$ gives us the following hypothesis: If $n \geq 1$ and $g \subset sp(1, n+1)_{\mathbb{H}p}$ is a holonomy algebra, then $g$ contains the ideal $B$. We will prove this hypothesis in an other paper.

In the following theorem we denote the real vector subspace $L \subset \mathbb{R}^{4n} = \mathbb{H}^n$ of the form

\[
L = \text{span}_{\mathbb{H}} \{e_1, \ldots, e_m\} \oplus \text{span}_{\mathbb{R}^{2m}} \{e_{m+1}, \ldots, e_{m+k}\} \oplus \text{span}_{\mathbb{R}} \{e_{m+k+1}, \ldots, e_n\}
\]

by $\mathbb{H}^m \oplus \mathbb{C}^k \oplus \mathbb{R}^{n-m-k}$. Let $u(k)$ be the subalgebra of $sp(\text{span}_{\mathbb{H}} \{e_{m+1}, \ldots, e_{m+k}\})$ that consists of the elements $Op\left(\begin{array}{cc} A & 0 \\ 0 & A \end{array}\right)$, where $A \in u(\text{span}_{\mathbb{R}^{2m}} \{e_{m+1}, \ldots, e_{m+k}\})$.
and we use the decomposition
\[
\text{span}_{\mathbb{H}} \{ e_{m+1}, \ldots, e_{m+k} \} = \text{span}_{\mathbb{R} \oplus i\mathbb{R}} \{ e_{m+1}, \ldots, e_{m+k} \} + j\text{span}_{\mathbb{R} \oplus i\mathbb{R}} \{ e_{m+1}, \ldots, e_{m+k} \}.
\]
Similarly, let \( \mathfrak{so}(n - m - k) \) be the subalgebra of \( \mathfrak{sp}(\text{span}_{\mathbb{R}} \{ e_{m+1}, \ldots, e_n \}) \) that consists of the elements
\[
\text{Op} \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{pmatrix}, \quad \text{where } A \in \mathfrak{so}(\text{span}_{\mathbb{R}} \{ e_{m+1}, \ldots, e_n \})
\]
and we use the decomposition \( H^{n-m-k} = \mathbb{R}^{n-m-k} \oplus i\mathbb{R}^{n-m-k} \oplus j\mathbb{R}^{n-m-k} \oplus k\mathbb{R}^{n-m-k} \). For a Lie algebra \( \mathfrak{h} \) we denote by \( \mathfrak{h}' \) the commutant \([\mathfrak{h}, \mathfrak{h}]\) of \( \mathfrak{h} \).

**Theorem 1.** Let \( n \geq 1 \). Any weakly irreducible subalgebra of \( \mathfrak{sp}(1, n+1)_{\mathbb{R}} \) that contains the ideal \( \mathcal{B} \) is conjugated by an element of \( \mathfrak{so}(4, 4n + 4) \) to one of the following subalgebras:

**Type I.** \( \mathfrak{g} = \{ (a_1 + a_2, A, X, b) \mid a_1, a_2 \in \mathbb{R}, A \in \mathfrak{h}, X \in \mathbb{H}^n, b \in \text{Im} \mathbb{H} \} \), where \( \mathfrak{h}_0 \subset \mathfrak{sp}(1) \) is a subalgebra of dimension 2 or 3, \( \mathfrak{h} \subset \mathfrak{sp}(n) \) is a subalgebra.

**Type II.** \( \mathfrak{g} = \{ (a_1 + ta_2 + \phi(A), A, X, b) \mid a_1, t \in \mathbb{R}, A \in \mathfrak{h}, X \in \mathbb{H}^n, b \in \text{Im} \mathbb{H} \} \), where \( a_2 \in \mathfrak{sp}(1), \mathfrak{h} \subset \mathfrak{sp}(n) \) is a subalgebra, \( \phi : \mathfrak{h} \to \mathfrak{sp}(1) \) is a homomorphism.

If \( a_2 \neq 0 \), then \( \text{rk} \phi \leq 1 \) and \([\text{Im} \phi, a_2] \subset \mathfrak{Ra}_2 \).

**Type III.** \( \mathfrak{g} = \{ (\varphi(a_2, A) + a_2, A, X, b) \mid a_2 \in \mathfrak{h}_0, A \in \mathfrak{h}, X \in \mathbb{H}^n, b \in \text{Im} \mathbb{H} \} \), where \( \mathfrak{h}_0 \subset \mathfrak{sp}(1) \) is a subalgebra of dimension 2 or 3, \( \mathfrak{h} \subset \mathfrak{sp}(n) \) is a subalgebra, \( \varphi \in \text{Hom}(\mathfrak{h}_0 \oplus \mathfrak{h}, \mathbb{R}), \varphi|_{\mathfrak{h}_0} = 0 \). In particular, if \( \text{dim} \mathfrak{h}_0 = 3 \), i.e. \( \mathfrak{h}_0 = \mathfrak{sp}(1) \), then \( \varphi|_{\mathfrak{h}_0} = 0 \).

