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WEAKLY IRREDUCIBLE SUBGROUPS OF $\mathrm{Sp}(1, n+1)$

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Abstract. Connected weakly irreducible not irreducible subgroups of $\mathrm{Sp}(1, n+1) \subset \mathrm{SO}(4, 4n+4)$ that satisfy a certain additional condition are classified. This will be used to classify connected holonomy groups of pseudo-hyper-Kählerian manifolds of index 4.

1. Introduction

The classification of connected holonomy groups of Riemannian manifolds is well known [4, 5, 6, 10]. A classification of holonomy groups of pseudo-Riemannian manifolds is an actual problem of differential geometry. Very recently were obtained classifications of connected holonomy groups of Lorentzian manifolds [3, 11, 9] and of pseudo-Kählerian manifolds of index 2 [7]. These groups are contained in $\mathrm{SO}(1, n+1)$ and $\mathrm{U}(1, n+1) \subset \mathrm{SO}(2, 2n+2)$, respectively. As the next step, we study connected holonomy groups contained in $\mathrm{Sp}(1, n+1) \subset \mathrm{SO}(4, 4n+4)$, i.e. holonomy groups of pseudo-hyper-Kählerian manifolds of index 4. By the Wu theorem [12] and the results of Berger for connected irreducible holonomy groups of pseudo-Riemannian manifolds [4], it is enough to consider only weakly irreducible not irreducible holonomy groups (each such group does not preserve any proper non-degenerate vector subspace of the tangent space, but preserves a degenerate subspace).

In the present paper we classify connected weakly irreducible not irreducible subgroups of $\mathrm{Sp}(1, n+1) \subset \mathrm{SO}(4, 4n+4)$ $(n \geq 1)$ that satisfy a natural condition. The case $n = 0$ will be considered separately. We generalize the method of [8, 7]. Let $G \subset \mathrm{Sp}(1, n+1)$ be a weakly irreducible not irreducible subgroup and $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)$ the corresponding subalgebra. The results of [7] allow us to expect that if $\mathfrak{g}$ is the holonomy algebra, then $\mathfrak{g}$ containes a certain 3-dimensional ideal $\mathcal{B}$. We will prove this in another paper. Consider the action of G on the space $\mathbb{H}^{1,n+1}$, then G acts on the boundary of the quaternionic hyperbolic space, which is diffeomorphic to the $4n+3$-dimensional sphere S^{4n+3} and G preserves a point of this space. We define a map $s_1 : S^{4n+3}\backslash\{point\} \to \mathbb{H}^n$ similar to the usual stereographic projection. Then any $f \in G$ defines the map $F(f) = s_1 \circ f \circ s_2 : \mathbb{H}^n \to \mathbb{H}^n$, where $s_2 : \mathbb{H}^n \to S^{4n+3}\backslash\{point\}$ is the inverse of the usual stereographic projection restricted to $\mathbb{H}^n \subset \mathbb{H}^n \oplus \mathbb{R}^3 = \mathbb{R}^{4n+3}$. We get that $F(G)$ is contained in the group

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$\operatorname{Sim}\mathbb{H}^n$ of similarity transformations of $\mathbb{H}^n$. We show that $F(G)$ preserves an affine subspace $L \subset \mathbb{R}^{4n} = \mathbb{H}^n$ such that the minimal affine subspace of $\mathbb{H}^n$ containing L is $\mathbb{H}^n$. Moreover, $F(G)$ does not preserve any proper affine subspace of L. Then $F(G)$ acts transitively on L [1]. We describe subspaces L with this property and using results of [7] we find all connected Lie subgroups $K \subset \operatorname{Sim}\mathbb{H}^n$ preserving L and acting transitively on L. Note that the kernel of the Lie algebra homomorphism $dF : \mathfrak{g} \to \mathcal{LA}(\operatorname{Sim}\mathbb{H}^n)$ coincides with the ideal $\mathcal{B}$. Consequently, $\mathfrak{g} = (dF)^{-1}(\mathfrak{k})$, where $\mathfrak{k} \subset \mathcal{LA}(\operatorname{Sim}\mathbb{H}^n)$ is the Lie algebra of one of the obtained Lie subgroups $K \subset \operatorname{Sim}\mathbb{H}^n$.

Note that we classify weakly irreducible not irreducible subgroups of $\mathrm{Sp}(1, n+1)$ up to conjugacy in $\mathrm{SO}(4, 4n+4)$. It is also possible to classify these subgroups up to conjugacy in $\mathrm{Sp}(1, n+1)$, see Remark 1.

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2. Preliminaries

First we summarize some facts about quaternionic vector spaces. Let $\mathbb{H}^m$ be an m-dimensional quaternionic vector space and $e_1, \dots, e_m$ a basis of $\mathbb{H}^m$. We identify an element $X \in \mathbb{H}^m$ with the column (X_t) of the left coordinates of X with respect to this basis, $X = \sum_{t=1}^m X_t e_t$.

Let $f\colon \mathbb{H}^m \to \mathbb{H}^m$ be an $\mathbb{H}$-linear map. Define the matrix Mat_f of f by the relation $fe_l = \sum_{t=1}^m (\mathrm{Mat}_f)_{tl} e_t$. Now if $X \in \mathbb{H}^m$, then $fX = (X^t \,\mathrm{Mat}_f^t)^t$ and because of the non-commutativity of the quaternions this is not the same as $\mathrm{Mat}_f X$. Conversely, to an $m \times m$ matrix A of the quaternions we put in correspondence the linear map $\mathrm{Op}\, A\colon \mathbb{H}^m \to \mathbb{H}^m$ such that $\mathrm{Op}\, A \cdot X = (X^t A^t)^t$. If $f, g\colon \mathbb{H}^m \to \mathbb{H}^m$ are two $\mathbb{H}$-linear maps, then $\mathrm{Mat}_{fg} = (\mathrm{Mat}_g^t \,\mathrm{Mat}_f^t)^t$. Note that the multiplications by the imaginary quaternions are not $\mathbb{H}$-linear maps. Also, for $a, b \in \mathbb{H}$ holds $\overline{ab} = \bar{b}\bar{a}$. Consequently, for two square quaternionic matrices we have $(\overline{AB})^t = \bar{B}^t \bar{A}^t$.

A pseudo-quaternionic-Hermitian metric g on $\mathbb{H}^m$ is a non-degenerate $\mathbb{R}$-bilinear map $g\colon \mathbb{H}^m \times \mathbb{H}^m \to \mathbb{H}$ such that $g(aX, Y) = ag(X, Y)$ and $\overline{g(Y, X)} = g(X, Y)$, where $a \in \mathbb{H}$, $X, Y \in \mathbb{H}^m$. Hence, $g(X, aY) = g(X, Y)\bar{a}$. There exists a basis $e_1, \dots, e_m$ of $\mathbb{H}^m$ and integers (r, s) with $r + s = m$ such that $g(e_t, e_l) = 0$ if $t \neq l$, $g(e_t, e_t) = -1$ if $1 \le t \le s$ and $g(e_t, e_t) = 1$ if $s+1 \le t \le m$. The pair (r, s) is called the signature of g. In this situation we denote $\mathbb{H}^m$ by $\mathbb{H}^{r,s}$. The realification of $\mathbb{H}^m$ gives us the vector space $\mathbb{R}^{4m}$ with the quaternionic structure (i, j, k). Conversely, a quaternionic structure on $\mathbb{R}^{4m}$, i.e. a triple (I, J, K) of endomorphisms of $\mathbb{R}^{4m}$ such that $I^2 = J^2 = K^2 = -\mathrm{id}$ and $K = IJ = -JI$, allows us to consider $\mathbb{R}^{4m}$ as $\mathbb{H}^m$. A pseudo-quaternionic-Hermitian metric g on $\mathbb{H}^m$ of signature (r, s) defines on $\mathbb{R}^{4m}$ the i, j, k-invariant pseudo-Euclidean metric η of signature $(4r, 4s)$, $\eta(X, Y) = \mathrm{Re}\, g(X, Y)$, $X, Y \in \mathbb{R}^{4m}$. Conversely, a I, J, K-invariant pseudo-Euclidean metric on $\mathbb{R}^{4m}$ defines a pseudo-quaternionic-Hermitian metric g on $\mathbb{H}^m$,

