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ON THE NON-INVARINACE
OF SPAN AND IMMERSION CO-DIMENSION FOR
MANIFOLDS

DIARMUID J. CROWLEY AND PETER D. ZVENGROWSKI

Abstract. In this note we give examples in every dimension $m \geq 9$ of piecewise linearly homeomorphic, closed, connected, smooth $m$-manifolds which admit two smoothness structures with differing spans, stable spans, and immersion co-dimensions. In dimension 15 the examples include the total spaces of certain 7-sphere bundles over $S^8$. The construction of such manifolds is based on the topological variance of the second Pontrjagin class: a fact which goes back to Milnor and which was used by Roitberg to give examples of span variation in dimensions $m \geq 18$.

We also show that span does not vary for piecewise linearly homeomorphic smooth manifolds in dimensions less than or equal to 8, or under connected sum with a smooth homotopy sphere in any dimension. Finally, we use results of Morita to show that in all dimensions $m \geq 19$ there are topological manifolds admitting two piecewise linear structures having different $PL$-spans.

1. Introduction

We shall use the notation $M$ for a closed, connected, topological manifold, $M_A, M_B, \ldots$ for $M$ together with a given piecewise linear (henceforth $PL$) structure, and $M_\alpha, M_\beta, \ldots$ for $M$ together with a given smoothness structure. Recall that for a smooth $m$-dimensional manifold $M_\alpha$, two basic and classical geometric invariants are its span and its immersion co-dimension. The span is the maximal number $r$ such that $M_\alpha$ admits $r$ pointwise linearly independent vector fields, while the immersion co-dimension is the least $k$ such that $M_\alpha$ immerses in $\mathbb{R}^{m+k}$. Clearly $0 \leq r \leq m$, and from the Whitney Immersion Theorem (together with the fact that a closed $m$-manifold cannot immere in dimension $m$), one has $1 \leq k \leq m - 1$. A fundamental question is whether these two invariants can differ for distinct smooth structures, $M_\alpha$ and $M_\beta$, on the same $PL$-manifold $M_A$. An affirmative answer was first given by Roitberg [22] in 1969, in all dimensions $m \geq 18$. In this paper we use smoothing theory to settle this question in all dimensions: we give an affirmative answer for dimensions $m \geq 9$ and show that span and immersion co-dimension are $PL$ invariants in dimensions less than or equal to 8.

Key words and phrases: span, stable span, manifolds, non-invariance.
Let us first fix some definitions and notation. For a vector bundle \( \xi \) over a space \( X \), we define
\[
\text{span}(\xi) := \max \{ r : \xi \approx r\varepsilon \oplus \eta \}
\]
where \( \approx \) denotes isomorphism of vector bundles, \( r\varepsilon \) denotes the trivial bundle of rank \( r \) and \( \eta \) is some other vector bundle over \( X \). This is the same as the maximal number of pointwise linearly independent sections of \( \xi \), and if \( \xi \) is of rank \( m \), then clearly \( 0 \leq \text{span}(\xi) \leq m \). We also write \( m - \text{span}(\xi) = \text{gd}(\xi) \), the geometric dimension of \( \xi \), and this clearly equals \( \text{rank}(\eta) \). Replacing isomorphism \( \approx \) by stable isomorphism \( \sim \) in the above definitions gives the corresponding notions of stable span and stable geometric dimension, written respectively \( \text{span}^0 \), \( \text{gd}^0 \). Writing \( \xi^0 \) for the stable vector bundle represented by \( \xi \) we also define \( \text{span}(\xi^0) := \text{span}^0(\xi) \) and similarly for geometric dimension. Evidently
\[
0 \leq \text{span}(\xi) \leq \text{span}^0(\xi) = \text{span}(\xi^0) \leq m, \quad m \geq \text{gd}(\xi) \geq \text{gd}^0(\xi) = \text{gd}(\xi^0) \geq 0.
\]
We remark that in the literature “geometric dimension” is often used to denote what we are calling “stable geometric dimension”. Let \( M_\alpha \) be a smooth \( m \)-dimensional manifold with underlying topological manifold \( M \). With the above definitions, the span (resp. stable span) of \( M_\alpha \) is simply the span (resp. stable span) of its tangent bundle \( \tau_\alpha = \tau(M_\alpha) \), i.e.
\[
\text{span}(M_\alpha) := \text{span}(\tau_\alpha), \quad \text{span}^0(M_\alpha) := \text{span}^0(\tau_\alpha).
\]
The manifold \( M \) is also a CW-complex of dimension \( m = \text{rank}(\tau) \), it is then useful to note that by standard stability properties of vector bundles (cf. \([8, \text{Ch. 9}]\)), \( \text{span}^0(M_\alpha) = \max \{ r : \tau_\alpha \oplus \varepsilon \approx (r + 1)\varepsilon \oplus \eta \} \). The notation \( M^{(k)} \) will be used, as usual, to denote the \( k \)-skeleton of \( M \).

Turning to the normal bundle \( \nu_\alpha^0 = \nu^0(M_\alpha) \) (which is a stable bundle), the Hirsch immersion theorem states that the immersion co-dimension \( k \) of \( M_\alpha \) is given by the formula \( k = \max \{ 1, \text{gd}(\nu^0_\alpha) \} \). The stable isomorphism \( \tau_\alpha^0 \oplus \nu_\alpha^0 \sim 0 \) suggests a possible relation between the stable span and the immersion co-dimension. For interesting inequalities relating these with the Lyusternik-Schnirel’man category of \( M \) we refer the reader to Korbaš and Szűcs, \([12]\).

