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PERRON TYPE INTEGRATION ON $n$-DIMENSIONAL INTERVALS AS AN EXTENSION OF INTEGRATION OF STEPFUNCTIONS BY STRONG EQUICONVERGENCE

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0. INTRODUCTION

The integral of a stepfunction (on an $n$-dimensional interval $I$) is given by an elementary formula so that a linear functional $\Lambda$ on the linear space $S$ of stepfunctions is defined in the natural way. The idea of obtaining an integration theory directly as an extension of $\Lambda$ from $S$ to a wider space has been used several times. Monotone convergence was used by E. J. McShane in [8] and by F. Riesz and B. Sz.-Nagy in [9] to develop Lebesgue integration. If $S$ is made a normed space by putting $\|x\| = \Lambda(|x|)$ for $x \in S$, then the Lebesgue integral is the continuous extension of $\Lambda$ to the completion of $S$ (cf. the approach to Bochner integration in [10]). In the case $n = 1$ Lee and Chew in [7] proved that for every Denjoy integrable (in the restricted sense) $f: I \to \mathbb{R}$ there exists a sequence of stepfunctions $f_k: I \to \mathbb{R}$ such that $f_k$ is control convergent to $f$ for $k \to \infty$ (and, consequently, $(D_+) \int_I f \, dt$ is the limit of $\Lambda(f_k)$). On the other hand, any $g: I \to \mathbb{R}$ which is the limit in the control convergence of a sequence of Denjoy integrable $g_k$ is itself Denjoy integrable and $(D_+) \int_I g_k \, dt \to (D_+) \int_I g \, dt$ for $k \to \infty$. In this paper an analogue to the result of Lee and Chew is proved in the multidimensional case. The concept of integral involved is the strong $D$-integral which was introduced by the authors in [3]; in the onedimensional case the strong $\rho$-integral reduces to the Henstock-Kurzweil integral which is equivalent to the Denjoy and Perron integrals. The paper is organized in three sections. In Section 1 the relevant notions and results from [3] are recalled and a suitable convergence concept (strong $\rho$-equi convergence) is introduced. In Section 2

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we formulate the main result and establish some auxiliary facts. Section 3 is devoted to the proof of the main result.

1. THE STRONG $\varrho$-EQUICONVERGENCE

The same notation and concepts as in [3] will be used throughout the paper, in particular

$$I = [a, b] = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n$$

is a nondegenerate compact interval and

$$\varrho: I \times (0, \infty) \to [0, 1)$$

fulfils

\begin{align}
\limsup_{\sigma \to 0^+} \varrho(t, \sigma) &< 1 \quad \text{for } t \in I, \\
\inf \{ \varrho(t, \sigma); t \in I, \sigma > 0 \} &> 0,
\end{align}

(cf. [3], (2.1), (3.1)).

As usual, $\partial J$, $\text{Int} J$, $m(J)$ and $d(J)$ denote the boundary, the interior, the measure and the diameter of $J \subset \mathbb{R}^n$.

1.1 Definition. A function $f: I \to \mathbb{R}^n$ is called strongly $\varrho$-integrable if there exists an additive interval function $F$ such that for every $\varepsilon > 0$ there is a gauge $\delta$ such that

\begin{equation}
\sum_{\Delta, M} |f(t)m(M) - F(M)| \leq \varepsilon
\end{equation}

holds for every $\delta$-fine $\varrho$-regular system $\Delta = \{(t, J)\}$ and every set $\mathbb{M} = \{M\}$ of intervals such that the inclusion $M \subset J$ defines a one-to-one correspondence between $\Delta$ and $\mathbb{M}$. For brevity, such a set of intervals will be called an associated (with $\Delta$) family.

Of course, a strongly $\varrho$-integrable function $f$ is $\varrho$-integrable and $F$ is its primitive. For $n = 1$ the two integrals coincide and, moreover, reduce to the Perron integral.

A convergence theorem concerning a pointwise convergent sequence of strongly $\varrho$-integrable functions, which was proved in [3], Theorem 4.6, is the starting point for further convergence results.
1.2. **Theorem.** Let \( f_j : I \to \mathbb{R} \) be strongly \( q \)-integrable for \( j \in \mathbb{N} \), \( F_j \) being the primitives, let \( f : I \to \mathbb{R} \). Assume that

\[
(1.4) \quad \text{for every } \xi > 0 \text{ there is a gauge } \omega \text{ such that}
\]

\[
\left| \sum_{\Delta, M} f_j(t)m(M) - F_j(M) \right| \leq \xi
\]

for any \( \omega \)-fine \( q \)-regular system \( \Delta = \{(t, J)\} \), any \( j \) and any associated family of intervals \( M \),

and that

\[
(1.5) \quad f_j(t) \to f(t) \quad \text{for } t \in I, j \to \infty.
\]

Then \( f \) is strongly \( q \)-integrable and \( F_j(K) \to F(K) \) for \( j \to \infty \), \( K \subset I \) being an interval, \( F \) being the primitive of \( f \).