$$g(X, Y) = \eta(X, Y) + i\eta(X, IY) + j\eta(X, JY) + k\eta(X, KY).$$

The Lie group $\mathrm{Sp}(r,s)$ and its Lie algebra $\mathfrak{sp}(r,s)$ are defined as follows

$$\begin{aligned}\mathrm{Sp}(r,s) &= \left\{ f \in \mathrm{Aut}(\mathbb{H}^{r,s}) \mid g(fX,fY) = g(X,Y) \text{ for all } X,Y \in \mathbb{H}^{r,s} \right\},\\ \mathfrak{sp}(r,s) &= \left\{ f \in \mathrm{End}(\mathbb{H}^{r,s}) \mid g(fX,Y) + g(X,fY) = 0 \text{ for all } X,Y \in \mathbb{H}^{r,s} \right\}.\end{aligned}$$

3. The main theorem

Definition 1. A subgroup $G \subset \mathrm{SO}(r,s)$ (or a subalgebra $\mathfrak{g} \subset \mathfrak{so}(r,s)$) is called weakly irreducible if it does not preserve any non-degenerate proper vector subspace of $\mathbb{R}^{r,s}$.

Let $\mathbb{R}^{4,4n+4}$ be a $(4n+8)$-dimensional real vector space endowed with a quaternionic structure $I,J,K \in \mathrm{End}(\mathbb{R}^{4,4n+4})$ and an I,J,K-invariant metric η of signature $(4,4n+4)$. We identify this space with the $(n+2)$-dimensional quaternionic space $\mathbb{H}^{1,n+1}$ endowed with the pseudo-quaternionic-Hermitian metric g of signature $(1,n+1)$ as above.

Obviously, if a Lie subgroup $G \subset \mathrm{Sp}(1,n+1)$ acts weakly irreducibly not irreducibly on $\mathbb{R}^{4,4n+4}$, then G acts weakly irreducibly not irreducibly on $\mathbb{H}^{1,n+1}$. The converse is not true, see Example 2 below. If G acts weakly irreducibly not irreducibly on $\mathbb{H}^{1,n+1}$, then G preserves a proper degenerate subspace $W \subset \mathbb{H}^{1,n+1}$. Consequently, G preserves the intersection $W \cap W^{\perp} \subset \mathbb{H}^{1,n+1}$, which is an isotropic quaternionic line.

Fix a Wit basis $p, e_1, \dots, e_n, q$ of $\mathbb{H}^{1,n+1}$, i.e. the Gram matrix of the metric g with respect to this basis has the form $\begin{pmatrix} 0 & 0 & 1 \\ 0 & E_n & 0 \\ 1 & 0 & 0 \end{pmatrix}$, where E_n is the n-dimensional identity matrix. Denote by $\mathrm{Sp}(1,n+1)_{\mathbb{H}p}$ the Lie subgroup of $\mathrm{Sp}(1,n+1)$ acting on $\mathbb{H}^{1,n+1}$ and preserving the quaternionic isotropic line $\mathbb{H}p$. Note that any weakly irreducible and not irreducible subgroup of $\mathrm{Sp}(1,n+1)$ is conjugated to a weakly irreducible subgroup of $\mathrm{Sp}(1,n+1)_{\mathbb{H}p}$. The Lie subalgebra $\mathfrak{sp}(1,n+1)_{\mathbb{H}p} \subset \mathfrak{sp}(1,n+1)$ corresponding to the Lie subgroup $\mathrm{Sp}(1,n+1)_{\mathbb{H}p} \subset \mathrm{Sp}(1,n+1)$ has the following form

$$\mathfrak{sp}(1,n+1)_{\mathbb{H}p} = \left\{ \mathrm{Op}\begin{pmatrix} \bar{a} & -\bar{X}^t & b \\ 0 & \mathrm{Mat}_h & X \\ 0 & 0 & -a \end{pmatrix} \;\middle|\; \begin{matrix} a \in \mathbb{H}, & X \in \mathbb{H}^n, \\ h \in \mathfrak{sp}(n), & b \in \mathrm{Im}\,\mathbb{H} \end{matrix} \right\}.$$

Let (a,A,X,b) denote the above element of $\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$. Define the following vector subspaces of $\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$:

$$\begin{aligned}\mathcal{A}_1 &= \{(a,0,0,0) \mid a \in \mathbb{R}\}, & \mathcal{A}_2 &= \{(a,0,0,0) \mid a \in \mathrm{Im}\,\mathbb{H}\},\\ \mathcal{N} &= \{(0,0,X,0) \mid X \in \mathbb{H}^n\}, & \mathcal{B} &= \{(0,0,0,b) \mid b \in \mathrm{Im}\,\mathbb{H}\}.\end{aligned}$$

Obviously, $\mathfrak{sp}(n)$ is a subalgebra of $\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$ with the inclusion

$$h \in \mathfrak{sp}(n) \mapsto \mathrm{Op}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{Mat}_h & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sp}(1,n+1)_{\mathbb{H}p}.$$

We obtain that $\mathcal{A}_1$ is a one-dimensional commutative subalgebra that commutes with $\mathcal{A}_2$ and $\mathfrak{sp}(n)$, $\mathcal{A}_2$ is a subalgebra isomorphic to $\mathfrak{sp}(1)$ and commuting with $\mathfrak{sp}(n)$, $\mathcal{B}$ is a commutative ideal, which commutes with $\mathfrak{sp}(n)$ and $\mathcal{N}$. Also,

$$\begin{aligned}
[(a,0,0,0),(0,0,X,b)] =&(0,0,aX,2\operatorname{Im} ab)\,,\\
[(0,0,X,0),(0,0,Y,0)] =&(0,0,0,2\operatorname{Im} g(X,Y))\,,\\
[(0,A,0,0),(0,0,X,0)] =&(0,0,(X^tA^t)^t,0)\,,
\end{aligned}$$

where $a\in\mathbb{H}$, $X,Y\in\mathbb{H}^n$, $A=\mathrm{Mat}_h$, $h\in\mathfrak{sp}(n)$, $b\in\operatorname{Im}\mathbb{H}$. Thus we have the decomposition

$$\mathfrak{sp}(1,n+1)_{\mathbb{H}p}=(\mathcal{A}_1\oplus\mathcal{A}_2\oplus\mathfrak{sp}(n))\ltimes(\mathcal{N}+\mathcal{B})\simeq(\mathbb{R}\oplus\mathfrak{sp}(1)\oplus\mathfrak{sp}(n))\ltimes(\mathbb{H}^n+\mathbb{R}^3)\,.$$

Now consider two examples.

Example 1. The subalgebra $\mathfrak{g}=\{(0,0,X,b)\mid X\in\mathbb{R}^n,\ b\in\operatorname{Im}\mathbb{H}\}\subset\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$ acts weakly irreducibly on $\mathbb{R}^{4,4n+4}$.