Now let \( M_A \) be the \( PL \)-manifold underlying \( M_\alpha \) and let \( C(M_\alpha) \) denote the finite set of concordance classes of smooth structures on \( M_A \) (see Section \([2]\)). We define the \textit{smooth span variation} of \( M_A \) to be to be the maximal difference of spans over all the smooth structures on \( M_A \) and similarly define the \textit{smooth stable span variation} of \( M_A \):
\[
\text{ssv}(M_A) := \max \{ \text{span}(M_\alpha) \mid [M_\alpha] \in C(M_A) \} - \min \{ \text{span}(M_\alpha) \mid [M_\alpha] \in C(M_A) \},
\]
\[
\text{ss}^0v(M_A) := \max \{ \text{span}^0(M_\alpha) \mid [M_\alpha] \in C(M_A) \} - \min \{ \text{span}^0(M_\alpha) \mid [M_\alpha] \in C(M_A) \}.
\]
Evidently \( \text{ssv}(M_A) \) and \( \text{ss}^0v(M_A) \) are invariants of the \( PL \)-homeomorphism type of \( M_A \). We also note that both span variations can be defined to give topological
Theorem 1.2. Theorem 1.1. Atiyah [1] says that the stable spherical fibration associated to the tangent bundle with non-zero Euler characteristic (whence variation amongst vector bundles in the kernel of the
Remark 1.3. All of the manifolds we find for Theorem 1.1 admit a smooth
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in closed. Analogously results hold for immersion co-dimension.
Theorem 1.1. In every dimension m ≥ 9 there are PL-manifolds MA for which
ssv(MA) ≥ 4 and ss0v(MA) ≥ 4.
Theorem 1.2.
(a) Let M be a topological manifold with dim(M) ≤ 8 which admits a PL-structure MA. Then ssv(MA) = ss0v(MA) = 0. If also H3(M; Z/2) = 0 then
ssv(M) = ss0v(M) = 0.
(b) For every oriented homotopy sphere Sm, and every locally oriented smooth manifold Mα, span(Mα) = span(Mα#S0m). In particular for every homotopy sphere span(S0m) = span(S0m).
Remark 1.3. All of the manifolds we find for Theorem 1.1 admit a smooth structure Mα which is parallelisable and another smooth structure Mβ with non-vanishing second Pontryagin class, p2(Mβ) ≠ 0. This explains the 4, since p2(ξ) = 0 for any vector bundle with stable geometric dimension less than 4. It was also stated in [19] that the second Pontryagin class is not a topological invariant for closed manifolds, and a recent proof appears in [15].

One can also define the span and stable span of CAT-manifolds for CAT = PL or Top as well as for smooth manifolds where CAT = O (see [25] for the topological case and also [21]). Let CAT(k) be the group of CAT-isomorphisms of \( \mathbb{R}^k \) fixing zero. An m-dimensional CAT manifold MA has a CAT-tangent bundle \( \tau(M_A) \) and a stable CAT-bundle \( \tau^0(M_A) \). The span of MA equals j if the principal CAT(m)-bundle associated to \( \tau(M_A) \) has a CAT(m - j) reduction but no CAT(m - j - 1)-reduction. The stable span of MA is j if the same is true of the principal CAT-bundle associated to \( \tau^0(M_A) \). Analogously to the case of smooth span variations, we obtain the PL-span variations of a topological manifold M by setting \( \mathcal{C}_{PL}(M) \) to be the finite set of concordance classes of PL-structures on M.
and defining
\[
\text{plsv}(M) := \max\{\text{span}(M_C) \mid [M_C] \in \mathcal{C}_{PL}(M)\} - \min\{\text{span}(M_C) \mid [M_C] \in \mathcal{C}_{PL}(M)\},
\]
\[
\text{pls}^0v(M) := \max\{\text{span}^0(M_C) \mid [M_C] \in \mathcal{C}_{PL}(M)\} - \min\{\text{span}^0(M_C) \mid [M_C] \in \mathcal{C}_{PL}(M)\}.
\]

In [18] Morita discovered topological manifolds $M$ in each dimension $m \geq 22$ which admit $PL$ structures $M_A$ and $M_B$ which cannot both be smoothed. It is a relatively simple matter to combine Morita’s results with a theorem of Wall [26] to prove

**Theorem 1.4.** In all dimensions $m \geq 19$ there are topological manifolds $M$ such that $\text{plsv}(M) > 0$ and $\text{pls}^0v(M) > 0$.

The remainder of the paper is organised as follows. In Section 2 we review the smoothing theory we need and prove Theorem 1.2. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.4. We now conclude the introduction with a list of open problems concerning span variation.

**Problem 1.5** (Problems about span variation and span). Let $M$ be a closed topological manifold. We state these problems for $\text{ssv}(M)$ and $\text{plsv}(M)$ for brevity but the analogous problems are open and interesting for $\text{ss}^0v(M)$ and $\text{pls}^0v(M)$, as well as for immersion co-dimension.

1. Relate $\text{ssv}(M)$ to other topological invariants of $M$.
2. For a dimension $m$, determine the largest $\text{ssv}(M)$ for an $m$-dimensional manifold.
3. If possible, find families of manifolds $M_i$ such that $\lim_{i \to \infty} \text{ssv}(M_i) = \infty$.
4. Find a manifold $M$ where the spherical fibration associated to $\tau(M)$ is non-trivial and $\text{ssv}(M) > 0$.
5. Determine the dimensions $m$ for which $\text{plsv}(M^m) = 0$ is always zero. This relates to the next problem.
6. Determine whether the assumption that $H^3(M; \mathbb{Z}/2) = 0$ can be removed from the second part of Theorem 1.2 (a).
7. Compute $\text{ssv}(M)$ for well known manifolds. In particular, for the total spaces of 7-bundles over $S^8$. This relates to the next problem.
8. Determine the span of stably parallelisable topological 15-manifolds. (Breeden and Kosinski calculated the span of stably parallelisable smooth manifolds in [3]. In [25] Varadarajan extended their result to stably parallelisable topological manifolds except in dimension 15.)