Our aim is to prove that any strongly \( q \)-integrable function \( g \) is the limit of a sequence of stepfunctions in a suitable convergence. Of course, \( f : I \to \mathbb{R} \) is called a stepfunction if there exist intervals \( J_1, J_2, \ldots, J_k \subset I \) such that \( \bigcup_i J_i = I \), \( \text{Int} J_i \cap \text{Int} J_l = \emptyset \) for \( i \neq l \) and if the restriction of \( f \) to any \( \text{Int} J_i \) is a constant function. The convergence from Theorem 1.2 cannot be directly applied to our purpose since the condition (1.5) is too restrictive. The assumption of pointwise convergence can be weakened to the assumption of convergence almost everywhere as a consequence of the following theorem, the proof of which is straightforward.

1.3 **Theorem.** Assume that there is \( N \subset I, m(N) = 0 \) such that

\[
(1.6) \quad g_j : I \to \mathbb{R} \quad \text{for } j \in \mathbb{N}, \quad g : I \to \mathbb{R},
\]

\[
(1.7) \quad f_j(t) = g_j(t), f(t) = g(t) \quad \text{for } t \in I \setminus N, \quad j \in \mathbb{N},
\]

\[
\quad f_j(t) = 0, f(t) = 0 \quad \text{for } t \in N, \quad j \in \mathbb{N}.
\]
Then the following two properties are equivalent:

(1.8) (i) \( g_j(t) \to g(t) \) for \( j \to \infty, \ t \in I \setminus N \),
(ii) there exists an additive interval function \( G_j \) on \( I \) for \( j \in \mathbb{N} \) and for every \( \eta > 0 \) there is a gauge \( \vartheta \) such that

\[
\sum_{\Delta, M} |g_j(t)m(M) - G_j(M)| \leq \eta
\]

for \( j \in \mathbb{N} \) and for any \( \vartheta \)-fine \( q \)-regular \( (I \setminus N) \)-tagged system \( \Delta = \{(t, J)\} \) and any associated family \( M \) of intervals \( M \), and

\[
\sum_{\Delta, M} |G_j(M)| \leq \eta
\]

for \( j \in \mathbb{N} \) and for any \( \vartheta \)-fine \( q \)-regular \( N \)-tagged system \( \Delta = \{(t, J)\} \) and any associated family \( M \) of intervals \( M \);

(1.9) \( f_j \) is strongly \( q \)-integrable for \( j \in \mathbb{N} \) and both (1.4) and (1.5) hold.

1.4. Remark. Let (1.8) hold. Then \( G_j \) is the primitive of \( g_j \) so that \( G_j' = g_j \) a.e. (cf. [3], Definition 2.6 and Theorem 2.8). Moreover, [3], Lemma 1.8 implies that (1.8) holds if \( N \) is replaced by \( N_1 \) provided \( N \subset N_1 \subset I, m(N_1) = 0 \), since \( \{g_j(t); j \in \mathbb{N}\} \) is bounded for \( t \in I \setminus N \).

1.5 Definition. Let \( g_j : I \to \mathbb{R} \) for \( j \in \mathbb{N}, g : I \to \mathbb{R} \). The sequence \( g_j \) is said to be strongly \( q \)-equiconvergent to \( g \) for \( j \to \infty \) if there exists \( N \subset I \) such that \( m(N) = 0 \) and (1.8) holds.

The next theorem is a direct consequence of Theorems 1.2 and 1.3.

1.6 Theorem ([3], Theorem 4.9). Let \( g_j : I \to \mathbb{R} \) for \( j \in \mathbb{N}, g : I \to \mathbb{R} \) and let \( g_j \) be strongly \( q \)-equiconvergent to \( g \) for \( j \to \infty \). Then \( g \) is strongly \( q \)-integrable and \( G_j(K) \to G(K) \) for \( j \to \infty \) and any interval \( K \subset I \) (\( G_j \) and \( G \) being the primitives of \( g_j \) and \( g \), respectively).