Proof. Assume the converse. Let $\mathfrak{g}$ preserve a non-degenerate proper vector subspace $L\subset\mathbb{R}^{4,4n+4}$. Suppose the projection of L to $\mathbb{H}q\subset\mathbb{H}^{1,n+1}=\mathbb{R}^{4,4n+4}$ is non-zero, then there is a vector $v\in L$ such that $v=v_0p+v_1+v_2q$, where $v_0,v_2\in\mathbb{H}$, $v_2\neq 0$ and $v_1\in\mathbb{H}^n$. Consider elements $\xi_1=(0,0,X,0)\in\mathfrak{g}$ with $g(X,X)=1$ and $\xi_2=(0,0,0,b)\in\mathfrak{g}$. Then, $\xi_1(\xi_1 v)=-v_2p\in L$ and $\xi_2 v=v_2bp\in L$. Since $v_2\neq 0$, we have $\mathbb{H}p\subset L$. It follows that $L^{\perp_\eta}\subset\mathbb{H}p\oplus\mathbb{H}^n$ and $L^{\perp_\eta}$ is a $\mathfrak{g}$-invariant non-degenerate proper subspace. Now we can assume that $\mathfrak{g}$ preserves a non-trivial non-degenerate vector subspace $L\subset\mathbb{H}p\oplus\mathbb{H}^n$. Let $v=v_0p+v_1\in L$, $v\neq 0$. If $v_1=0$, then L is degenerate. If $v_1\neq 0$, then there is $X\in\mathbb{R}^n$ with $g(v_1,X)\neq 0$. We get $(0,0,X,0)v=-g(v_1,X)p\in L$. Hence L is degenerate. Thus we have a contradiction. □

Example 2. The subalgebra $\mathfrak{g}=\{(0,0,X,0)|\ X\in\mathbb{R}^n\}\subset\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$ acts weakly irreducibly on $\mathbb{H}^{1,n+1}$ and not weakly irreducibly on $\mathbb{R}^{4,4n+4}$.

Proof. The proof of the first statement is similar to the proof of Example 1. Clearly, the subalgebra $\mathfrak{g}$ preserves the non-degenerate vector subspace $\operatorname{span}_{\mathbb{R}}\{p,e_1,\dots,e_n,q\}\subset\mathbb{R}^{4,4n+4}$. □

The classification of the holonomy algebras contained in $\mathfrak{u}(1,n+1)$ [7] gives us the following hypothesis: *If* $n\geq 1$ *and* $\mathfrak{g}\subset\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$ *is a holonomy algebra, then* $\mathfrak{g}$ *containes the ideal* $\mathcal{B}$. We will prove this hypothesis in an other paper.

In the following theorem we denote the real vector subspace $L\subset\mathbb{R}^{4n}=\mathbb{H}^n$ of the form

$$L=\operatorname{span}_{\mathbb{H}}\{e_1,\dots,e_m\}\oplus\operatorname{span}_{\mathbb{R}\oplus i\mathbb{R}}\{e_{m+1},\dots,e_{m+k}\}\oplus\operatorname{span}_{\mathbb{R}}\{e_{m+k+1},\dots,e_n\}$$

by $\mathbb{H}^m\oplus\mathbb{C}^k\oplus\mathbb{R}^{n-m-k}$. Let $\mathfrak{u}(k)$ be the subalgebra of $\mathfrak{sp}(\operatorname{span}_{\mathbb{H}}\{e_{m+1},\dots,e_{m+k}\})$ that consists of the elements $\mathrm{Op}\begin{pmatrix}A&0\\0&A\end{pmatrix}$, where $A\in\mathfrak{u}\big(\operatorname{span}_{\mathbb{R}\oplus i\mathbb{R}}\{e_{m+1},\dots,e_{m+k}\}\big)$

and we use the decomposition

$$\operatorname{span}_{\mathbb{H}}\{e_{m+1},\dots,e_{m+k}\} \\ = \operatorname{span}_{\mathbb{R}\oplus i\mathbb{R}}\{e_{m+1},\dots,e_{m+k}\} + j\operatorname{span}_{\mathbb{R}\oplus i\mathbb{R}}\{e_{m+1},\dots,e_{m+k}\}\,.$$

Similarly, let $\mathfrak{so}(n-m-k)$ be the subalgebra of $\mathfrak{sp}\big(\operatorname{span}_{\mathbb{H}}\{e_{m+k+1},\dots,e_n\}\big)$ that consists of the elements

$$\operatorname{Op}\begin{pmatrix} A & 0 & 0 & 0\\ 0 & A & 0 & 0\\ 0 & 0 & A & 0\\ 0 & 0 & 0 & A\end{pmatrix}, \quad \text{where} \quad A\in\mathfrak{so}\big(\operatorname{span}_{\mathbb{R}}\{e_{m+k+1},\dots,e_n\}\big)$$

and we use the decomposition $\mathbb{H}^{n-m-k} = \mathbb{R}^{n-m-k}\oplus i\mathbb{R}^{n-m-k}\oplus j\mathbb{R}^{n-m-k}\oplus k\mathbb{R}^{n-m-k}$. For a Lie algebra $\mathfrak{h}$ we denote by $\mathfrak{h}'$ the commutant $[\mathfrak{h},\mathfrak{h}]$ of $\mathfrak{h}$.

Theorem 1. *Let $n\geq 1$. Any weakly irreducible subalgebra of $\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$ that contains the ideal $\mathcal{B}$ is conjugated by an element of $\mathrm{SO}(4,4n+4)$ to one of the following subalgebras:*