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2. A rapid review of smoothing theory

Recall the notation established in the introduction: $M_\alpha$ is a closed, connected smooth manifold with underlying PL-manifold $M_A$ and underlying topological manifold $M$. In this section we review the implications of Cairns-Hirsch smoothing theory for the question of whether the smooth span of $M_\alpha$ depends upon the choice of smooth structure $\alpha$. We use [16] as our reference for smoothing theory and for further details relating to this brief review.

A concordance between smooth structures $M_\alpha$ and $M_\beta$ is a smooth structure on $M_A \times [0,1]$, compatible with the PL structure of $M_A \times [0,1]$, which restricts to $M_\alpha$ on $M_A \times \{0\}$ and to $M_\beta$ on $M_A \times \{1\}$. The set of concordance classes of smooth structures on $M_A$ is denoted by $\mathcal{C}(M_A)$, and $[M_\alpha] \in \mathcal{C}(M_A)$ will denote the equivalence class of $M_\alpha$, i.e. the set of all $M_\beta$ refining $M_A$ that are concordant to $M_\alpha$. We are interested in the difference a choice of smooth structure can make to the smooth tangent bundle considered as an abstract vector bundle up to isomorphism. Notice that if $M_\alpha$ and $M_\beta$ are concordant, then their tangent bundles are stably equivalent. The following lemma implies that this remains true unstably.

**Lemma 2.1.** Let $M_\alpha$ and $M_\beta$ be smooth structures on the topological manifold $M$. Then $\tau(M_\alpha) \sim \tau(M_\beta)$ if and only if $\tau(M_\alpha) \approx \tau(M_\beta)$.

**Proof.** One implication is trivial, so let $\tau(M_\alpha)$ and $\tau(M_\beta)$ be classified by $f_\alpha : M \to BO(m)$ and $f_\beta : M \to BO(m)$, and suppose these bundles are stably equivalent. Then they agree over $M^{(m-1)}$. Now let $O_{\alpha,\beta} \in H^m(M; K)$ be the obstruction to a homotopy $f_\alpha \simeq f_\beta$, where $K = \text{Ker}(\pi_{m-1}(O(m)) \to \pi_{m-1}(O)) \cong 0$, $\mathbb{Z}/2$, $\mathbb{Z}$, corresponding to $m \in \{1, 3, 7\}$, or $m$ odd and $m \notin \{1, 3, 7\}$, or $m$ even, respectively. We now show this obstruction vanishes in turn for the cases: $m$ is odd, $m$ is even with $M$ orientable, and $m$ is even with $M$ non-orientable.

If $m = 2r + 1$ is odd, it follows from [9] that there are either one or two isomorphism classes of rank $m$ vector bundles over $M$, stably equivalent to $\tau(M_\alpha)$, this number being called the James-Thomas number. If the James-Thomas number is one then automatically $\tau(M_\alpha) \approx \tau(M_\beta)$. On the other hand, if this number is two, then the two isomorphism classes are distinguished by the Browder-Dupont invariant $b_B$, cf. [24]. But according to [24], $b_B(\tau(M_\alpha))$ and $b_B(\tau(M_\beta))$ must both equal the mod-2 Kervaire semi-characteristic $\chi_2(M) := \sum_{i=0}^r \text{rank}(H^i(M; \mathbb{Z}/2)) \pmod{2}$, so $O_{\alpha,\beta} = 0$.

If $m$ is even and $M$ is orientable then $O_{\alpha,\beta}$ lies in $H^m(M; \mathbb{Z})$, where the coefficients are untwisted. In this case $O_{\alpha,\beta}$ measures the difference in the Euler classes of the bundles $\tau(M_\alpha)$ and $\tau(M_\beta)$, but these are both determined by the Euler characteristic of $M$ and hence the same. Thus $O_{\alpha,\beta}$ vanishes.

If $m$ is even and non-orientable let $\omega : \pi_1(M) \to \mathbb{Z}/2 = \{1, -1\}$ be the first Stiefel-Whitney class. In this case $O_{\alpha,\beta} \in H^m(M; \mathbb{Z})$ where the coefficients are twisted and $\mathbb{Z}$ denotes the $\mathbb{Z}[\pi_1(M)]$-module with $g \in \pi_1(M)$ acting via multiplication by $\omega(g)$. By twisted Poincaré duality (see, for example, [5, §5]), $H^m(M; \mathbb{Z}) \cong H_0(M; \mathbb{Z}) \cong \mathbb{Z}$. Now let $p : \tilde{M} \to M$ denote the orientation double cover of $M$ and $\tilde{M}_\alpha^\gamma, \tilde{M}_\beta^\gamma$ the corresponding smooth structures on $\tilde{M}$ induced via $p$. Of course
the classifying map for $\tau(\tilde{M}_\alpha)$ is $f_\alpha \circ p$ and similarly for the classifying map of $\tau(\tilde{M}_\beta)$. We write $O_{\alpha,\beta}$ for the obstruction to a homotopy of the classifying map for $\tau(\tilde{M}_\alpha)$ to that of $\tau(\tilde{M}_\beta)$, which is zero by the oriented case. The covering map $p$ induces $p^*: H^m(M; \tilde{Z}) \to H^m(\tilde{M}; \tilde{Z})$ where the latter coefficients are untwisted and we have that $p^*(O_{\alpha,\beta}) = O_{\alpha,\beta}$. Since $p^*$ is induced by a double covering it is isomorphic to $\times 2: \tilde{Z} \to \tilde{Z}$ and we conclude that $O_{\alpha,\beta} = 0$. □