The concept of strong \( q \)-equiconvergence plays the crucial role in Theorem 1.3; observe that \( \{g_j(t); j \in \mathbb{N}\} \) need not be bounded if \( t \in N \).
Let $q_k : I \times (0, \infty) \rightarrow (0,1)$ fulfil (1.1), (1.2) for $k \in \mathbb{N}$ and let

\begin{equation}
q_k(t, \sigma) \geq q_{k+1}(t, \sigma) \quad \text{for } k \in \mathbb{N}, t \in I, \sigma > 0.
\end{equation}

**2.1 Theorem** (Main Result). Let $q : I \rightarrow \mathbb{R}$ be strongly $q_k$-integrable for $k \in \mathbb{N}$. Then there exists a sequence of stepfunctions $g_j : I \rightarrow \mathbb{R}$, $j \in \mathbb{N}$ such that $g_j$ is strongly $q_k$-equiconvergent to $q$ for $j \rightarrow \infty$ and every $k \in \mathbb{N}$.

The proof will be given in Section 3. Now we will only establish some auxiliary results.

**2.2 Remark.** If $q_k(t, \sigma) = q(t, \sigma)$ for $k \in \mathbb{N}$, then by Theorem 2.1 any strongly $q$-integrable function can be obtained as the limit of a strongly $q$-convergent sequence of stepfunctions $g_j$, $j \in \mathbb{N}$ and the primitive $G$ of $q$ is the limit of the sequence $G_j$ of the primitives of $g_j$. If we put $q_k(t, \sigma) = \frac{1}{k+1}$ for $t \in I$, $\sigma > 0, k \in \mathbb{N}$, then in an analogous way any $q$ may be obtained which is strongly $q$-integrable for every constant function $\varrho$, $\varrho \in (0,1)$. Such a $q$ need not be Perron integrable, see [5].

**2.3 Remark.** Let $n = 1$. The strong $q$-integral reduces to the Henstock-Kurweil integral (cf. [3], Note 4.3) independently of $q$ and the Henstock-Kurzweil integral is known to be equivalent both to the Perron integral and to the Denjoy integral. Lee and Chew in [6] introduced the control convergence for sequences of Denjoy integrable functions and proved the corresponding convergence theorem. In [7] they proved that every Denjoy integrable function is the limit of a control convergent sequence of stepfunctions. Another concept of convergence was introduced and studied by R. A. Gordon, [1], [2]; it follows from his results that any control convergent sequence is equiconvergent.

**2.4. Lemma.** Let $K \subset \mathbb{R}^n$ be a nondegenerate compact interval, let $0 < A < 1$ and reg $K \geq A$. Denote by $\Omega(S, r)$ the neighbourhood of a set $S$ with radius $r$.

Then there exists a constant $\kappa = \kappa(n) > 0$ such that

\begin{equation}
m(\Omega(\partial K, \zeta d(K))) \leq \kappa A^{1-n} \zeta m(K)
\end{equation}

provided $0 < \zeta < \frac{1}{2} A$.

**Proof.** Without loss of generality, let us assume

\begin{equation}
K = [0, a_1] \times [0, a_2] \times \ldots \times [0, a_n],
d(K) = a_1 \geq a_2 \geq \ldots \geq a_n \geq a_1 A.
\end{equation}
Then
\[
\Omega(\partial K, \zeta d(K)) = [-\zeta a_1, a_1 + \zeta a_1] \times [-\zeta a_1, a_2 + \zeta a_1] \times \ldots \times [-\zeta a_1, a_n + \zeta a_1] \\
\setminus \left[\zeta a_1, a_1 - \zeta a_1\right] \times \left[\zeta a_1, a_2 - \zeta a_1\right] \times \ldots \times \left[\zeta a_1, a_n - \zeta a_1\right],
\]
\[
m(\Omega(\partial K, \zeta d(K))) = (a_1 + 2\zeta a_1)(a_2 + 2\zeta a_1) \ldots (a_n + 2\zeta a_1) \\
- (a_1 - 2\zeta a_1)(a_2 - 2\zeta a_1) \ldots (a_n - 2\zeta a_1) \\
= 2 \sum (2\zeta a_1)a_{i_1} \ldots a_{i_{n-1}} + 2 \sum (2\zeta a_1)^3a_{i_1} \ldots a_{i_{n-3}} + \ldots \\
\leq \kappa a_1^n \zeta \leq \kappa A^{1-n} \zeta m(K).
\]

\[\square\]

2.5. Lemma. Let \(K, H_1, H_2, \ldots, H_p\) be nondegenerate compact intervals in \(\mathbb{R}^n\), let \(0 < A < 1, \zeta > 0, \text{ reg } H_i \geq A\) and \(d(H_i) \geq \zeta d(K)\) for \(i = 1, 2, \ldots, p\). Let \(H_1, H_2, \ldots, H_p\) be nonoverlapping.
Then
\[
\#\{H_i; H_i \cap K \neq \emptyset\} \leq 3^n A^{1-n} \max\{1, \zeta^{-n}\}.
\]