- **Type I.** $\mathfrak{g}=\big\{(a_1+a_2,A,X,b)\mid a_1\in\mathbb{R},\ a_2\in\mathfrak{h}_0,\ A\in\mathfrak{h},\ X\in\mathbb{H}^n,\ b\in\operatorname{Im}\mathbb{H}\big\}$, *where $\mathfrak{h}_0\subset\mathfrak{sp}(1)$ is a subalgebra of dimension 2 or 3, $\mathfrak{h}\subset\mathfrak{sp}(n)$ is a subalgebra.*
- **Type II.** $\mathfrak{g}=\big\{(a_1+ta_2+\phi(A),A,X,b)\mid a_1,t\in\mathbb{R},\ A\in\mathfrak{h},\ X\in\mathbb{H}^n,\ b\in\operatorname{Im}\mathbb{H}\big\}$, *where $a_2\in\mathfrak{sp}(1)$, $\mathfrak{h}\subset\mathfrak{sp}(n)$ is a subalgebra, $\phi\colon\mathfrak{h}\to\mathfrak{sp}(1)$ is a homomorphism.*
  If $a_2\neq 0$, then $\operatorname{rk}\phi\leq 1$ and $[\operatorname{Im}\phi,a_2]\subset\mathbb{R}a_2$.
- **Type III.** $\mathfrak{g}=\big\{(\varphi(a_2,A)+a_2,A,X,b)\mid a_2\in\mathfrak{h}_0,\ A\in\mathfrak{h},\ X\in\mathbb{H}^n,\ b\in\operatorname{Im}\mathbb{H}\big\}$, *where $\mathfrak{h}_0\subset\mathfrak{sp}(1)$ is a subalgebra of dimension 2 or 3, $\mathfrak{h}\subset\mathfrak{sp}(n)$ is a subalgebra, $\varphi\in\operatorname{Hom}(\mathfrak{h}_0\oplus\mathfrak{h},\mathbb{R})$, $\varphi|_{\mathfrak{h}_0'\oplus\mathfrak{h}'}=0$. In particular, if $\dim\mathfrak{h}_0=3$, i.e. $\mathfrak{h}_0=\mathfrak{sp}(1)$, then $\varphi|_{\mathfrak{h}_0}=0$.*
- **Type IV.** $\mathfrak{g}=\big\{(\varphi(t,A)+ta_2+\phi(A),A,X,b)\mid t\in\mathbb{R},\ A\in\mathfrak{h},\ X\in\mathbb{H}^n,\ b\in\operatorname{Im}\mathbb{H}\big\}$, *where $a_2\in\mathfrak{sp}(1)$, $\mathfrak{h}\subset\mathfrak{sp}(n)$ is a subalgebra, $\varphi\in\operatorname{Hom}(\mathbb{R}\oplus\mathfrak{h},\mathbb{R})$, $\varphi|_{\mathfrak{h}'}=0$, $\phi\colon\mathfrak{h}\to\mathfrak{sp}(1)$ is a homomorphism. If $a_2\neq 0$, then $\operatorname{rk}\phi\leq 1$ and $[\operatorname{Im}\phi,a_2]\subset\mathbb{R}a_2$. If $a_2\neq 0$ and $\phi\neq 0$, then $\varphi|_{\mathbb{R}}=0$.*
- **Type V.** $\mathfrak{g}=\big\{(a_1+a_2i,A,X,b)\mid a_1,a_2\in\mathbb{R},\ A\in\mathfrak{h},\ X\in\mathbb{H}^m\oplus\mathbb{C}^{n-m},\ b\in\operatorname{Im}\mathbb{H}\big\}$, *where $0\leq m<n$, $\mathfrak{h}\subset\mathfrak{sp}(m)\oplus\mathfrak{u}(n-m)$ is a subalgebra.*
- **Type VI.** $\mathfrak{g}=\big\{(a_1+\phi(A)i,A,X,b)\mid a_1\in\mathbb{R},\ A\in\mathfrak{h},\ X\in\mathbb{H}^m\oplus\mathbb{C}^k\oplus\mathbb{R}^{n-m-k},\ b\in\operatorname{Im}\mathbb{H}\big\}$, *where $0\leq m<n$, $0\leq k\leq n-m$, $\mathfrak{h}\subset\mathfrak{sp}(m)\oplus\mathfrak{u}(k)\oplus\mathfrak{so}(n-m-k)$ is a subalgebra, $\phi\in\operatorname{Hom}(\mathfrak{h},\mathbb{R})$, $\phi|_{\mathfrak{h}'}=0$. If $n-m-k\geq 1$, then $\phi=0$.*
- **Type VII.** $\mathfrak{g}=\big\{(\varphi(a_2,A)+a_2i,A,X,b)\mid a_2\in\mathbb{R},\ A\in\mathfrak{h},\ X\in\mathbb{H}^m\oplus\mathbb{C}^{n-m},\ b\in\operatorname{Im}\mathbb{H}\big\}$, *where $0\leq m<n$, $\mathfrak{h}\subset\mathfrak{sp}(m)\oplus\mathfrak{u}(n-m)$ is a subalgebra, $\varphi\in\operatorname{Hom}(\mathbb{R}\oplus\mathfrak{h},\mathbb{R})$, $\varphi|_{\mathfrak{h}'}=0$.*
- **Type VIII.** $\mathfrak{g}=\big\{(\varphi(A)+\phi(A)i,A,X,b)\mid A\in\mathfrak{h},\ X\in\mathbb{H}^m\oplus\mathbb{C}^k\oplus\mathbb{R}^{n-m-k},\ b\in\operatorname{Im}\mathbb{H}\big\}$, *where $0\leq m<n$, $0\leq k\leq n-m$, $\mathfrak{h}\subset\mathfrak{sp}(m)\oplus\mathfrak{u}(k)\oplus\mathfrak{so}(n-m-k)$ is a subalgebra, $\varphi,\phi\in\operatorname{Hom}(\mathfrak{h},\mathbb{R})$, $\varphi|_{\mathfrak{h}'}=\phi|_{\mathfrak{h}'}=0$. If $n-m-k\geq 1$, then $\phi=0$.*

Type IX. $\mathfrak{g} = \{(0, A, \psi(A) + X, b) \mid A \in \mathfrak{h},\ X \in W,\ b \in \operatorname{Im}\mathbb{H}\}$. *Here* $0 \le m \le n$ *and* $0 \le k \le n - m$. *For* $L = \mathbb{H}^m \oplus \mathbb{C}^k \oplus \mathbb{R}^{n-m-k} \subset \mathbb{R}^{4n} = \mathbb{H}^n$ *we have an* η*-orthogonal decomposition* $L = W \oplus U$, $\mathfrak{h} \subset \mathfrak{sp}(W \cap iW \cap jW \cap kW)$ *is a subalgebra and* $\psi\colon \mathfrak{h} \to W$ *is a surjective linear map with* $\psi|_{\mathfrak{h}'} = 0$.

4. Relation with the group of similarity transformations of $\mathbb{H}^n$

Let $\mathbb{H}^n$ be the n-dimensional quaternionic vector space endowed with a quaternionic-Hermitian metric g. For elements $a_1 \in \mathbb{R}_+$, $a_2 \in \mathrm{Sp}(1)$, $f \in \mathrm{Sp}(n)$ and $X \in \mathbb{H}^n$ consider the following transformations of $\mathbb{H}^n$: $d(a_1)\colon Y \mapsto a_1 Y$ (real dilation), $a_2\colon Y \mapsto a_2 Y$ (quaternionic dilation), $f\colon Y \mapsto fY$ (rotation), $t(Y)\colon Y \mapsto Y + X$ (translation), here $Y \in \mathbb{H}^n$. Note that the elements $a_2 \in \mathrm{Sp}(1)$ act on $\mathbb{H}^n$ as $\mathbb{R}$-linear (but not $\mathbb{H}$-linear) isomorphism. These transformations generate the Lie group $\operatorname{Sim}\mathbb{H}^n$ of similarity transformations of $\mathbb{H}^n$. We get the decomposition

$$\operatorname{Sim}\mathbb{H}^n = (\mathbb{R}_+ \times \mathrm{Sp}(1) \cdot \mathrm{Sp}(n)) \rightthreetimes \mathbb{H}^n.$$

The Lie group $\operatorname{Sim}\mathbb{H}^n$ is a Lie subgroup of the connected Lie group $\operatorname{Sim}^0\mathbb{R}^{4n}$ of similarity transformations of $\mathbb{R}^{4n}$, $\operatorname{Sim}^0\mathbb{R}^{4n} = (\mathbb{R}_+ \times \mathrm{SO}(4n)) \rightthreetimes \mathbb{R}^{4n}$.
The corresponding Lie algebra $\mathcal{LA}(\operatorname{Sim}\mathbb{H}^n)$ to the Lie group $\operatorname{Sim}\mathbb{H}^n$ has the following decomposition

$$\mathcal{LA}(\operatorname{Sim}\mathbb{H}^n) = (\mathbb{R} \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)) \ltimes \mathbb{H}^n.$$

Let $p, e_1, \dots, e_n, q$ be the basis of $\mathbb{H}^{1,n+1}$ as above. Consider also the basis $e_0, e_1, \dots, e_n, e_{n+1}$, where $e_0 = \frac{\sqrt{2}}{2}(p - q)$ and $e_{n+1} = \frac{\sqrt{2}}{2}(p + q)$. With respect to this basis the Gram matrix of g has the form $\begin{pmatrix} -1 & 0 \\ 0 & E_{n+1} \end{pmatrix}$.

The subset of the $(n+1)$-dimensional quaternionic projective space $\mathbb{PH}^{1,n+1}$ that consists of all quaternionic isotropic lines is called the *boundary* of the quaternionic hyperbolic space and is denoted by $\partial\mathbf{H}^{n+1}_{\mathbb{H}}$.