**Type IV.** \( \mathfrak{g} = \{ (\varphi(t, A) + ta_2 + \phi(A), A, X, b) \mid t \in \mathbb{R}, A \in \mathfrak{h}, X \in \mathbb{H}^n, b \in \text{Im} \mathbb{H} \} \), where \( a_2 \in \mathfrak{sp}(1), \mathfrak{h} \subset \mathfrak{sp}(n) \) is a subalgebra, \( \varphi \in \text{Hom}(\mathfrak{R} \oplus \mathfrak{h}, \mathfrak{R}), \varphi|_{\mathfrak{h}'} = 0 \). \( \phi : \mathfrak{h} \to \mathfrak{sp}(1) \) is a homomorphism. If \( a_2 \neq 0 \), then \( \text{rk} \phi \leq 1 \) and \([\text{Im} \phi, a_2] \subset \mathfrak{Ra}_2 \). If \( a_2 \neq 0 \) and \( \phi \neq 0 \), then \( \varphi|_{\mathfrak{R}} = 0 \).

**Type V.** \( \mathfrak{g} = \{ (a_1 + a_2, A, X, b) \mid a_1, a_2 \in \mathbb{R}, A \in \mathfrak{h}, X \in \mathbb{H}^m \oplus \mathbb{C}^{n-m}, b \in \text{Im} \mathbb{H} \} \), where \( 0 \leq m < n, \mathfrak{h} \subset \mathfrak{sp}(m) \oplus \mathfrak{u}(n-m) \) is a subalgebra.

**Type VI.** \( \mathfrak{g} = \{ (a_1 + \phi(A)i, A, X, b) \mid a_1 \in \mathbb{R}, A \in \mathfrak{h}, X \in \mathbb{H}^m \oplus \mathbb{C}^k \oplus \mathbb{R}^{n-m-k}, b \in \text{Im} \mathbb{H} \} \), where \( 0 \leq m < n, 0 \leq k \leq n-m, \mathfrak{h} \subset \mathfrak{sp}(m) \oplus \mathfrak{u}(k) \oplus \mathfrak{so}(n-m-k) \) is a subalgebra, \( \phi \in \text{Hom}(\mathfrak{h}, \mathfrak{R}), \phi|_{\mathfrak{h}'} = 0 \). If \( n - m - k \geq 1 \), then \( \phi = 0 \).

**Type VII.** \( \mathfrak{g} = \{ (\varphi(a_2, A + a_2i, A, X, b) \mid a_2 \in \mathbb{R}, A \in \mathfrak{h}, X \in \mathbb{H}^m \oplus \mathbb{C}^{n-m}, b \in \text{Im} \mathbb{H} \} \), where \( 0 \leq m < n, \mathfrak{h} \subset \mathfrak{sp}(m) \oplus \mathfrak{u}(n-m) \) is a subalgebra, \( \phi \in \text{Hom}(\mathfrak{R} \oplus \mathfrak{h}, \mathfrak{R}), \phi|_{\mathfrak{h}'} = 0 \).

**Type VIII.** \( \mathfrak{g} = \{ (\varphi(A) + \phi(A)i, A, X, b) \mid A \in \mathfrak{h}, X \in \mathbb{H}^m \oplus \mathbb{C}^k \oplus \mathbb{R}^{n-m-k}, b \in \text{Im} \mathbb{H} \} \), where \( 0 \leq m < n, 0 \leq k \leq n-m, \mathfrak{h} \subset \mathfrak{sp}(m) \oplus \mathfrak{u}(k) \oplus \mathfrak{so}(n-m-k) \) is a subalgebra, \( \phi, \phi \in \text{Hom}(\mathfrak{h}, \mathfrak{R}), \phi|_{\mathfrak{h}'} = 0 \). If \( n - m - k \geq 1 \), then \( \phi = 0 \).
Thus we identify the space $LA$. The corresponding Lie algebra $H$ is given by the system of equations:

$$A \leftrightarrow a_1 Y \ (\text{real dilation}), \ a_2: Y \rightarrow a_2 Y \ (\text{quaternionic dilation}), \ f: Y \rightarrow fY \ (\text{rotation}), \ t(Y): Y \rightarrow Y + X \ (\text{translation}), \text{ here } Y \in H.$$

Let the elements $a_2 \in \text{Sp}(1)$ act on $H$ as $\mathbb{R}$-linear (but not $H$-linear) isomorphism. These transformations generate the Lie group $\text{Sim} H^n$ of similarity transformations of $H^n$. We get the decomposition

$$\text{Sim} H^n = (\mathbb{R} \times \text{Sp}(1) \cdot \text{Sp}(n)) \times H^n.$$ 

The Lie group $\text{Sim} H^n$ is a Lie subgroup of the connected Lie group $\text{Sim}^0 \mathbb{R}^{4n}$ of similarity transformations of $\mathbb{R}^{4n}$, $\text{Sim}^0 \mathbb{R}^{4n} = (\mathbb{R} \times \text{SO}(4n)) \times \mathbb{R}^{4n}$.