Let us now define the following sets of isomorphism classes of vector bundles and stable vector bundles:

$$Tv(M_A) := \{[\tau(M_\alpha)] \mid [M_\alpha] \in C(M_A)\}$$

and

$$T^0v(M_A) := \{[\tau^0(M_\alpha)] \mid [M_\alpha] \in C(M_A)\}.$$  

Observe that Lemma 2.1 shows that there is a bijection $T^0v(M_A) \equiv Tv(M_A)$. We first show that $Tv(M_A)$ is a singleton in dimensions $m \leq 4$.

**Lemma 2.2.** Let $h: M_\alpha \to N_\beta$ be a homotopy equivalence between smooth $m$-manifolds with $m \leq 4$. Then $h$ preserves the tangent bundles; i.e. $h^*(\tau(N_\beta)) \approx \tau(M_\alpha)$.

**Proof.** By Lemma 2.1 it is enough to show that $h^*(\tau^0(N_\beta)) \sim \tau^0(M_\alpha)$. Let $f_\alpha: M \to BO$ and $g_\beta: N \to BO$ classify the stable tangent bundles of $M_\alpha$ and $N_\beta$, and $p: BO \to BG$ be the canonical fibration, and let $i: G/O \to BO$ be the inclusion of a fibre. By [1], $h$ preserves the stable spherical fibrations underlying $\tau^0(M_\alpha)$ and $\tau^0(N_\beta)$ and so $p \circ f_\alpha$ is homotopic to $p \circ g_\beta \circ h$. As $p$ is an isomorphism on $\pi_1$ and $\pi_2$ and as $\pi_3(BO) = 0$, $f_\alpha$ and $g_\beta \circ h$ agree on $M^{(3)}$. Hence the lemma holds in dimensions $m \leq 3$.

Now assume that $\dim(M) = 4$. There is a cohomology class $O_{\alpha,\beta} \in H^4(M; \pi_4(BO))$ which is the obstruction to a homotopy from $f_\alpha$ to $g_\beta \circ h$. The coefficients are untwisted since $\pi_1(BO)$ acts trivially on $\pi_4(BO)$. Moreover we see that $O_{\alpha,\beta}$ lies in the image of the map from $H^4(M; \pi_4(G/O))$. If $M$ is not orientable then $H^4(M; \pi_4(G/O))$ and $H^4(M; \pi_4(BO))$ are both isomorphic to $\mathbb{Z}/2$ but the map $\pi_4(G/O) \to \pi_4(BO)$ is multiplication by 24, and since $O_{\alpha,\beta}$ lifts to $H^4(M; \pi_4(G/O))$ it must vanish. If $M$ and $N$ are orientable then orient them so that $h$ is orientation preserving and repeat the above argument replacing $BO$ and $BG$ respectively by $BSO$ and $BSG$, and using the classifying maps of the oriented tangent bundles. The class $O_{\alpha,\beta}$ is now detected by the difference of the Pontrjagin classes $p_1(\tau^0(M_\alpha)) - h^*(p_1(\tau^0(N_\beta)))$ but by the signature theorem these classes agree since $h$ is an orientation preserving homotopy equivalence from $M$ to $N$. Hence $\tau^0(M_\alpha)$ and $h^*(\tau^0(M_\beta))$ may be oriented so that they become isomorphic oriented stable vector bundles and so, in particular, they are isomorphic. □

We now recall how smoothing theory calculates $T^0v(M_A)$ and hence $Tv(M_A)$ in dimensions $m \geq 5$. Fixing a smooth structure, $M_\alpha$, makes $C(M_A)$ into a pointed set denoted $\tilde{C}(M_\alpha)$. A fundamental result of smoothing theory is the following
**Theorem 2.3** (Cairns-Hirsch, see [16, Theorem 7.2]). Let $M_\alpha$ be a smooth manifold of dimension at least 5, then there is a bijection

$$\Psi_\alpha : C(M_\alpha) \equiv [M, PL/O]$$

which takes the base point $[M_\alpha]$ to the homotopy class of the constant map.

Recall that $PL/O$ has a commutative $H$-space structure which makes the fibration $PL/O \to BO \to BPL$ into a sequence of $H$-space maps where $BO$ and $BPL$ have compatible commutative $H$-space structures coming from the Whitney sum of bundles [16][p 92]. Associated to this fibration we have the long exact Puppe sequence of abelian groups, for any space $X$,

$$\cdots \to [X, PL] \to [X, PL/O] \xrightarrow{\partial_X} [X, BO] \to [X, BPL].$$

When $X = M$ is homeomorphic to a smooth manifold $M_\alpha$, $\partial_M$ computes the difference a smooth structure makes to the isomorphism class of the stable tangent bundle. That is, for the appropriate choice of $\Psi_\alpha$,

$$\partial_M(\Psi_\alpha(M_\beta)) = [\tau^0(M_\alpha)] - [\tau^0(M_\beta)] \in \tilde{KO}(M) = [M, BO].$$

Combining Lemma 2.2, the fact that $PL/O$ is 6-connected and the above identity we deduce

**Lemma 2.4.** The group $\text{Im}(\partial_M)$ acts freely and transitively on $T^0v(M_\alpha)$.

Applying Lemma 2.1 we immediately obtain

**Corollary 2.5.** If $\partial_M = 0$ then $Tv(M_\alpha)$ and $T^0v(M_\alpha)$ are singletons and so $ssv(M_\alpha) = ss^0v(M_\alpha) = 0$.