Let $h_0, \dots, h_{n+1}$, where $h_s = x_s + iy_s + jz_s + kw_s \in \mathbb{H}$ $(0 \le s \le n+1)$ be the coordinates on $\mathbb{H}^{1,n+1}$ with respect to the basis $e_0, \dots, e_{n+1}$. Denote by $\mathbb{H}^n$ and $\mathbb{H}^{n+1}$ the subspaces of $\mathbb{H}^{1,n+1}$ spanned by the vectors $e_1, \dots, e_n$ and $e_1, \dots, e_{n+1}$, respectively. Note that the intersection $(e_0 + \mathbb{H}^{n+1}) \cap \{X \in \mathbb{H}^{1,n+1} \mid g(X, X) = 0\}$ is given by the system of equations:

$$x_0 = 1\,, \quad y_0 = 0\,, \quad z_0 = 0\,, \quad w_0 = 0\,,$$

$$x_1^2 + y_1^2 + z_1^2 + w_1^2 + \cdots + x_{n+1}^2 + y_{n+1}^2 + z_{n+1}^2 + w_{n+1}^2 = 1\,,$$

i.e. this set is the $(4n+3)$-dimensional unite sphere S^{4n+3}. Moreover, each isotropic line intersects this set at a unique point, e.g. $\mathbb{H}p$ intersects it at the point $\sqrt{2}p$. Thus we identify the space $\partial\mathbf{H}^{n+1}_{\mathbb{H}}$ with the sphere S^{4n+3}. Any $f \in \mathrm{Sp}(1, n+1)_{\mathbb{H}p}$ takes quaternionic isotropic lines to quaternionic isotropic lines and preserves the quaternionic isotropic line $\mathbb{H}p$. Hence it acts on $\partial\mathbf{H}^{n+1}_{\mathbb{H}} \setminus \{\mathbb{H}p\} = S^{4n+3} \setminus \{\sqrt{2}p\}$.

Consider the connected Lie subgroups $A_1, A_2, \mathrm{Sp}(n)$ and P of $\mathrm{Sp}(1, n+1)_{\mathbb{H}p}$ corresponding to the subalgebras $\mathcal{A}_1, \mathcal{A}_2, \mathfrak{sp}(n)$ and $\mathcal{N} + \mathcal{B}$ of the Lie algebra

$\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$. With respect to the basis $p, e_1, \dots, e_n, q$ these groups have the following matrix form:

$$A_1 = \left\{ \mathrm{Op}\begin{pmatrix} a_1 & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & a_1^{-1} \end{pmatrix} \middle| \; a_1 \in \mathbb{R}_+ \right\},$$

$$A_2 = \left\{ \mathrm{Op}\begin{pmatrix} e^{-a_2} & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & e^{-a_2} \end{pmatrix} \middle| \; a_2 \in \mathrm{Im}\,\mathbb{H} \right\},$$

$$\mathrm{Sp}(n) = \left\{ \mathrm{Op}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Mat}_f & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; f \in \mathrm{Sp}(n) \right\},$$

$$P = \left\{ \mathrm{Op}\begin{pmatrix} 1 & -\bar{Y}^t & b - \frac{1}{2}Y^t\bar{Y} \\ 0 & E_n & Y \\ 0 & 0 & 1 \end{pmatrix} \middle| \; \begin{matrix} Y \in \mathbb{H}^n, \\ b \in \mathrm{Im}\,\mathbb{H} \end{matrix} \right\}.$$

We have the decomposition

$$\mathrm{Sp}(1,n+1)_{\mathbb{H}p} = (A_1 \times A_2 \times \mathrm{Sp}(n)) \ltimes P \simeq (\mathbb{R}_+ \times \mathrm{Sp}(1) \times \mathrm{Sp}(n)) \ltimes (\mathbb{H}^n \cdot \mathbb{R}^3).$$

Let $s_1 \colon S^{4n+3} \setminus \{\sqrt{2}p\} \to e_0 + \mathbb{H}^n$ be the map defined as the usual stereographic projection, but using quaternionic lines. More precisely, for $s \in S^{4n+3} \setminus \{\sqrt{2}p\}$ we define $s_1(s)$ to be the point of the intersection of $e_0 + \mathbb{H}^n$ with the quaternionic line passing through the points $\sqrt{2}p$ and s. It is easy to see that this intersection consists of a single point. Let $s_2 \colon e_0 + \mathbb{H}^n \to S^{4n+3} \setminus \{\sqrt{2}p\}$ be the restriction to $e_0 + \mathbb{H}^n$ of the inverse to the usual stereographic projection from $S^{4n+3} \setminus \{\sqrt{2}p\}$ to $e_0 + \mathbb{H}^n \oplus (\mathrm{Im}\,\mathbb{H})e_{n+1}$. Note that $s_1 \circ s_2 = \mathrm{id}_{e_0+\mathbb{H}^n}$, but unlike in the usual case, s_1 is not surjective. We have $s_2 \circ s_1|_{\mathrm{Im}\, s_2} = \mathrm{id}_{\mathrm{Im}\, s_2}$. Also, let e_0 and $-e_0$ denote the translations $\mathbb{H}^n \to e_0 + \mathbb{H}^n$ and $e_0 + \mathbb{H}^n \to \mathbb{H}^n$, respectively.

For $f \in \mathrm{Sp}(1,n+1)_{\mathbb{H}p}$ define the map

$$F(f) = (-e_0) \circ s_1 \circ f \circ s_2 \circ e_0 : \mathbb{H}^n \to \mathbb{H}^n.$$

Now we will show that F is a surjective homomorphism from the Lie group $\mathrm{Sp}(1,n+1)_{\mathbb{H}p}$ to the Lie group $\mathrm{Sim}\,\mathbb{H}^n$ and $\ker F = \mathbb{Z}_2 \times B$, where $\mathbb{Z}_2 = \{\mathrm{id}, -\mathrm{id}\} \in \mathrm{Sp}(1,n+1)_{\mathbb{H}p}$ and B is the connected Lie subgroup of $\mathrm{Sp}(1,n+1)_{\mathbb{H}p}$ corresponding to the ideal $\mathcal{B} \subset \mathfrak{sp}(1,n+1)_{\mathbb{H}p}$. First of all, the computations show that for $a_1 \in \mathbb{R}$, $a_2 \in \mathrm{Im}\,\mathbb{H}$, $f \in \mathrm{Sp}(n)$ and $Y \in \mathbb{H}^n$ it holds

$$F\left(\mathrm{Op}\begin{pmatrix} a_1 & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & a_1^{-1} \end{pmatrix}\right) = d(a_1) \in \mathbb{R}_+ \subset \mathrm{Sim}\,\mathbb{H}^n,$$

$$F\left(\mathrm{Op}\begin{pmatrix} e^{-a_2} & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & a^{-a_2} \end{pmatrix}\right) = e^{a_2} \in \mathrm{Sp}(1) \subset \mathrm{Sim}\,\mathbb{H}^n,$$

$$F\left(\operatorname{Op}\begin{pmatrix}1 & 0 & 0\\ 0 & \mathrm{Mat}_f & 0\\ 0 & 0 & 1\end{pmatrix}\right) = f \in \mathrm{Sp}(n) \subset \mathrm{Sim}\,\mathbb{H}^n\,,$$

$$F\left(\operatorname{Op}\begin{pmatrix}1 & -\bar{Y}^t & b-\frac{1}{2}Y^t\bar{Y}\\ 0 & E_n & Y\\ 0 & 0 & 1\end{pmatrix}\right) = t\left(-\frac{\sqrt{2}}{2}Y\right) \in \mathbb{H}^n \subset \mathrm{Sim}\,\mathbb{H}^n.$$