Proof. Assume that \(K\) has the form (2.3) from the proof of Lemma 2.4. If \(H_i \cap K \neq \emptyset\) then
\[
m(H_i \cap [-\zeta a_1, a_1 + \zeta a_1]^n) \geq (\zeta a_1)^n A^{n-1},
\]
hence
\[
\#\{H_i, H_i \cap K \neq \emptyset\} \leq (a_1 + 2\zeta a_1)^n / \zeta^n a_1^n A^{n-1} \leq 3^n A^{1-n} \max\{1, \zeta^{-n}\}.
\]
\[\square\]

Put
\[
V(t, \nu) = [t_1 - \nu, t_1 + \nu] \times \ldots \times [t_n - \nu, t_n + \nu] \quad \text{for } t \in \mathbb{R}^n, \nu > 0.
\]

The authors proved in [4], Corollary 2 and Theorem 1, the following result:
Let \(G\) be an additive function of interval on \(I, g \in \mathbb{R}, t \in \text{Int } I\). Let \(G\) be regularly differentiable to \(g\) at \(t\). Then for every \(\epsilon > 0\) there is \(r > 0\) such that
\[
|G(J) - gm(J)| \leq \epsilon (2\nu)^n
\]
for every interval \(J \subset V(t, \nu)\), where \(\nu \leq r\).
(2.4) can be rewritten in the following way. For \( L = [c_1, d_1] \times \ldots \times [c_n, d_n], s \in \mathbb{R}^n \) let \( \psi(s, L) \) be the smallest \( \nu \) such that \( L \subset V(s, \nu) \) (i.e. \( \psi(s, L) = \max\{|d_i - c_i|, |c_i - s_i|, |d_i - s_i|; i = 1, 2, \ldots, n\} \). Obviously \( \psi(t, J) \) can be substituted for \( \nu \) in (2.4) so that (2.4) can be replaced by

\[
(2.5) \quad \left| G(J) - gm(J) \right| (2\psi(t, J))^{-n} \leq \varepsilon,
\]

provided \( \psi(t, J) \leq r \).

From this result we prove in a standard manner:

**2.6. Lemma.** Let \( F \) be an additive function of interval on \( I \). Denote by \( D_F \) the set of such \( t \in I \) that \( F \) is regularly differentiable to (some) \( F'(t) \in \mathbb{R} \) at \( t \) and put \( N_F = I \setminus D_F \). Assume that \( m(N_F) = 0 \) and that \( F \) is continuous at any interval \( L \subset \text{Int} \ I \) (i.e. for every \( \varepsilon > 0 \) there is \( \eta > 0 \) such that \( |F(K) - F(L)| \leq \varepsilon \) for every interval \( K \subset I \) satisfying \( m(K \setminus L) + m(L \setminus K) \leq \eta \)). Put

\[
f(t) = F'(t) \quad \text{for} \quad t \in D_F.
\]

Then \( f \) is measurable and there exist

\[
N \subset I, \quad N \supset N_F \cup \partial I, \quad m(N) = 0, \quad \xi \in (0, 1/4),
\]

\[
\eta: [0, \xi] \to [0, 1) \quad \text{increasing}, \quad \eta(0) = 0, \quad \eta(\sigma) > \sigma \quad \text{for} \quad \sigma \in (0, \xi], \quad \lim_{\sigma \to 0^+} \eta(\sigma) = 0,
\]

\[
\omega: I \setminus N \to (0, \xi] \quad \text{measurable}, \quad V(t, \omega(t)) \subset I \quad \text{for} \quad t \in I \setminus N,
\]

such that

\[
(2.6) \quad |F(K) - f(t)m(K)| \leq \eta(\nu)\nu^n
\]

for every \( t \in I \setminus N, \nu \in (0, \omega(t)], K \subset \text{Int} \ V(t, \nu) \) (\( K \) being an interval).

**Proof.** Let \( I = [a_1, b_1] \times \ldots \times [a_n, b_n] \). \( f \) is measurable since

\[
f(t) = \lim_{\sigma \to 0^+} F(V(t, \sigma))(2\sigma)^{-n} \quad \text{for} \quad t \in D_F \cap \text{Int} \ I \quad \text{and} \quad F(V(t, \sigma)) \]

is continuous with respect to \( t \) on \( (a_1 + \sigma, b_1 - \sigma) \times \ldots \times (a_n + \sigma, b_n - \sigma) \). For \( M = [\alpha_1, \beta_1] \times \ldots \times [\alpha_n, \beta_n] \), \( t = (t_1, \ldots, t_n) \in \mathbb{R}^n \) put

\[
M(t) = [\alpha_1 + t_1, \beta_1 + t_1] \times \ldots \times [\alpha_n + t_n, \beta_n + t_n].
\]