The corresponding Lie algebra $\mathcal{LA}(\text{Sim} H^n)$ to the Lie group $\text{Sim} H^n$ has the following decomposition

$$\mathcal{LA}(\text{Sim} H^n) = (\mathbb{R} \oplus \text{sp}(1) \oplus \text{sp}(n)) \times H^n.$$ 

Let $p, e_1, \ldots, e_n, q$ be the basis of $H_{1,n+1}^+$ as above. Consider also the basis $e_0, e_1, \ldots, e_n, e_{n+1}$, where $e_0 = \sqrt{2} (p - q)$ and $e_{n+1} = \sqrt{2} (p + q)$. With respect to this basis the Gram matrix of $g$ has the form

$$\begin{pmatrix}
-1 & 0 \\
0 & E_{n+1}
\end{pmatrix}.$$

The subset of the $(n+1)$-dimensional quaternionic projective space $\mathbb{P}H_{1,n+1}^+$ that consists of all quaternionic isotropic lines is called the boundary of the quaternionic hyperbolic space and is denoted by $\partial H_{1,n+1}^+$. Let $h_0, \ldots, h_{n+1}$, where $h_s = x_s + iy_s + jz_s + kw_s \in H \ (0 \leq s \leq n + 1)$ be the coordinates on $H_{1,n+1}^+$ with respect to the basis $e_0, \ldots, e_{n+1}$. Denote by $H^n$ and $H^{n+1}$ the subspaces of $H_{1,n+1}^+$ spanned by the vectors $e_1, \ldots, e_n$ and $e_1, \ldots, e_{n+1}$, respectively. Note that the intersection $(e_0 + H^{n+1}) \cap \{X \in H_{1,n+1}^+ | g(X, X) = 0\}$ is given by the system of equations:

$$x_0 = 1, \quad y_0 = 0, \quad z_0 = 0, \quad w_0 = 0,$$

$$x_1^2 + y_1^2 + z_1^2 + w_1^2 + \cdots + x_{n+1}^2 + y_{n+1}^2 + z_{n+1}^2 + w_{n+1}^2 = 1,$$

i.e., this set is the $(4n+3)$-dimensional unite sphere $S^{4n+3}$. Moreover, each isotropic line intersects this set at a unique point, e.g. $H^p$ intersects it at the point $\sqrt{2}p$. Thus we identify the space $\partial H_{1,n+1}^+$ with the sphere $S^{4n+3}$. Any $f \in \text{Sp}(1, n+1)_{\mathbb{H}p}$ takes quaternionic isotropic lines to quaternionic isotropic lines and preserves the quaternionic isotropic line $H_p$. Hence it acts on $\partial H_{1,n+1}^+ \setminus \{H_p\} = S^{4n+3} \setminus \{\sqrt{2}p\}$.

Consider the connected Lie subgroups $A_1, A_2, \text{Sp}(n)$ and $P$ of $\text{Sp}(1, n+1)_{\mathbb{H}p}$ corresponding to the subalgebras $A_1, A_2, \text{sp}(n)$ and $N + B$ of the Lie algebra $\text{Sp}(n)$.
We have the decomposition $a$ to the ideal $Sp$. Now we will show that $F$ consists of a single point. Let line passing through the points $s$ define projection, but using quaternionic lines. More precisely, for $s$ Let $A_1 = \{ \text{Op} \left( \begin{array}{ccc} a_1 & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & a_1^{-1} \end{array} \right) \mid a_1 \in \mathbb{R}_+ \}$, $A_2 = \{ \text{Op} \left( \begin{array}{ccc} e^{-a_2} & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & e^{-a_2} \end{array} \right) \mid a_2 \in \text{Im } \mathbb{H} \}$, $Sp(n) = \{ \text{Op} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & \text{Mat}_f & 0 \\ 0 & 0 & 1 \end{array} \right) \mid f \in Sp(n) \}$, $P = \{ \text{Op} \left( \begin{array}{ccc} 1 & -Y^t & b - \frac{1}{2} Y^t \bar{Y} \\ 0 & E_n & \bar{Y} \\ 0 & 0 & 1 \end{array} \right) \mid Y \in \mathbb{H}^n, b \in \text{Im } \mathbb{H} \}$. We have the decomposition $Sp(1, n + 1)_{\mathbb{H}_p} = (A_1 \times A_2 \times Sp(n)) \lhd P \simeq (\mathbb{R}_+ \times Sp(1) \times Sp(n)) \lhd (\mathbb{H}^n \cdot \mathbb{R}^3)$. Let $s_1: S^{4n+3} \setminus \{ \sqrt{2}p \} \to e_0 + \mathbb{H}^n$ be the map defined as the usual stereographic projection, but using quaternionic lines. More precisely, for $s \in S^{4n+3} \setminus \{ \sqrt{2}p \}$ we define $s_1(s)$ to be the point of the intersection of $e_0 + \mathbb{H}^n$ with the quaternionic line passing through the points $\sqrt{2}p$ and $s$. It is easy to see that this intersection consists of a single point. Let $s_2: e_0 + \mathbb{H}^n \to S^{4n+3} \setminus \{ \sqrt{2}p \}$ be the restriction to $e_0 + \mathbb{H}^n$ of the inverse to the usual stereographic projection from $S^{4n+3} \setminus \{ \sqrt{2}p \}$ to $e_0 + \mathbb{H}^n \oplus (\text{Im } \mathbb{H})e_{n+1}$. Note that $s_1 \circ s_2 = \text{id}_{e_0 + \mathbb{H}^n}$, but unlike in the usual case, $s_1$ is not surjective. We have $s_2 \circ s_1|_{\text{Im } s_2} = \text{id}_{\text{Im } s_2}$. Also, let $e_0$ and $-e_0$ denote the translations $\mathbb{H}^n \to e_0 + \mathbb{H}^n$ and $e_0 + \mathbb{H}^n \to \mathbb{H}^n$, respectively.