**Proof of Theorem 1.2.** Lemma 2.2 implies both parts in dimensions $m \leq 4$. So we now assume that $m \geq 5$ and start with part (b). If $M = S^m$, then it is known [?] that $\pi_m(PL) \to \pi_m(PL/O)$ is surjective and so $\partial_{S^m} = 0$. It follows that every exotic sphere gives rise to the same tangent bundle as the usual one (a fact already observed in [20]). Now for any smooth locally oriented manifold $M_\alpha$ and any homotopy $m$-sphere $S^m_\sigma$ we have $M_{\alpha+\sigma} := M_\alpha \sharp S^m_\sigma$. Using smoothing theory we identify the smooth structure $\alpha + \sigma$ as follows. Identify $C(S^m) = \pi_m(PL/O)$ using the standard smooth structure $S^m_0$ on the sphere so that $\sigma \in \pi_m(PL/O)$ corresponds to the exotic sphere $S^m_\sigma$ under the bijection $\Psi_0$, and let $c : M \to S^m$ be the collapse map taking an open $m$-disc in $M$ homeomorphically onto $S^m \setminus \{\text{pt}\}$ and all points outside the open $m$-disc to pt. By definition we have that $\Psi_0^{-1}(c^*\sigma) = M_{\alpha+\sigma}$. Now the induced maps $c^* : \pi_m(PL/O) \to [M, PL/O]$ and $c^* : \pi_m(BO) \to [M, BO]$ give rise to the following commutative diagram:

$$\pi_m(PL/O) \xrightarrow{\partial_{S^m}} \pi_m(BO)$$

$$\downarrow c^* \quad \quad \quad \quad \quad \downarrow c^*$$

$$[M, PL/O] \xrightarrow{\partial_M} [M, BO].$$
It follows that
\[ \partial_M(\Psi_\alpha(M_{\alpha+\sigma})) = \partial_M(c^*(\sigma)) = c^*(\partial_{S^m}(\sigma)) = c^*(0) = 0. \]

Thus \( \tau^0(M_\alpha) \sim \tau^0(M_{\alpha+\sigma}) \). By Lemma 2.1 we have that \( \tau(M_\alpha) \approx \tau(M_{\alpha+\sigma}) \) and so \( \text{span}(M_\alpha) = \text{span}(M_{\alpha+\sigma}) \). This concludes the proof of part (b).

We now prove part (a). For the PL-statement, since \( m \geq 5 \) we apply Theorem 2.3. As \( PL/O \) is 6-connected, if \( M_A \) is 5 or 6 dimensional then \( M_A \) admits a unique smooth structure. If \( M_A \) is of dimension 7 then Theorem 2.3 implies that all smooth structures are obtained from a fixed one by connected sum with a homotopy 7-sphere and so part (b) don't alter the span. If \( M \) is 8-dimensional it suffices, by Corollary 2.5, to show that \( \partial_M = 0 \). As usual, let \( M \) be the topological manifold underlying \( M_A \) and let \( M^{(6)} \) be the 6-skeleton of a CW-decomposition for \( M \) containing just one 8-cell. Such a decomposition exists by [27]. As \( PL/O \) is 6-connected, \( [M/M^{(6)}, PL/O] \to [M, BO] \) is surjective and thus the image of \( \partial_M \) lies in \( \text{Im}([M/M^{(6)}, BO] \to [M, BO]) \). If \( M \) is orientable then \( M/M^{(6)} \simeq (\vee S^7) \vee S^8 \) is homotopy equivalent to a wedge of 7-spheres and an 8-sphere, then \( \partial_M \) splits as the sum of \( \partial_{S^7} \)'s and \( \partial_{S^8} \) but these are zero. If \( M \) is not orientable then \( M/M^{(6)} \simeq M(\mathbb{Z}/2, 7) \vee (\vee S^7) \) is homotopy equivalent to a degree 7 Moore space wedged with a wedge of 7-spheres. Since the short exact sequence \( \pi_7(O) \to \pi_7(PL) \to \pi_7(PL/O) \) (see Section 2) splits at the prime 2 it again follows that \( \partial_M = 0 \).

It remains to prove that \( \text{ssv}(M) = 0 \) if \( H^3(M; \mathbb{Z}/2) = 0 \), in dimensions \( 5 \leq m \leq 8 \). In dimensions \( m \geq 5 \) there is a smoothing theory for \( PL \)-structures on topological manifolds which is analogous to the smoothing theory for smooth structures on \( PL \)-manifolds we sketched above. In particular the set of concordance classes of \( PL \)-structures on \( M \), \( C_{PL}(M) \), corresponds bijectively with \( [M, TOP/PL] \). Moreover, the fundamental work of [11] shows that \( TOP/PL \) is homotopy equivalent to the Eilenberg-MacLane space \( K(\mathbb{Z}/2, 3) \). Hence the assumption that \( H^3(M; \mathbb{Z}/2) = 0 \) ensures that there is a unique concordance class \( [M_A] \) of \( PL \) structures on \( M \). Thus the span variations for \( M \) and the span variations for \( M_A \) are zero by the \( PL \) case.