For \( \sigma \in \left(0, \frac{1}{2} \min\{b_i - a_i; i = 1, 2, \ldots, n\}\right) \), \( t \in [a_1 + \sigma, b_1 - \sigma] \times \ldots \times [a_n + \sigma, b_n - \sigma] \), put

\[
(2.7) \quad \varphi_{\sigma}(t) = \sup \{|F(M(t)) - f(t)m(M(t))| \cdot (\psi(0, M))^{-n}; \quad M \subset V(0, \sigma), \alpha_i, \beta_i \text{ rational}\}.
\]

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\[ \varphi_\sigma : [a_1 + \sigma, b_1 - \sigma] \times \ldots \times [a_n + \sigma, b_n - \sigma] \to \mathbb{R} \] is measurable and it follows from (2.5) that
\[ \varphi_\sigma(t) \searrow 0 \quad \text{for } \sigma \searrow 0, t \in D_F \cap \text{Int } I. \]

For \( \lambda > 0 \) put
\[ E(\sigma, \lambda) = \{ t \in \text{Int } I ; V(t, \sigma) \subset I, \varphi_\sigma(t) \leq \lambda \}. \]

\( E(\sigma, \lambda) \) has the following properties:
\[ E(\sigma, \lambda) \text{ is measurable}, \]
\[ E(\sigma_1, \lambda) \supset E(\sigma_2, \lambda) \quad \text{for } 0 < \sigma_1 \leq \sigma_2, \]
\[ \bigcup_{\sigma > 0} E(\sigma, \lambda) = D_F \cap \text{Int } I, \]
\[ \lim_{\sigma \to 0} m(E(\sigma, \lambda)) = m(I). \]

Therefore, for \( i \in \mathbb{N} \) there exists \( \sigma_i > 0 \) such that
\[ \sigma_1 < 1/4, \quad 0 < \sigma_{i+1} < \frac{1}{2} \sigma_i, \quad m(I \setminus E(\sigma_i, 2^{-i})) \leq 2^{-i}. \]

Put
\[ N = I \setminus \bigcap_{i=1}^{\infty} E(\sigma_i, 2^{-i}) = I \setminus \bigcap_{j=1}^{\infty} E(\sigma_i, 2^{-i}). \]

It can be seen that \( m(N) = 0, \ N \supset N_F \cup \partial I. \)

Let \( \xi = \sigma_1. \) For \( \nu \in (0, \xi] \) there exists a unique \( i \in \mathbb{N} \) such that \( \nu \in (\sigma_{i+1}, \sigma_i] \); put \( \eta(\nu) = 2^{-i}. \) Since \( \sigma_i < 2^{-i-1} \) for \( i \in \mathbb{N}, \) we have \( \eta(\sigma_i) > \sigma_i \) for \( i \in \mathbb{N} \) and obviously \( \eta(\nu) > \nu \) for \( \nu \in (0, \xi] \) and \( \lim_{\nu \to 0^+} \eta(\sigma) = 0. \)

For \( t \in I \setminus N \) let \( h(t) \) be the smallest \( j \) such that \( t \in \bigcap_{i=j}^{\infty} E(\sigma_i, 2^{-i}). \) Put \( \omega(t) = \sigma_{h(t)}; \omega \) is measurable.

If \( t \in I \setminus N, \nu \in (0, \omega(t)], \) then \( \nu \in (\sigma_{k+1}, \sigma_k] \) for some \( k \geq h(t) \) so that \( t \in E(\sigma_k, 2^{-k}). \) Assume that \( M = [a_1, \beta_1] \times \ldots \times [a_n, \beta_n] \subset V(0, \nu), \) \( \alpha_i, \beta_i \) being rationals (i.e. \( \psi(0, M) < \nu \)). By (2.7) we have
\[ \eta(\nu) = 2^{-k} \geq |F(M(t)) - f(t)m(M(t))| \left( \psi(0, M) \right)^{-n} \]
\[ \geq |F(M(t)) - f(t)m(M(t))| \nu^{-n}. \]