For $f \in Sp(1, n + 1)_{\mathbb{H}_p}$ define the map

$$F(f) = (-e_0) \circ s_1 \circ f \circ s_2 \circ e_0 : \mathbb{H}^n \to \mathbb{H}^n.$$ 

Now we will show that $F$ is a surjective homomorphism from the Lie group $Sp(1, n + 1)_{\mathbb{H}_p}$ to the Lie group $\text{Sim } \mathbb{H}^n$ and $\ker F = \mathbb{Z}_2 \times B$, where $\mathbb{Z}_2 = \{ \text{id}, - \text{id} \} \in Sp(1, n + 1)_{\mathbb{H}_p}$. $B$ is the connected Lie subgroup of $Sp(1, n + 1)_{\mathbb{H}_p}$ corresponding to the ideal $B \subset \mathfrak{sp}(1, n + 1)_{\mathbb{H}_p}$. First of all, the computations show that for $a_1 \in \mathbb{R}$, $a_2 \in \text{Im } \mathbb{H}$, $f \in Sp(n)$ and $Y \in \mathbb{H}^n$ it holds

$$F \left( \text{Op} \left( \begin{array}{ccc} a_1 & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & a_1^{-1} \end{array} \right) \right) = d(a_1) \in \mathbb{R}_+ \subset \text{Sim } \mathbb{H}^n,$$

$$F \left( \text{Op} \left( \begin{array}{ccc} e^{-a_2} & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & a^{-a_2} \end{array} \right) \right) = e^{a_2} \in Sp(1) \subset \text{Sim } \mathbb{H}^n.$$
We have
\[ F \left( \text{Op} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Mat}_f & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = f \in \text{Sp}(n) \subset \text{Sim} \mathbb{H}^n, \]
\[ F \left( \text{Op} \begin{pmatrix} 1 & -\tilde{Y}^t & b - \frac{1}{2} Y^t \tilde{Y} \\ 0 & E_n & \tilde{Y} \\ 0 & 0 & 1 \end{pmatrix} \right) = t \left( -\frac{\sqrt{2}}{2} Y \right) \in \mathbb{H}^n \subset \text{Sim} \mathbb{H}^n. \]

It follows that if \( f_1, f_2 \in P \), then \( F(f_1 f_2) = F(f_1) F(f_2) \), i.e. \( F|_P \) is a homomorphism from \( P \) to \( \text{Sim} \mathbb{H}^n \). It can easily be checked that any \( f \in A_1 \times A_2 \times \text{Sp}(n) \) considered as a map from \( S^{4n+3} \setminus \{ \sqrt{2}p \} \) to itself preserves \( \text{Im} s_2 \subset S^{4n+3} \setminus \{ \sqrt{2}p \} \). Hence if \( f_1 \) is from \( P \) or \( A_1 \times A_2 \times \text{Sp}(n) \) and \( f_2 \in A_1 \times A_2 \times \text{Sp}(n) \), then
\[ F(f_1 f_2) = (-e_0) \circ s_1 \circ f_1 \circ f_2 \circ s_2 \circ e_0 \]
\[ = (-e_0) \circ s_1 \circ f_1 \circ s_2 \circ e_0 \circ (-e_0) \circ s_1 \circ f_2 \circ s_2 \circ e_0 = F(f_1) F(f_2), \]

since \( s_2 \circ s_1 |_{\text{Im} s_2} = \text{id}_{\text{Im} s_2} \). Therefore it is enough to prove that \( F(f_1 f_2) = F(f_1) F(f_2) \), for \( f_1 \in A_1 \times A_2 \times \text{Sp}(n) \) and \( f_2 \in P \). Let
\[
f_1 = \text{Op} \begin{pmatrix} a_1 e^{-a_2} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a_1^{-1} e^{-a_2} \end{pmatrix} \in A_1 \times A_2 \times \text{Sp}(n),
\]
\[
f_2 = \text{Op} \begin{pmatrix} 1 & -\tilde{Y}^t & b - \frac{1}{2} Y^t \tilde{Y} \\ 0 & E_n & \tilde{Y} \\ 0 & 0 & 1 \end{pmatrix} \in P.
\]

Then \( f_1 f_2 f_1^{-1} = f'_2 \in P \), where
\[
f'_2 = \text{Op} \begin{pmatrix} 1 & -((A^{-1})^t \tilde{Y} a_1 e^{-a_2})^t & a_1^2 e^{a_2} (b - \frac{1}{2} Y^t \tilde{Y}) e^{-a_2} \\ 0 & E_n & a_1 e^{a_2} (\tilde{Y}^t A^t)^t \\ 0 & 0 & 1 \end{pmatrix}.
\]