It follows that if $f_1, f_2 \in P$, then $F(f_1f_2) = F(f_1)F(f_2)$, i.e. $F|_P$ is a homomorphism from P to $\mathrm{Sim}\,\mathbb{H}^n$. It can easily be checked that any $f \in A_1 \times A_2 \times \mathrm{Sp}(n)$ considered as a map from $S^{4n+3}\setminus\{\sqrt{2}p\}$ to itself preserves $\mathrm{Im}\, s_2 \subset S^{4n+3}\setminus\{\sqrt{2}p\}$. Hence if f_1 is from P or $A_1 \times A_2 \times \mathrm{Sp}(n)$ and $f_2 \in A_1 \times A_2 \times \mathrm{Sp}(n)$, then

$$\begin{aligned} F(f_1f_2) &= (-e_0)\circ s_1 \circ f_1 \circ f_2 \circ s_2 \circ e_0\\ &= (-e_0)\circ s_1 \circ f_1 \circ s_2 \circ e_0 \circ (-e_0)\circ s_1 \circ f_2 \circ s_2 \circ e_0 = F(f_1)F(f_2)\,, \end{aligned}$$

since $s_2 \circ s_1|_{\mathrm{Im}\, s_2} = \mathrm{id}_{\mathrm{Im}\, s_2}$. Therefore it is enough to prove that $F(f_1f_2) = F(f_1)F(f_2)$, for $f_1 \in A_1 \times A_2 \times \mathrm{Sp}(n)$ and $f_2 \in P$. Let

$$f_1 = \operatorname{Op}\begin{pmatrix}a_1e^{-a_2} & 0 & 0\\ 0 & A & 0\\ 0 & 0 & a_1^{-1}e^{-a_2}\end{pmatrix} \in A_1 \times A_2 \times \mathrm{Sp}(n)\,,$$

$$f_2 = \operatorname{Op}\begin{pmatrix}1 & -\bar{Y}^t & b-\frac{1}{2}Y^t\bar{Y}\\ 0 & E_n & Y\\ 0 & 0 & 1\end{pmatrix} \in P\,.$$

Then $f_1f_2f_1^{-1} = f_2' \in P$, where

$$f_2' = \operatorname{Op}\begin{pmatrix}1 & -((A^{-1})^t\bar{Y}a_1e^{-a_2})^t & a_1^2e^{a_2}(b-\frac{1}{2}Y^t\bar{Y})e^{-a_2}\\ 0 & E_n & a_1e^{a_2}(Y^tA^t)^t\\ 0 & 0 & 1\end{pmatrix}.$$

We have

$$\begin{aligned} F(f_1f_2) = F(f_2'f_1) = F(f_2')F(f_1) &= t\left(-\frac{\sqrt{2}}{2}a_1e^{a_2}(Y^tA^t)^t\right)a_1e^{a_2}\operatorname{Op}A\\ &= t\left(-\frac{\sqrt{2}}{2}a_1e^{a_2}\operatorname{Op}A\cdot Y\right)a_1e^{a_2}\operatorname{Op}A\\ &= a_1e^{a_2}\operatorname{Op}A\cdot t\left(-\frac{\sqrt{2}}{2}Y\right) = F(f_1)F(f_2)\,, \end{aligned}$$

since for any $f \in \mathbb{R}_+ \times \mathrm{SO}(4n)$ and $X \in \mathbb{R}^{4n}$ it holds $ft(X)f^{-1} = t(fX)$ or $t(fX)f = ft(X)$. Thus F is the homomorphism from the Lie group $\mathrm{Sp}(1, n+1)_{\mathbb{H}p}$ to the Lie group $\mathrm{Sim}\,\mathbb{H}^n$. Obviously, F is surjective. The claim is proved.

Let $L \subset \mathbb{R}^{4n}$ be a vector (affine) subspace. We call the subset $L \subset \mathbb{H}^n$ *a real vector* (*affine*) *subspace.*

Theorem 2. *Let $G \subset \mathrm{Sp}(1, n+1)_{\mathbb{H}p}$ act weakly irreducibly on $\mathbb{H}^{1,n+1}$. Then if $F(G) \subset \mathrm{Sim}\,\mathbb{H}^n$ preserves a proper real affine subspace $L \subset \mathbb{H}^n$, then the minimal affine subspace of $\mathbb{H}^n$ containing L is $\mathbb{H}^n$.*

Proof. First we prove that the subgroup $F(G)\subset \mathrm{Sim}\,\mathbb{H}^n$ does not preserve any proper affine subspace of $\mathbb{H}^n$. Assume that $F(G)$ preserves a vector subspace $L\subset\mathbb{H}^n$. Choosing the basis $e_1,\dots,e_n$ of $\mathbb{H}^n$ in a proper way, we can suppose that $L=\mathbb{H}^m=\mathrm{span}_{\mathbb{H}}\{e_1,\dots,e_m\}$. Consequently, $F(G)\subset(\mathbb{R}_+\times(\mathrm{Sp}(1)\cdot(\mathrm{Sp}(m)\times \mathrm{Sp}(n-m))))\leftthreetimes\mathbb{H}^m$. Hence, $G\subset(\mathbb{R}_+\times\mathrm{Sp}(1)\times\mathrm{Sp}(m)\times\mathrm{Sp}(n-m))\leftthreetimes(\mathbb{H}^m\cdot\mathbb{R}^3)$ and G preserves the non-degenerate vector subspace $\mathrm{span}_{\mathbb{H}}\{e_{m+1},\dots,e_n\}\subset\mathbb{H}^{1,n+1}$. Now suppose that $F(G)$ preserves an affine subspace $L\subset\mathbb{H}^n$. Let $L=Y+L_0$, where $Y\in L$ and $L_0\subset\mathbb{H}^n$ is the vector subspace corresponding to L. We may assume that $L_0=\mathbb{H}^m=\mathrm{span}_{\mathbb{H}}\{e_1,\dots,e_m\}$. Consider $f=\mathrm{Op}\begin{pmatrix}1&\sqrt{2}\bar{Y}^t&-Y^t\bar{Y}\\0&E_n&-\sqrt{2}Y\\0&0&1\end{pmatrix}\in P$ and the subgroup $\tilde{G}=f^{-1}Gf\subset\mathrm{Sp}(1,n+1)_{\mathbb{H}p}$. For $F(\tilde{G})$ we get that $F(\tilde{G})=-t(Y)F(G)t(Y)$. By the above $\tilde{G}$ preserves the non-degenerate vector subspace $\mathrm{span}_{\mathbb{H}}\{e_{m+1},\dots,e_n\}\subset\mathbb{H}^{1,n+1}$. Hence G preserves the non-degenerate vector subspace $f(\mathrm{span}_{\mathbb{H}}\{e_{m+1},\dots,e_n\})\subset\mathbb{H}^{1,n+1}$. Since G is weakly irreducible, we get $m=n$.