Since \( F \) is continuous at any interval \( L \subset \text{Int } I, \) we obtain
\[ |F(M(t)) - f(t)m(M(t))| \leq \eta(\nu) \nu^n \]
for every \( M = [a_1, \beta_1] \times \ldots \times [a_n, \beta_n] \subset \text{Int } V(0, \nu), \) \( \alpha_i, \beta_i \) being reals, \( i = 1, 2, \ldots, n. \) (2.6) holds since any \( \text{Int } K \subset V(t, \nu) \) is equal to some \( M(t) \) with \( M \subset \text{Int } V(0, \nu). \) \( \square \)
2.7. Corollary. If \( t \in I \setminus N, t \in H \subset V(t, \omega(t)), K \subset H, H, K \) being intervals, then

\[
|F(K) - f(t)m(K)| \leq \eta(d(H))(d(H))^n;
\]

if, moreover, \( \text{reg} \, H \geq A \) with \( 0 < A < 1 \), then

\[
|F(K) - f(t)m(K)| \leq A^{1-n}\eta(d(H))m(H).
\]

The last inequality follows from the fact that the longest edge of \( H \) has the length \( d(H) \) while all the others have lengths not less than \( Ad(H) \).

2.8. Lemma. If \( g: I \to \mathbb{R} \) is Lebesgue integrable, then \( g \) is strongly \( \varrho \)-integrable.

Proof. \( g \) is \( \varrho \)-integrable by [3], Note 1.5. Let \( G \) be the primitive of \( g \). By [3], Theorem 3.2 we conclude that \( m(I \setminus D_G) = 0 \) and \( G' = g \) a.e. Since \( (L) \int_K f = (\varrho) \int_K f \) for every interval \( K \subset I \), the absolute continuity of the Lebesgue integral implies that (4.2) from [3] holds. Take into account that the correct version of condition (B) in [3], Theorem 4.12, is

\[
\text{(B)} \quad F \text{ is additive, } m(I \setminus D_F) = 0, (4.2) \text{ holds and } F' = f \text{ a.e.}
\]

(by a misprint the incorrect (4.4) appears instead of the correct (4.2) in condition B of [3], Theorem 4.12). Thus (B) from [3], Theorem 4.12, is fulfilled and it follows that \( g \) is strongly \( \varrho \)-integrable and \( G \) is its primitive.

3. Proof of main result

Let \( g: I \to \mathbb{R} \) be strongly \( \varrho_k \)-integrable for \( k \in \mathbb{N} \). For any interval \( L \subset I \) we put \( F(L) = (\varrho_k) \int_L g \); the right hand side is independent of \( k \) (cf. (2.1)) and \( F \) is called the primitive of \( g \). \( F \) is an additive function of interval on \( I \) and it is continuous at any interval \( L \subset \text{Int} \, I \) by [3], Theorem 2.1 (in [3], Theorem 2.1 the correct form of the assumption on \( L \) is \( L \subset I \) and the corresponding form of continuity of \( F \) at \( L \) is described even if \( L \not\subset \text{Int} \, I \)). By [3], Theorem 2.8 and Definition 2.6 \( F \) is regularly differentiable to \( F'(t) \) at every \( t \in D_F, m(N_F) = 0 \) where \( N_F = I \setminus D_F \) and \( F' = g \) a.e. The assumptions of Lemma 2.6 being fulfilled, let \( f, N, \xi, \eta, \nu, \omega \) have the same meaning as in Lemma 2.6 so that, in particular, (2.6) holds. If necessary the set \( N \) can be enlarged so that

\[
(3.1) \quad f(t) = g(t) \quad \text{for } t \in I \setminus N, m(N) = 0.
\]
Moreover, by [3], Theorem 4.5 we conclude that simultaneously

\begin{equation}
\sum_{\Xi, \mathcal{M}} |F(M)| \leq \lambda
\end{equation}

for every \( \gamma \)-fine \( \varrho \)-regular \( N \)-tagged system \( \Xi = \{(s, K)\} \) and any associated family of intervals \( \mathcal{M} \).

Let us choose a sequence \( \{\xi_k\} \) such that

\begin{equation}
\xi \geq \xi_1 \geq \xi_2 \geq \ldots > 0, \quad \lim_{k \to \infty} \xi_k = 0,
\end{equation}

\( ([0, \xi] \) being the domain of \( \eta )\). There is a measurable \( \omega_1 : I \setminus N \to (0, 1] \) such that

\begin{equation}
|f(t)| \leq \left[ \eta(2\omega_1(t)) \right]^{-\frac{1}{4n}}
\end{equation}

for \( t \in I \setminus N \). Let us set

\begin{equation}
\delta_k(t) = \min \left\{ \frac{1}{2} \xi_k, \omega_1(t), \omega(t) \right\}
\end{equation}

for \( t \in I \setminus N, \ k = 1, 2, 3, \ldots \), where \( \omega \) is from Lemma 2.6.