We have
\[
F(f_1 f_2) = F(f'_2 f_1) = F(f'_2) F(f_1) = t \left( -\frac{\sqrt{2}}{2} a_1 e^{a_2} (Y^t A^t)^t \right) a_1 e^{a_2} \text{Op} A
\]
\[
= t \left( -\frac{\sqrt{2}}{2} a_1 e^{a_2} \text{Op} A \cdot Y \right) a_1 e^{a_2} \text{Op} A
\]
\[
= a_1 e^{a_2} \text{Op} A \cdot t \left( -\frac{\sqrt{2}}{2} Y \right) = F(f_1) F(f_2),
\]
since for any \( f \in \mathbb{R}^+ \times \text{SO}(4n) \) and \( X \in \mathbb{R}^{4n} \) it holds \( f t(X) f^{-1} = t(f X) \) or \( t(f X) f = f t(X) \). Thus \( F \) is the homomorphism from the Lie group \( \text{Sp}(1, n+1)_{\mathbb{H}_p} \) to the Lie group \( \text{Sim} \mathbb{H}^n \). Obviously, \( F \) is surjective. The claim is proved.

Let \( L \subset \mathbb{R}^{4n} \) be a vector (affine) subspace. We call the subset \( L \subset \mathbb{H}^n \) a real vector (affine) subspace.

**Theorem 2.** Let \( G \subset \text{Sp}(1, n+1)_{\mathbb{H}_p} \) act weakly irreducibly on \( \mathbb{H}^{1,n+1} \). Then if \( F(G) \subset \text{Sim} \mathbb{H}^n \) preserves a proper real affine subspace \( L \subset \mathbb{H}^n \), then the minimal affine subspace of \( \mathbb{H}^n \) containing \( L \) is \( \mathbb{H}^n \).
**Proof.** First we prove that the subgroup \( F(G) \subset \text{Sim} \mathbb{H}^n \) does not preserve any proper affine subspace of \( \mathbb{H}^n \). Assume that \( F(G) \) preserves a vector subspace \( L \subset \mathbb{H}^n \). Choosing the basis \( e_1, \ldots, e_n \) of \( \mathbb{H}^n \) in a proper way, we can suppose that \( L = \mathbb{H}^m = \text{span}_\mathbb{H}\{e_1, \ldots, e_m\} \). Consequently, \( F(G) \subset (\mathbb{R}_+ \times (\text{Sp}(1) \times (\text{Sp}(m) \times \text{Sp}(n-m))) ) \backslash \mathbb{H}^m \). Hence, \( G \subset (\mathbb{R}_+ \times \text{Sp}(1) \times \text{Sp}(m) \times \text{Sp}(n-m)) \backslash (\mathbb{H}^m, \mathbb{R}^3) \) and \( G \) preserves the non-degenerate vector subspace \( \text{span}_\mathbb{H}\{e_{m+1}, \ldots, e_n\} \subset \mathbb{H}^{1,n+1} \). Now suppose that \( F(G) \) preserves an affine subspace \( L \subset \mathbb{H}^n \). Let \( Y = L \cup L_0 \), where \( Y \subset L \) and \( L_0 \subset \mathbb{H}^n \) is the vector subspace corresponding to \( L \). We may assume that \( L_0 = \mathbb{H}^m = \text{span}_\mathbb{H}\{e_1, \ldots, e_m\} \). Consider \( f = \text{Op}(1 \sqrt{2}Y^t - Y^t \sqrt{2}Y, 0 E_n - \sqrt{2}Y, 0 0 1) \in P \) and the subgroup \( \tilde{G} = f^{-1}GF \subset \text{Sp}(1, n+1)_{\mathbb{H}^p} \). For \( F(\tilde{G}) \) we get that \( F(\tilde{G}) = -t(Y)F(G)t(Y) \). By the above \( \tilde{G} \) preserves the non-degenerate vector subspace \( \text{span}_\mathbb{H}\{e_{m+1}, \ldots, e_n\} \subset \mathbb{H}^{1,n+1} \). Hence \( G \) preserves the non-degenerate vector subspace \( f(\text{span}_\mathbb{H}\{e_{m+1}, \ldots, e_n\}) \subset \mathbb{H}^{1,n+1} \). Since \( G \) is weakly irreducible, we get \( m = n \).

Let \( F(G) \) preserve a real affine subspace \( L \subset \mathbb{H}^n \) and let \( L_0 \subset \mathbb{H}^n \) be the corresponding real vector subspace. Consider the vector subspace \( (\text{span}_\mathbb{H} L_0)^{\perp} \subset \mathbb{H}^n \). As above, it can be proved that \( G \) preserves the non-degenerate vector subspace \( f((\text{span}_\mathbb{H} L_0)^{\perp}) \subset \mathbb{H}^{1,n+1} \). Since \( G \) is weakly irreducible, we have \( (\text{span}_\mathbb{H} L_0)^{\perp} = 0 \) and \( \text{span}_\mathbb{H} L_0 = \mathbb{H}^n \). The theorem is proved.