We remark that our proof in fact shows

**Corollary 2.6.** Let \( M_A \) be a \( PL \)-manifold of dimension \( m \leq 8 \). Then \( |Tv(M_A)| = 1 \).

Turning our attention now to higher dimensions, if there is a \( PL \)-manifold \( M_A \) with \( \partial_M \neq 0 \) and which admits a parallelisable smooth structure \( M_\alpha \), i.e. \( \tau(M_\alpha) \approx m_\varepsilon \), then there will be a smooth structure \( M_\beta \) such that \( \tau^0(M_\beta) \) is non-trivial and so \( \text{span}(M_\beta) \leq \text{span}^0(M_\beta) < m \). However, \( \text{span}(M_\alpha) = \text{span}^0(M_\alpha) = m \), so in such a case both \( \text{ssv}(M_A) > 0 \) and \( \text{ss}^0v(M_A) > 0 \). In the next section we produce examples of this sort.

### 3. PL-Manifolds with Varying Smooth Spans

In this section we give examples of \( PL \)-manifolds \( M_A \) in dimensions 9 and higher with \( \text{ssv}(M_A) \geq 4 \) and \( \text{ss}^0v(M_A) \geq 4 \). Let \( M(C_k, 1) = S^1 \cup_k e^2 \) be the degree 1 Moore space with first homology group cyclic of order \( k \). As \( M(C_k, 1) \) is a 2-dimensional complex it can be embedded into \( \mathbb{R}^5 \); we take an embedding
into $\mathbb{R}^{10}$ and then take a regular neighbourhood of $M(C_k, 1)$, $T^{10}_\alpha(k)$, which is a compact, smooth, parallelisable 10-manifold with boundary. Here $\alpha$ is the induced smoothness structure coming from the standard one on $\mathbb{R}^{10}$. Let $N^9_\alpha(k)$ be the boundary of $T^{10}_\alpha(k)$. We see that $N^9_\alpha(k)$ is a closed, connected, smooth stably parallelisable 9-manifold and we write $N^9_\alpha(k)$ for the underlying PL-manifold.

Before starting the next theorem, we recall (following [3]) the definitions of the semi-characteristic $\chi^*(M)$ and the reduced semi-characteristic $\hat{\chi}(M)$ of a manifold $M$. If $\dim(M)$ is even then $\chi^*(M)$ is the half-integer $\chi(M)/2$ where $\chi(M)$ is as usual the Euler characteristic of $M$. If $\dim(M)$ is odd then $\chi^*(M) \in \mathbb{Z}/2$ is equal to $\chi_2(M)$, the mod-2 Kervaire semi-characteristic (defined in the proof of Lemma 2.1). The reduced semi-characteristic is defined to be $\hat{\chi}(M) = 1 - \chi^*(M)$ and satisfies $\hat{\chi}(M_0 \sharp M_1) = \hat{\chi}(M_0) + \hat{\chi}(M_1)$. For example: $\hat{\chi}(S^1 \times S^m) = 1$ if $m \geq 1$ $\chi(N^9_\alpha(k)) = 0$. We also orient the manifolds $N^9_\alpha(k)$ and use the notation $M \#_j T = M \# T \# \cdots \# T$ for the connected sum of $M$ with $j$ copies of an oriented manifold $T$, for any choice of $CAT = O, PL, Top$.

**Theorem 3.1.**

1. Let $n \geq 0$ and $W^n_B$ be any closed, oriented PL-$n$-manifold admitting a stably parallelisable smooth structure. Assume that 7 divides $k$ and set $l = \chi^*(N^9_\alpha(k) \times W^n_B)$. Then for all $j \geq 0$

$$ss^0 v((N^9_\alpha(k) \times W^n_B)^{j}(S^1 \times S^{n+8})) \geq 4 \text{ and } ssv((N^9_\alpha(k) \times W^n_B)^{j}(S^1 \times S^{n+8})) \geq 4,$$

where we regard $S^1 \times S^{n+8}$ as a PL manifold.

2. Let $\xi$ be a linear 7-sphere bundle over $S^8$ and let $P^{15}_A$ be the PL-manifold underlying the total space of $\xi$. If the total space of $\xi$ is stably parallelisable and 14 divides the Euler class of $\xi$, $e(\xi) \in H^8(S^8; \mathbb{Z}) \cong \mathbb{Z}$, then $ssv(P^{15}_A) \geq 4$ and $ss^0 v(P^{15}_A) \geq 4$.

**Remark 3.2.** Of course in part (1) above one may take $W^0_B$ to be a point, and $W^n_B = S^n$, $n > 0$. Furthermore, $l \in \mathbb{Z}$ because $\text{span}^0(N^9_\alpha(k) \times W^n_B) = 9 + n > 0$ implies $\chi(N^9_\alpha(k) \times W^n_B)$ is even. The idea of taking neighbourhoods of appropriate Moore spaces to find examples of homeomorphic smooth manifolds with differing tangent bundles goes back to Milnor [17, Roitberg [22 doubled compact neighbourhoods of Moore spaces of degree at least 7 to exhibit smooth span variation for closed manifolds in dimensions 18 and higher. We are able to get examples down to dimension 9 by using a degree 1 Moore space so that a “dual” Moore space appears in dimension 7. In (2), note that $E(\xi)$ has a standard smoothness structure because it is a linear 7-sphere bundle.