Referring to (3.2) let us choose \( \delta_k(t) \) for \( t \in N \) such that

\begin{equation}
\delta_k(t) \leq \frac{1}{2} \xi_k
\end{equation}

and

\begin{equation}
\sum_{\Xi, \mathcal{M}} |F(M)| \leq \xi_k
\end{equation}

provided \( \Xi = \{(s, K)\} \) is a \( \delta \)-fine \( \varrho \)-regular \( N \)-tagged system and \( \mathcal{M} = \{M\} \) is an associated family of intervals (i.e., the inclusion \( M \subset K \) defines a one-to-one correspondence between \( \mathcal{M} \) and \( \Xi \)).

For the basic interval \( I \) let us write

\[ I = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]. \]

If \( K = [c_1, d_1] \times [c_2, d_2] \times \ldots \times [c_n, d_n] \subset I \), then we write

\[ K^0 = [c, d_1]^0 \times [c_2, d_2]^0 \times \ldots \times [c_n, d_n]^0 \]
where
\[
[c_i, d_i]^0 = \begin{cases} 
[c_i, d_i] & \text{if } d_i < b_i \\
[c_i, d_i] & \text{if } d_i = b_i.
\end{cases}
\]

Now we can define the desired sequence of stepfunctions \( g_k \).

For \( k \in \mathbb{N} \) let us choose a \( \delta_k \)-fine \( \vartheta_1 \)-regular partition \( \Delta_k = \{(t, J)\} \) of the interval \( I \), and for \( s \in I \) let us set
\[
g_k(s) = \frac{F(J)}{m(J)}
\]
where \( J \) is such that \( (t, J) \in \Delta_k \) for some \( t \) and \( s \in J^0 \). (Evidently, there is a unique \( J \) with the property.)

The function \( g_k \) is integrable (cf. Lemma 2.8); let \( G_k \) be its primitive function, \( k \in \mathbb{N} \). For any interval \( M \subset I \) we have
\[
G_k(M) = \sum_{(t, J) \in \Delta_k} \frac{F(J)}{m(J)} m(J \cap M).
\]

The result to be established can be formulated as follows.

**3.1. Theorem.** For every \( i \in \mathbb{N} \) the sequence \( \{g_k\} \) is strongly \( \vartheta_i \)-equiconvergent to \( g \).

It is a consequence of the following two propositions.

**3.2. Proposition.** For every \( \varepsilon > 0 \) and \( i \in \mathbb{N} \) there are \( l_1 \in \mathbb{N} \) and \( \vartheta_1 : N \to (0, 1] \) such that
\[
\Sigma_1 = \sum_{\Theta, M} |G_k(M)| \leq \varepsilon
\]
for every \( \vartheta_1 \)-fine \( \vartheta_i \)-regular \( N \)-tagged system \( \Theta = \{(u, L)\} \), every associated family \( \mathbb{M} = \{M\} \) and every \( k \geq l_1 \).

**3.3. Proposition.** For every \( \varepsilon > 0 \) and \( i \in \mathbb{N} \) there are \( l_2 \in \mathbb{N} \) and \( \vartheta_2 : I \setminus N \to (0, 1] \) such that
\[
\Sigma_2 = \sum_{\Theta, M} |G_k(M) - g_k(u)m(M)| \leq \varepsilon
\]
for every \( \vartheta_2 \)-fine \( \vartheta_i \)-regular \( I \setminus N \)-tagged system \( \Theta = \{(u, L)\} \), every associated family \( \mathbb{M} \) and every \( k \geq l_2 \). Moreover,
\[
g_k(s) \to g(s) \quad \text{for } s \in I \setminus N, \ k \to \infty.
\]
3.4. Convention. Since $q_k$ fulfil (1.1), (1.2) and (2.1), for every $k \in \mathbb{N}$ there is $A_k$, $0 < A_k \leq q_k(t, \sigma) \leq q_1(t, \sigma)$, and we may assume $A_{k+1} \leq A_k$ for $k \in \mathbb{N}$. Hence $\text{reg} J \geq A_1 \geq A_i$ for $(t, J) \in \Delta_k$ and any $k, l \in \mathbb{N}$, and $\text{reg} L \geq A_i$ for $(u, L) \in \Theta$ since $\Theta$ is $q_i$-regular. The index $i \in \mathbb{N}$ is fixed throughout the proofs of Propositions 3.2, 3.3 and $A_1 \geq A_i$. Therefore we may and will write $A$ instead of $A_1$ and $A_i$, which implies that $\Delta_k$, $k \in \mathbb{N}$, as well as $\Theta$ are $A$-regular. To simplify the formulas we will also assume (without loss of generality) that $m(I) \leq 1$.