5. Proof of the main theorem

First of all, from Example 1 it follows that the algebras of Types I–VIII act weakly irreducibly on \( \mathbb{R}^{4,4n+4} \). For the algebras of Type IX it can be proved in the same way. Therefore we must only prove that any subalgebra \( g \subset \text{sp}(1, n+1)_{\mathbb{H}^p} \) that acts weakly irreducibly on \( \mathbb{R}^{4,4n+4} \) and contains the ideal \( \mathcal{B} \) is conjugated (by an element from \( \text{SO}(4, 4n+4) \)) to one of the algebras of Types I–IX. Suppose that \( g \subset \text{sp}(1, n+1)_{\mathbb{H}^p} \) acts weakly irreducibly on \( \mathbb{R}^{4,4n+4} \) and contains the ideal \( \mathcal{B} \). Let \( G \subset \text{Sp}(1, n+1)_{\mathbb{H}^p} \) be the corresponding connected Lie subgroup. By Theorem 2 \( F(G) \) preserves a real affine subspace \( L \subset \mathbb{H}^n \) such that the minimal affine subspace of \( \mathbb{H}^n \) containing \( L \) is \( \mathbb{H}^n \). We already know that \( G \) is conjugated to a subgroup \( \tilde{G} \subset \text{Sp}(1, n+1)_{\mathbb{H}^p} \) such that \( F(\tilde{G}) \) preserves a real vector subspace \( L_0 \subset \mathbb{H}^n \) with \( \text{span}_\mathbb{H} L_0 = \mathbb{H}^n \). Hence we can assume that \( F(G) \) preserves a real vector subspace \( L \subset \mathbb{H}^n \) and \( \text{span}_\mathbb{H} L = \mathbb{H}^n \). Moreover, assume that \( F(G) \) does not preserve any proper affine subspace of \( L \). Then \( F(G) \) acts transitively on \( L \). The connected transitively acting groups of similarity transformations of the Euclidean spaces are well known. In [7] these groups were divided into three types. We describe real subspaces \( L \subset \mathbb{H}^n \) with \( \text{span}_\mathbb{H} L = \mathbb{H}^n \) and subalgebras \( \mathfrak{t} \subset \mathcal{L}(\text{Sim} \mathbb{H}^n) \) such that the corresponding connected Lie subgroups \( K \subset \text{Sim} \mathbb{H}^n \) preserve \( L \) and act transitively on \( L \). Then the algebra \( g \) must be of the form \( (dF)^{-1}(\mathfrak{t}) \) for a subalgebra \( \mathfrak{t} \).

Now we describe real vector subspaces \( L \subset \mathbb{H}^n \) with \( \text{span}_\mathbb{H} L = \mathbb{H}^n \). Let \( L \) be such a subspace. Put \( L_1 = L \cap iL \cap jL \cap kL \), i.e. \( L_1 \) is the maximal quaternionic vector
subspace in \( L \). Let \( L_2 \) be the orthogonal complement to \( L_1 \) in \( L \), then \( L = L_1 \oplus L_2 \) and \( L_2 \cap iL_2 \cap jL_2 \cap kL_2 = 0 \). Now let \( L_3 = L_2 \cap iL_2 \), i.e. \( L_3 \) is the maximal \( i \)-invariant real vector subspace in \( L_2 \). Let \( L_4 \) be its orthogonal complement in \( L_2 \), then \( L_2 = L_3 \oplus L_4 \). Similarly, define the spaces \( L_5, L_6, L_7, L_8 \subset L \) such that \( L_5 = L_4 \cap jL_4 \), \( L_4 = L_5 \oplus L_6 \), \( L_7 = L_6 \cap kL_6 \) and \( L_6 = L_7 \oplus L_8 \). By construction, we get the orthogonal decomposition \( L = L_1 \oplus L_3 \oplus L_5 \oplus L_7 \oplus L_8 \) and there exists a \( g \)-orthogonal basis \( e_1, \ldots, e_n \) of \( \mathbb{H}^n \) such that this decomposition has the form

\[
L = \text{span}_H \{ e_1, \ldots, e_m \} \oplus \text{span}_{R\oplus iR} \{ e_{m+1}, \ldots, e_{m_1} \} \oplus \text{span}_{R\oplus jR} \{ e_{m+1}, \ldots, e_{m_2} \} \\
+ \text{span}_{R\oplus kR} \{ e_{m+1}, \ldots, e_{m_3} \} \oplus \text{span}_R \{ e_{m+1}, \ldots, e_n \}.
\]

Obviously, there is an \( f \in \text{SO}(n) \) such that

\[
fL = \text{span}_H \{ e_1, \ldots, e_m \} \oplus \text{span}_{R\oplus iR} \{ e_{m+1}, \ldots, e_{m+k} \} \oplus \text{span}_R \{ e_{m+k+1}, \ldots, e_n \},
\]

where \( m + k = m_3 \). Since we consider the subgroups of \( \text{Sp}(1,n+1)_{\mathbb{H}p} \) up to conjugacy in \( \text{SO}(4,4n+4) \), we can assume that \( L \) has the form (2). We will write for short

\[
L = \mathbb{H}^m \oplus \mathbb{C}^k \oplus \mathbb{R}^{n-m-k}.
\]