**Remark 3.3.** Total spaces as in Theorem 3.1 (2) exist: in the notation of [23, §2] take any 7-sphere bundle $\xi_{h,j} \in \pi_7(SO(8)) \cong \mathbb{Z} \oplus \mathbb{Z}$ with $(h,j) = (7k, 7k)$ and $k \neq 0$. By [23] the corresponding total spaces are almost parallelisable and hence stably parallelisable since $\pi_{14}(O) = 0$ (or cf. [14, Ch. 9 (8.5)]). We do not resolve whether the non-stably parallelisable smooth structures in this case are also realised as the total spaces of 7-sphere bundles over $S^8$. 
Proof of Theorem 3.1. Let $M^m_A$ be any manifold satisfying the hypotheses of the theorem. By assumption $M_A$ admits a stably parallelisable smooth structure $M_\alpha$, so $\text{span}^0(M_\alpha) = m$. If, in addition, the semi-characteristic $\chi^*(M)$ vanishes then \cite{3} asserts that $\text{span}(M_\alpha) = m$ and it is a simple matter (using the addition formula for the reduced semicharacteristic $\hat{\chi}$ under connected sums, as well as $\hat{\chi}(S^1 \times S^{n+8}) = 1$) to check that the additional hypotheses in the theorem ensure that the semi-characteristic vanishes. We will show that each $M_A$ admits a smooth structure $M_\beta$ with non-zero second Pontrjagin class, $p_2(M_\beta) \neq 0$. The theorem then follows since any smooth $m$-manifold with stable span greater than $m-4$ has vanishing second Pontrjagin class, which shows

$$\text{span}(M_\beta) \leq \text{span}^0(M_\beta) \leq m - 4.$$ 

It remains to show the existence of a smooth structure $\beta$ with $p_2(M_\beta) \neq 0$. We may therefore specialize to the case where $M^m_A$ is one of $N^3_\lambda(k)$ or $P^15_A$ using the product formula for the Pontrjagin classes of the manifolds in Theorem \cite{3} (1). First recall \cite{4, 7} that the homotopy exact sequence

$$0 \to \pi_7(O) \to \pi_7(PL) \to \pi_7(PL/O) \to 0$$

is isomorphic to

$$0 \to \mathbb{Z} \xrightarrow{(7,1)} \mathbb{Z} \oplus \mathbb{Z}/4 \xrightarrow{(-1)} \mathbb{Z}/28 \to 0.$$ 

We denote the Bockstein homomorphism associated to the first short exact sequence by $Bk$. We shall relate $Bk$ to $\partial_M : [M, PL/O] \to [M, BO]$.

Since $M_\alpha$ is stably parallelisable and $PL/O$ is 6-connected it follows for any smooth structure, $M_\gamma$, that $\tau^0(M_\gamma)$ is trivial when restricted to $M^{(6)}$. Further, since $\pi_7(BO) = 0$, we can extend this statement to $M^{(7)}$. Thus the primary obstruction to the triviality of $\tau^0(M_\gamma)$, $\text{Ob}_O(\tau^0(M_\gamma))$, lies in $H^8(M; \pi_7(O))$ and there is a commutative diagram

$$\begin{array}{ccc}
[M, PL/O] & \xrightarrow{\partial_M} & \text{Im}(\partial_M) \\
\downarrow_{\text{Ob}_{PL/O}} & & \downarrow_{\text{Ob}_O} \\
H^7(M; \pi_7(PL/O)) & \xrightarrow{Bk} & H^8(M; \pi_7(O))
\end{array}$$

where we have used $\Psi_\alpha$ to identify $C(M_A) \equiv [M, PL/O]$ and $\text{Ob}_{PL/O} : [M, PL/O] \to H^7(M; \pi_7(PL/O))$ as the primary obstruction to a null-homotopy. Now for all the $M$ to which we have specialized, $H^8(M; \pi_7(O)) \cong H^8(M; \mathbb{Z})$ contains a cyclic summand of order $7^a$ with $a \geq 1$. Let $y$ be a generator for this summand. We claim that there is an element $x \in [M, PL/O]$ such that $Bk \circ \text{Ob}_{PL/O}(x) = 7^{a-1}y$.

Firstly we observe that $\text{Ob}_{PL/O}$ is onto the 7-torsion in $H^7(M; \pi_7(PL/O))$ since the Atiyah-Hirzeburch spectral sequence to compute $[M, PL/O]$ gives an exact sequence

$$\cdots \to [M, PL/O] \xrightarrow{\text{Ob}_{PL/O}} H^7(M; \pi_7(PL/O)) \to H^m(M; \pi_{m-1}PL/O) \to \cdots$$
and $H^m(M;\pi_{m-1}PL/O) \cong \pi_{m-1}(PL/O)$ is prime to 7 ($m = 9$ or 15, and $\pi_8(PL/O) \cong \pi_{14}(PL/O) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$). Secondly, from the coefficient sequence above, we see that when restricted to the summand generated by $y$, the map $H^8(M;\pi_7(PL/O)) \to H^8(M;\pi_7(PL/O))$ is isomorphic to multiplication by 7. It follows that $7^a - 7y \neq 0$ lies in the image of $B_k$ and since it is 7-torsion it also lies in the image of $B_k \circ \text{Ob}_{PL/O}$.

From the claim and the commutativity of the above diagram we have an $x \in [M,PL/O]$ such that $\text{Ob}_{O} \circ \partial_M(x) = 7^a - y$. Setting $\beta = \Psi^{-1}_a(x)$ we obtain a smooth structure $\beta$ on $M_A$ with $\text{Ob}_{O}(\tau^0(M_\beta)) = 7^a - y$. Finally, Kervaire [10] has shown that $p_2 = 6 \cdot \text{Ob}_O$ for vector bundles which are trivial over $M^{(7)}$ and hence $p_2(M_\beta) = 6 \cdot \text{Ob}_O(\tau^0(M_\beta)) = 6 \cdot 7^a - y \neq 0$.