Let $F(G)$ preserve a real affine subspace $L\subset\mathbb{H}^n$ and let $L_0\subset\mathbb{H}^n$ be the corresponding real vector subspace. Consider the vector subspace $(\mathrm{span}_{\mathbb{H}} L_0)^\perp\subset\mathbb{H}^n$. As above, it can be proved that G preserves the non-degenerate vector subspace $f((\mathrm{span}_{\mathbb{H}} L_0)^\perp)\subset\mathbb{H}^{1,n+1}$. Since G is weakly irreducible, we have $(\mathrm{span}_{\mathbb{H}} L_0)^\perp=0$ and $\mathrm{span}_{\mathbb{H}} L_0=\mathbb{H}^n$. The theorem is proved. □

5. Proof of the main theorem

First of all, from Example 1 it follows that the algebras of Types I–VIII act weakly irreducibly on $\mathbb{R}^{4,4n+4}$. For the algebras of Type IX it can be proved in the same way. Therefore we must only prove that any subalgebra $\mathfrak{g}\subset\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$ that acts weakly irreducibly on $\mathbb{R}^{4,4n+4}$ and contains the ideal $\mathcal{B}$ is conjugated (by an element from $\mathrm{SO}(4,4n+4)$) to one of the algebras of Types I–IX. Suppose that $\mathfrak{g}\subset\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$ acts weakly irreducibly on $\mathbb{R}^{4,4n+4}$ and contains the ideal $\mathcal{B}$. Let $G\subset\mathrm{Sp}(1,n+1)_{\mathbb{H}p}$ be the corresponding connected Lie subgroup. By Theorem 2, $F(G)$ preserves a real affine subspace $L\subset\mathbb{H}^n$ such that the minimal affine subspace of $\mathbb{H}^n$ containing L is $\mathbb{H}^n$. We already know that G is conjugated to a subgroup $\tilde{G}\subset\mathrm{Sp}(1,n+1)_{\mathbb{H}p}$ such that $F(\tilde{G})$ preserves a real vector subspace $L_0\subset\mathbb{H}^n$ with $\mathrm{span}_{\mathbb{H}} L_0=\mathbb{H}^n$. Hence we can assume that $F(G)$ preserves a real vector subspace $L\subset\mathbb{H}^n$ and $\mathrm{span}_{\mathbb{H}} L=\mathbb{H}^n$. Moreover, assume that $F(G)$ does not preserve any proper affine subspace of L. Then $F(G)$ acts transitively on L [1]. The connected transitively acting groups of similarity transformations of the Euclidean spaces are well know. In [7] these groups were divided into three types. We describe real subspaces $L\subset\mathbb{H}^n$ with $\mathrm{span}_{\mathbb{H}} L=\mathbb{H}^n$ and subalgebras $\mathfrak{k}\subset\mathcal{LA}(\mathrm{Sim}\,\mathbb{H}^n)$ such that the corresponding connected Lie subgroups $K\subset\mathrm{Sim}\,\mathbb{H}^n$ preserve L and act transitively on L. Then the algebra $\mathfrak{g}$ must be of the form $(dF)^{-1}(\mathfrak{k})$ for a subalgebra $\mathfrak{k}$.

Now we describe real vector subspaces $L\subset\mathbb{H}^n$ with $\mathrm{span}_{\mathbb{H}} L=\mathbb{H}^n$. Let L be such a subspace. Put $L_1=L\cap iL\cap jL\cap kL$, i.e. L_1 is the maximal quaternionic vector subspace in L. Let L_2 be the orthogonal complement to L_1 in L, then $L = L_1 \oplus L_2$ and $L_2 \cap iL_2 \cap jL_2 \cap kL_2 = 0$. Now let $L_3 = L_2 \cap iL_2$, i.e. L_3 is the maximal i-invariant real vector subspace in L_2. Let L_4 be its orthogonal complement in L_2, then $L_2 = L_3 \oplus L_4$. Similarly, define the spaces $L_5, L_6, L_7, L_8 \subset L$ such that $L_5 = L_4 \cap jL_4$, $L_4 = L_5 \oplus L_6$, $L_7 = L_6 \cap kL_6$ and $L_6 = L_7 \oplus L_8$. By construction, we get the orthogonal decomposition $L = L_1 \oplus L_3 \oplus L_5 \oplus L_7 \oplus L_8$ and there exists a g-orthogonal basis $e_1, \dots, e_n$ of $\mathbb{H}^n$ such that this decomposition has the form

$$\begin{aligned} L = \operatorname{span}_{\mathbb{H}}\{e_1, \dots e_m\} &\oplus \operatorname{span}_{\mathbb{R}\oplus i\mathbb{R}}\{e_{m+1}, \dots e_{m_1}\} \oplus \operatorname{span}_{\mathbb{R}\oplus j\mathbb{R}}\{e_{m_1+1}, \dots e_{m_2}\} \\ &\oplus \operatorname{span}_{\mathbb{R}\oplus k\mathbb{R}}\{e_{m_2+1}, \dots e_{m_3}\} \oplus \operatorname{span}_{\mathbb{R}}\{e_{m_3+1}, \dots e_n\}. \end{aligned} \tag{1}$$

Obviously, there is an $f \in \mathrm{SO}(n)$ such that

$$fL = \operatorname{span}_{\mathbb{H}}\{e_1, \dots e_m\} \oplus \operatorname{span}_{\mathbb{R}\oplus i\mathbb{R}}\{e_{m+1}, \dots e_{m+k}\} \oplus \operatorname{span}_{\mathbb{R}}\{e_{m+k+1}, \dots e_n\}, \tag{2}$$

where $m + k = m_3$. Since we consider the subgroups of $\mathrm{Sp}(1, n+1)_{\mathbb{H}p}$ up to conjugacy in $\mathrm{SO}(4, 4n+4)$, we can assume that L has the form (2). We will write for short

$$L = \mathbb{H}^m \oplus \mathbb{C}^k \oplus \mathbb{R}^{n-m-k}.$$

Suppose that a subgroup $K \subset \operatorname{Sim}\mathbb{H}^n$ preserves L. Since $K \subset \operatorname{Sim}\mathbb{H}^n \subset \operatorname{Sim}^0\mathbb{R}^{4n} = (\mathbb{R}_+ \times \mathrm{SO}(4n)) \ltimes \mathbb{R}^{4n}$, we have $K \subset (\mathbb{R}_+ \times \mathrm{SO}(L) \times \mathrm{SO}(L^\perp)) \ltimes L$. But $K \subset \operatorname{Sim}\mathbb{H}^n$, hence $\mathrm{pr}_{\mathrm{SO}(4n)} K \subset \mathrm{Sp}(1)\cdot\mathrm{Sp}(n)$. Consequently, $\mathrm{pr}_{\mathrm{SO}(4n)} K = \mathrm{pr}_{\mathrm{Sp}(1)\cdot\mathrm{Sp}(n)} K \subset \mathrm{Sp}(1)\cdot\mathrm{Sp}(n) \cap \mathrm{SO}(L) \times \mathrm{SO}(L^\perp)$. For the corresponding subalgebra $\mathfrak{k} \subset \mathcal{LA}(\operatorname{Sim}\mathbb{H}^n)$, we have $\mathrm{pr}_{\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)} \mathfrak{k} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^\perp)$. Considering the matrices of the elements of these algebras in the basis of $\mathbb{R}^{4n}$, we obtain

$$\mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^\perp) = \begin{cases} \mathfrak{sp}(1) \oplus \mathfrak{sp}(n), & \text{if } m = n; \\ \mathfrak{sp}(m) \oplus \mathfrak{u}(n-m) \oplus i\mathbb{R}, & \text{if } 0 \le m < n, \\ & \quad n - m = k; \\ \mathfrak{sp}(m) \oplus \mathfrak{u}(k) & \\ \qquad \oplus \mathfrak{so}(n-m-k), & \text{if } 0 \le m < n, \\ & \quad n-m-k \ge 1. \end{cases}$$

The action of the Lie algebras $\mathfrak{u}(n-m)$ and $\mathfrak{so}(n-m-k)$ on $\mathbb{C}^{n-m}$ and $\mathbb{R}^{n-m-k}$, respectively, is described in Section 3.

Let E be a Euclidean space. In [7] subalgebras $\mathfrak{k} \subset \mathcal{LA}(\operatorname{Sim}E)$ corresponding to connected transitively acting subgroups of $\operatorname{Sim}E$ were divided into the following three types:

Type $\mathbb{R}$. $\mathfrak{k} = (\mathbb{R} \oplus \mathfrak{h}) \ltimes E$, where $\mathfrak{h} \subset \mathfrak{so}(E)$ is a subalgebra.