Suppose that a subgroup \( K \subset \text{Sim} \mathbb{H}^n \) preserves \( L \). Since \( K \subset \text{Sim} \mathbb{H}^n \subset \text{Sim}^0 \mathbb{R}^{4n} = (\mathbb{R}_+ \times \text{SO}(4n)) \ltimes \mathbb{R}^{4n} \), we have \( K \subset (\mathbb{R}_+ \times \text{SO}(L) \times \text{SO}(L^\perp)) \ltimes L \). But \( K \subset \text{Sim} \mathbb{H}^n \), hence \( \text{pr}_{\text{SO}(4n)} K \subset \text{Sp}(1) \cdot \text{Sp}(n) \). Consequently, \( \text{pr}_{\text{SO}(4n)} K = \text{pr}_{\text{Sp}(1) \cdot \text{Sp}(n)} K \subset \text{Sp}(1) \cdot \text{Sp}(n) \cap \text{SO}(L) \times \text{SO}(L^\perp) \). For the corresponding subalgebra \( \mathfrak{k} \subset \mathcal{L}A(\text{Sim} \mathbb{H}^n) \), we have \( \text{pr}_{\text{Sp}(1) \cdot \text{Sp}(n)} \mathfrak{k} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^\perp) \). Considering the matrices of the elements of these algebras in the basis of \( \mathbb{R}^{4n} \), we obtain

\[
\mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^\perp) = \begin{cases} 
\mathfrak{sp}(1) \oplus \mathfrak{sp}(n), & \text{if } m = n; \\
\mathfrak{sp}(m) \oplus \mathfrak{u}(n-m) \oplus i\mathbb{R}, & \text{if } 0 \leq m < n, \\
\mathfrak{sp}(m) \oplus \mathfrak{u}(k), & \text{if } m = k; \\
\oplus \mathfrak{so}(n-m-k), & \text{if } 0 \leq m < n, \\
\mathfrak{so}(n-m-k), & \text{if } n-m-k \geq 1.
\end{cases}
\]

The action of the Lie algebras \( \mathfrak{u}(n-m) \) and \( \mathfrak{so}(n-m-k) \) on \( \mathbb{C}^{n-m} \) and \( \mathbb{R}^{n-m-k} \), respectively, is described in Section 3.

Let \( E \) be a Euclidean space. In \( \mathfrak{h} \) subalgebras \( \mathfrak{k} \subset \mathcal{L}A(\text{Sim} E) \) corresponding to connected transitively acting subgroups of \( \text{Sim} E \) were divided into the following three types:

**Type \( \mathbb{R} \).** \( \mathfrak{k} = (\mathbb{R} \oplus \mathfrak{h}) \ltimes E \), where \( \mathfrak{h} \subset \mathfrak{so}(E) \) is a subalgebra.

**Type \( \varphi \).** \( \mathfrak{k} = \{ \varphi(A) + A|A \in \mathfrak{h} \} \ltimes E \), where \( \mathfrak{h} \subset \mathfrak{so}(E) \) is a subalgebra, \( \varphi \in \text{Hom}(\mathfrak{h}, \mathbb{R}) \), \( \varphi|_{\mathfrak{h}^\perp} = 0 \).

**Type \( \psi \).** \( \mathfrak{k} = \{ A + \psi(A)|A \in \mathfrak{h} \} \ltimes U \), where we have an orthogonal decomposition \( E = W \oplus U \), \( \mathfrak{h} \subset \mathfrak{so}(W) \) is a subalgebra, \( \psi : \mathfrak{h} \to W \) is surjective linear map, \( \psi|_{\mathfrak{h}^\perp} = 0 \).

Suppose that \( m = n \), i.e. \( L = \mathbb{H}^n \). If \( \mathfrak{k} \) is of Type \( \mathbb{R} \), then \( \mathfrak{k} = (\mathbb{R} \oplus \mathfrak{h}) \ltimes L \), where \( \mathfrak{h} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \) is a subalgebra. If \( \mathfrak{h} \subset \mathfrak{sp}(n) \), then \( (dF)^{-1}(\mathfrak{k}) \) is of Type II with \( a_2 = 0 \) and \( \phi = 0 \). Let \( \mathfrak{h} \) have the form \( \mathfrak{h}_0 \oplus \mathfrak{h}_1 \), where \( \mathfrak{h}_0 \subset \mathfrak{sp}(1) \) and
We should generalize Type IX assuming that we should consider in addition the real vector subspace Type V or VI. If addition add two types of Lie algebras: \( g_m \subset h \): the above, since \( g \) is of Type VII or VIII. If \( \phi \) is of Type II, then \( (dF)^{-1}(t) \) can be obtained from the above, since \( t \) is obtained from \( (\mathbb{R} \oplus h) \times L \) by twisting between \( h \) and \( \mathbb{R} \). We will get that \( (dF)^{-1}(t) \) is of Type III or IV. Let \( h, h \) be a non-zero element.