4. TOPOLOGICAL MANIFOLDS WITH VARYING PL SPANS

In this section we prove Theorem [4.4]. We assume that the reader is familiar with the simply connected surgery exact sequences for smooth and PL-manifolds.

In every dimension $m \geq 22$, Morita [18, Theorem 6.1] defines a simply connected topological manifold $M = M^{m}(K)$ by embedding a 10-skeleton $K$ of $PL/O \simeq K(\mathbb{Z}/2, 3)$ in $\mathbb{R}^m$, $m \geq 22$, taking a regular neighbourhood $T = T^{m}(K)$ of $K$ and letting $M$ be the trivial double of $T$: $M = T \cup_{\text{Id}} T$. The manifold $M$ admits two PL structures, $M_A$ and $M_B$, such that $M_A$ admits a stably parallelisable smooth structure and $M_B$ is not smoothable (we explain this below). We first explain how to find examples of this type in dimensions 19 and higher. We observe that $M^{m}(K)$ is the boundary $T^{m}(K) \times [0, 1]$ and hence is a closed, stably parallelisable, topological manifold which contains $K$ as a retract. We observe also that these properties along with $K \to M$ being an 8-equivalence are all that is required in Morita’s arguments to show that PL-structures $A$ and $B$ exist as above. Now by [20] $K$ embeds into $\mathbb{R}^{19}$. Let $T^{19}(K)$ be a regular neighbourhood of such an embedding and let $M^{19}(K)$ be the boundary of $T^{19}(K) \times [0, 1]$. Then $M^{19}(K)$ is a closed, stably parallelisable, topological manifold containing $K$ as an 8-connected retract and hence admits PL structures $A$ and $B$ as above. We first prove the following

Lemma 4.1. For all the manifolds $M = M^{m}(K)$, $m \geq 19$, $M_A$ is stably parallelisable and $M_B$ is not smoothable. Hence $\text{pl}^3\nu(M) > 0$.

Proof. Morita’s arguments show the following. Consider the PL-structure, in the sense of surgery theory, $f : M_B \to M$, $f$ the identity map. This gives an element $[f]$ in the PL-structure set of $M$. As $M$ is simply connected, the PL-structure set injects into the normal invariant set and so we obtain an element $[f] \in [M,G/PL]$ (where we use $\text{Id}_M : M_A \to M$ as the base point to identify the normal invariants of $M$ with $[M,G/PL]$). Morita showed that $[f]$ does not belong to the image of the canonical map $q : [M,G/O] \to [M,G/PL]$.

Similarly to Section 2 the map $\delta^{PL}_M : [M,G/PL] \to [M,BPL]$ maps $[f]$ to the difference of the stable PL-tangent bundles $\tau^0(M_A) - \tau^0(M_B) \in KPL(M) = [M,BPL]$ and a similar statement holds for $\delta^{O}_M : [M,G/O] \to [M,BO]$ and the
smooth normal invariant set. There is a commuting diagram of long exact sequences

\[ \cdots \rightarrow [M, G] \rightarrow [M, G/O] \xrightarrow{\delta^0_M} [M, BO] \xrightarrow{BJ} [M, BG] \rightarrow \cdots \]

\[ \cdots \rightarrow [M, G] \rightarrow [M, G/PL] \xrightarrow{\delta^p_M} [M, BPL] \rightarrow [M, BG] \rightarrow \cdots \]

where \( BJ \) denotes the map induced on classifying spaces by the \( J \)-homomorphism \( J : O \rightarrow G \). Suppose that \( \tau^0(M_B) \) has a smooth reduction. Since \( \tau^0(M_A) \) is trivial this means that \( \delta^0_M(\alpha) \) lifts to \( x \in [M, BO] \). As \( BJ(x) \) is defined by the stable spherical fibration of \( M \) and this is trivial we conclude that \( x \in \text{Im}(\delta^0_M) \). Now a simple diagram chase ensures that \( y \in [M, G/O] \) can be chosen such that \( q(y) = [f] \), contradicting Morita's results. Hence \( \tau^0(M_B) \) cannot be smoothed, so it must be non-trivial and \( \text{span}^0(M_A) = m \), so \( \text{pl}^0v(M) > 0 \). \( \square \)

**Proof of Theorem 1.4.** Let \( M = M^{19}(K) \) and let \( M_0 \) be a stably parallelisable smooth structure refining \( M_A \). By the Bredon-Kosinski theorem we know that \( \tau(M_\alpha) \) is trivial if and only if \( \chi_2(M) = 0 \). However, we do not know \( \chi_2(M) \) so similarly to Theorem 3.1 we let \( N_\alpha = M_\alpha\#(S^1 \times S^{18}) \) where \( l = \chi_2(M) \) is 1 or 0. It follows that \( N_\alpha \) is stably parallelisable and that \( \chi_2(N) = 0 \). Thus \( N_\alpha \) is parallelisable and so \( N_A = M_A\#(S^1 \times S^{18}) \) is too. The manifold \( N \) also admits the \( PL \)-structure \( N_B = M_B\#(S^1 \times S^{18}) \) which is not smoothable. Hence \( \text{pl}v(N) > 0 \) and \( \text{pl}^0v(N) > 0 \). In dimensions \( m > 19 \) we take \( Q = N \times S^n \) for \( n > 0 \), for then \( Q \) admits a \( PL \)-structure \( Q_A = N_A \times S^n \) which is parallelisable and another \( PL \)-structure \( Q_B = N_B \times S^n \) which is not smoothable. Hence \( \text{pl}v(Q) > 0 \) and \( \text{pl}^0v(Q) > 0 \). \( \square \)

**References**

NON-INVARINANCE OF SPAN