Type φ. $\mathfrak{k} = \{\varphi(A) + A | A \in \mathfrak{h}\} \ltimes E$, where $\mathfrak{h} \subset \mathfrak{so}(E)$ is a subalgebra, $\varphi \in \operatorname{Hom}(\mathfrak{h}, \mathbb{R})$, $\varphi|_{\mathfrak{h}'} = 0$.

Type ψ. $\mathfrak{k} = \{A + \psi(A) | A \in \mathfrak{h}\} \ltimes U$, where we have an orthogonal decomposition $E = W \oplus U$, $\mathfrak{h} \subset \mathfrak{so}(W)$ is a subalgebra, $\psi : \mathfrak{h} \to W$ is surjective linear map, $\psi|_{\mathfrak{h}'} = 0$.

Suppose that $m = n$, i.e. $L = \mathbb{H}^n$. If $\mathfrak{k}$ is of Type $\mathbb{R}$, then $\mathfrak{k} = (\mathbb{R} \oplus \mathfrak{h}) \ltimes L$, where $\mathfrak{h} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ is a subalgebra. If $\mathfrak{h} \subset \mathfrak{sp}(n)$, then $(dF)^{-1}(\mathfrak{k})$ is of Type II with $a_2 = 0$ and $\phi = 0$. Let $\mathfrak{h}$ have the form $\mathfrak{h}_0 \oplus \mathfrak{h}_1$, where $\mathfrak{h}_0 \subset \mathfrak{sp}(1)$ and

$\mathfrak{h}_1 \subset \mathfrak{sp}(n)$. If $\dim \mathfrak{h}_0 = 1$, then $(dF)^{-1}(\mathfrak{k})$ is of Type II with $\phi = 0$ and $\mathfrak{h}$ changed to $\mathfrak{h}_1$. If $\dim \mathfrak{h}_0 = 2$ or 3, then $(dF)^{-1}(\mathfrak{k})$ is of Type I with $\mathfrak{h}$ changed to $\mathfrak{h}_1$. Suppose that $\mathfrak{h} \neq \mathrm{pr}_{\mathfrak{sp}(1)} \mathfrak{h} \oplus \mathrm{pr}_{\mathfrak{sp}(n)} \mathfrak{h}$. If $\mathfrak{h} \cap \mathfrak{sp}(1) = 0$, then $(dF)^{-1}(\mathfrak{k})$ is of Type II with $a_2 = 0$. Now let $\dim \mathfrak{h} \cap \mathfrak{sp}(1) = 1$ and let $a_2 \in \mathfrak{h} \cap \mathfrak{sp}(1)$ be a non-zero element. Obviously, $\mathfrak{h} = \{A + \phi(A) | A \in \mathrm{pr}_{\mathfrak{sp}(n)} \mathfrak{h}\} + \mathbb{R}a_2$, where $\phi\colon \mathrm{pr}_{\mathfrak{sp}(n)} \mathfrak{h} \to \mathfrak{sp}(1)$ is a homomorphism, $\phi \neq 0$ and $\mathrm{Im}\, \phi \cap \mathbb{R}a_2 = 0$. For $A + \phi(A) \in \mathfrak{h}$, we have $[A + \phi(A), a_2] = [\phi(A), a_2] \in \mathfrak{h} \cap \mathfrak{sp}(1)$. Hence, $[\phi(A), a_2] \subset \mathbb{R}a_2$. If $\mathrm{rk}\, \phi = 1$, then $(dF)^{-1}(\mathfrak{k})$ is of Type II. If $\mathrm{rk}\, \phi = 2$, then there exist $A_1, A_2 \in \mathrm{pr}_{\mathfrak{sp}(n)} \mathfrak{h}$ such that $\phi(A_1), \phi(A_2)$ and a_2 span $\mathfrak{sp}(1)$. But this is impossibly, since $\mathfrak{sp}(1)' = \mathfrak{sp}(1)$. In the same way, if $\dim \mathfrak{h} \cap \mathfrak{sp}(1) = 2$ and $\mathfrak{h} = \{A + \phi(A)\} + (\mathfrak{h} \cap \mathfrak{sp}(1))$, then $\phi = 0$. If $\mathfrak{k} = \{\varphi(A) + A | A \in \mathfrak{h}\} \ltimes L$ is of Type φ, then all $(dF)^{-1}(\mathfrak{k})$ can be obtained from the above, since $\mathfrak{k}$ is obtained from $(\mathbb{R} \oplus \mathfrak{h}) \ltimes L$ by twisting between $\mathfrak{h}$ and $\mathbb{R}$. We will get that $(dF)^{-1}(\mathfrak{k})$ is of Type III or IV. Let $\mathfrak{k}$ be of Type ψ, i.e. $\mathfrak{k} = \{A + \psi(A)\} \ltimes U$, where $L = W \oplus U$ is an orthogonal decomposition, $\mathfrak{h} \subset \mathfrak{so}(W)$ is a subalgebra and $\psi : \mathfrak{h} \to W$ is surjective linear map, $\psi|_{\mathfrak{h}'} = 0$. Since $\mathfrak{h} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$, we have $\mathfrak{h} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(W) = \mathfrak{sp}(W \cap iW \cap jW \cap kW)$. We obtain Type IX for $m = n$. The case $m < n$ can be consider similarly. If $\mathfrak{k}$ is of Type $\mathbb{R}$, then $\mathfrak{g}$ is of Type V or VI. If $\mathfrak{k}$ is of Type φ, then $\mathfrak{g}$ is of Type VII or VIII. If $\mathfrak{k}$ is of Type ψ, then $\mathfrak{g}$ is of Type IX. The theorem is proved. $\square$

Remark 1. It is also possible to classify weakly irreducible subalgebras of $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ containing the ideal $\mathcal{B}$ up to conjugacy by elements of $\mathrm{Sp}(1, n+1)$. For this we should consider in addition the real vector subspace $L \subset \mathbb{H}^n$ of the form (1) such that at least two of the inequalities $m < m_1 < m_2 < m_3$ hold. Note that

$$\begin{aligned} \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^\perp) &= \mathfrak{sp}(\mathrm{span}_{\mathbb{H}}\{e_1, \dots e_m\}) \\ &\oplus \mathfrak{u}(\mathrm{span}_{\mathbb{R}\oplus i\mathbb{R}}\{e_{m+1}, \dots e_{m_1}\}) \oplus \mathfrak{u}(\mathrm{span}_{\mathbb{R}\oplus j\mathbb{R}}\{e_{m_1+1}, \dots e_{m_2}\}) \\ &\oplus \mathfrak{u}(\mathrm{span}_{\mathbb{R}\oplus k\mathbb{R}}\{e_{m_2+1}, \dots e_{m_3}\}) \oplus \mathfrak{so}(\mathrm{span}_{\mathbb{R}}\{e_{m_3+1}, \dots e_n\}) \,. \end{aligned}$$

We should generalize Type IX assuming that L has the form (1) and we should in addition add two types of Lie algebras:

Type X. $\mathfrak{g} = \big\{(a_1, A, X, b) \mid a_1 \in \mathbb{R},\ A \in \mathfrak{h},\ X \in L,\ b \in \mathrm{Im}\, \mathbb{H}\big\}$, where $\mathfrak{h} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^\perp)$ is a subalgebra.

Type XI. $\mathfrak{g} = \big\{(\varphi(A), A, X, b) \mid A \in \mathfrak{h},\ X \in L,\ b \in \mathrm{Im}\mathbb{H}\big\}$, where $\mathfrak{h} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}(L^\perp)$ is a subalgebra, $\varphi \in \mathrm{Hom}(\mathfrak{h}, \mathbb{R})$, $\varphi|_{\mathfrak{h}'} = 0$.

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