\[
\begin{align*}
\phi &\neq \text{pr}_{\mathfrak{sp}(1)}(h) \oplus \text{pr}_{\mathfrak{sp}(n)}(h). \\
\text{If } \mathfrak{sp}(1) \cap \mathfrak{sp}(1) &= 0, \text{ then } (dF)^{-1}(t) \text{ is of Type II with } a_2 = 0. \text{ Now let } \dim \mathfrak{h} \cap \mathfrak{sp}(1) = 1 \text{ and let } a_2 \in \mathfrak{h} \cap \mathfrak{sp}(1) \text{ be a non-zero element. Obviously, } \mathfrak{h} = \{A + \phi(A) | A \in \text{pr}_{\mathfrak{sp}(n)}(h)\} + \mathbb{R}a_2, \text{ where } \phi : \text{pr}_{\mathfrak{sp}(n)}(h) \rightarrow \mathfrak{sp}(1) \text{ is a homomorphism, } \phi \neq 0 \text{ and } \Im \phi \cap \mathbb{R}a_2 = 0. \text{ For } A + \phi(A) \in \mathfrak{h}, \text{ we have } [A + \phi(A), a_2] = [\phi(A), a_2] \in \mathfrak{h} \cap \mathfrak{sp}(1). \text{ Hence, } [\phi(A), a_2] \subset \mathbb{R}a_2. \text{ If } \text{rk } \phi = 1, \text{ then } (dF)^{-1}(t) \text{ is of Type II. If } \text{rk } \phi = 2, \text{ then there exist } A_1, A_2 \in \text{pr}_{\mathfrak{sp}(n)}(h) \text{ such that } \phi(A_1), \phi(A_2) \text{ and } a_2 \text{ span } \mathfrak{sp}(1). \text{ But this is impossibly, since } \mathfrak{sp}(1)' = \mathfrak{sp}(1). \text{ In the same way, if } \dim \mathfrak{h} \cap \mathfrak{sp}(1) = 2 \text{ and } \mathfrak{h} = \{A + \phi(A) \} + (\mathfrak{h} \cap \mathfrak{sp}(1)), \text{ then } \phi = 0. \text{ If } \mathfrak{t} = \{\varphi(A) + A | A \in h\} \times L \text{ is of Type } \varphi, \text{ then all } (dF)^{-1}(t) \text{ can be obtained from the above, since } \mathfrak{t} \text{ is obtained from } (\mathbb{R} \oplus h) \times L \text{ by twisting between } h \text{ and } \mathbb{R}. \text{ We will get that } (dF)^{-1}(t) \text{ is of Type III or IV. Let } \mathfrak{t} \text{ be of Type } \psi, \text{ i.e. } \mathfrak{t} = \{A + \psi(A)\} \times U, \text{ where } L = W \oplus U \text{ is an orthogonal decomposition, } h \subset \mathfrak{so}(W) \text{ is a subalgebra and } \psi : h \rightarrow W \text{ is surjective linear map, } \psi|_h = 0. \text{ Since } h \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n), \text{ we have } h \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(W) = \mathfrak{sp}(W \cap iW \cap jW \cap kW). \text{ We obtain Type IX for } m = n. \text{ The case } m < n \text{ can be consider similarly. If } \mathfrak{t} \text{ is of Type } \mathbb{R}, \text{ then } g \text{ is of Type V or VI. If } \mathfrak{t} \text{ is of Type } \varphi, \text{ then } g \text{ is of Type VII or VIII. If } \mathfrak{t} \text{ is of Type } \psi, \text{ then } g \text{ is of Type IX. The theorem is proved.} \end{align*}
\]

**Remark 1.** It is also possible to classify weakly irreducible subalgebras of \( \mathfrak{sp}(1, n + 1)_{\mathbb{H}^p} \) containing the ideal \( B \) up to conjugacy by elements of \( \mathfrak{sp}(1, n + 1) \). For this we should consider in addition the real vector subspace \( L \subset \mathbb{H}^n \) of the form \([1]\) such that at least two of the inequalities \( m < m_1 < m_2 < m_3 \) hold. Note that

\[
\begin{align*}
\mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^\perp) &= \mathfrak{sp}(\text{span}_{\mathbb{R}}\{e_1, \ldots, e_m\}) \\
&\oplus \mathfrak{u}(\text{span}_{\mathbb{R} \oplus i\mathbb{R}}\{e_{m+1}, \ldots, e_{m_1}\}) \oplus \mathfrak{u}(\text{span}_{\mathbb{R} \oplus j\mathbb{R}}\{e_{m+1}, \ldots, e_{m_2}\}) \\
&\oplus \mathfrak{u}(\text{span}_{\mathbb{R} \oplus k\mathbb{R}}\{e_{m_2+1}, \ldots, e_{m_3}\}) \oplus \mathfrak{so}(\text{span}_{\mathbb{R}}\{e_{m_3+1}, \ldots, e_n\}).
\end{align*}
\]

We should generalize Type IX assuming that \( L \) has the form \([1]\) and we should in addition add two types of Lie algebras:

**Type X.** \( g = \{(a_1, A, X, b) | a_1 \in \mathbb{R}, A \in h, X \in L, b \in \text{Im } H\} \), where \( h \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^\perp) \) is a subalgebra.

**Type XI.** \( g = \{\varphi(A), A, X, b) | A \in h, X \in L, b \in \text{Im } H\} \), where \( h \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^\perp) \) is a subalgebra, \( \varphi \in \text{Hom}(h, \mathbb{H}), \varphi|_h = 0. \)

**References**


Department of Algebra and Geometry, Masaryk University
Kotlářská 2, 611 37 Brno, Czech Republic
E-mail: bezvitnaya@math.muni.cz