Proof of Proposition 3.2. Given $\varepsilon > 0$ and $i \in \mathbb{N}$, let us choose $j \in \mathbb{N}$ such that

\begin{equation}
(3.13) \quad j \geq i, \quad \xi_j(3 + 2 \cdot 3^n A^{1-n}) < \frac{1}{2} \varepsilon
\end{equation}

and denote

\begin{equation}
(3.14) \quad r(u) = \min\{k \in \mathbb{N}; \xi_k \leq \delta_j(u)\} \quad \text{for } u \in \mathbb{N}.
\end{equation}

For every $k \in \mathbb{N}$ there is an open set $U_k \subset \mathbb{R}^n$ such that $N \subset U_k$ and

\begin{equation}
(3.15) \quad m(U_k) \leq \xi_j \beta_k, \quad \beta_k = \frac{\min\{m(J); (t, J) \in \Delta_k\}}{\max\{1 + |F(J)|; (t, J) \in \Delta_k\}}.
\end{equation}

For every $k \in \mathbb{N}$ there is a gauge $\mu_k : N \to (0, 1]$ such that

\begin{equation}
(3.16) \quad V(u, \mu_k(u)) \subset U_k
\end{equation}

for $u \in N$. We choose a gauge $\varphi_1 : N \to (0, 1]$ satisfying the condition

\begin{equation}
(3.17) \quad \varphi_1(u) \leq \mu_k(u) \quad \text{for } k < r(u), \ u \in N,
\end{equation}

\begin{equation}
(3.18) \quad \varphi_1(u) \leq \delta_j(u) \quad \text{for } u \in N.
\end{equation}

Now we start estimates leading to (3.10). Let $\Theta = \{(u, L)\}$ be a $\varphi_1$-fine $q_1$-regular $N$-tagged system and let $\mathbb{M}$ be an associated family of intervals. For $k \in \mathbb{N}$ we have

$$
\Sigma_1 \leq \Gamma_1 + \Gamma_2 = \sum_{\Theta, M} |G_k(M)| + \sum_{L \not\in \Theta, M \neq \emptyset} |G_k(M)|.
$$

By virtue of (3.9) we obtain

$$
\Gamma_1 \leq \Gamma_3 + \Gamma_4 = \sum_{\Delta} \sum_{\Theta, M} |F(J)| \frac{m(M \cap J)}{m(J)}
$$

\begin{equation}
+ \sum_{\Delta} \sum_{\Theta, M} |F(J)| \frac{m(M \cap J)}{m(J)}.
\end{equation}
If \((t, J) \in \Delta_k\), \((u, L) \in \Theta\), \(k < r(u)\), \(L \subset J\) then \(M \cap J \subset L \subset U_k \cap J\) since \(u \in N\) (cf. \((3.16), (3.17)\)), and consequently (cf. \((3.15)\))

\[
\beta_k^{-1} \sum_{\Delta_k} \sum_{\begin{array}{l}
\exists (t, J) \in \Delta_k, L \subset J \\
k < r(u)
\end{array}} m(L) \leq \beta_k^{-1} \sum_{\Delta_k} m(J \cap U_k).
\]

We proceed to \(\Gamma_4\). For \((t, J) \in \Delta_k\) let \(\Omega(t, J)\) be the set of \((u, L) \in \Theta\) such that \(L \subset J, k \geq r(u)\). We have

\[
\Gamma_4 \leq \sum_{\Delta_k} |F(J)| \sum_{\Omega(t, J)} \frac{m(J \cap L)}{m(J)} \leq \sum_{\begin{array}{l}
\Delta_k \\
k \geq r(u)
\end{array}} |F(J)|.
\]

Obviously \(u \in N\) since \(\Theta\) is \(N\)-tagged. In the last sum we have \(u \in L \subset J\), \(d(J) \leq 2\delta_k(t) \leq \xi_j \leq \delta_j(u)\) by \((3.6)\) and \((3.14)\), hence \(J \subset V(u, \delta_j(u))\). From \((3.7)\) we conclude

\[
(3.19) \quad \Gamma_4 \leq \xi_j.
\]

(We apply \((3.7)\) for a system of pairs \((u, J)\) which is \(\delta_j\)-fine, \(\varphi_1\)-regular and \(N\)-tagged, putting \(M = J\).)

Now we shall estimate \(\Gamma_2\). Using \((3.9)\) we obtain

\[
\Gamma_2 \leq \Gamma_5 + \Gamma_6 = \sum_{\Theta, M} |F(M)|
\]

\[
+ \sum_{\Theta, M} \left| \sum_{\begin{array}{l}
\Delta_k \\
L \neq \emptyset, \forall (t, J) \in \Delta_k
\end{array}} \left( \frac{F(J)}{m(J)} m(M \cap J) - F(M \cap J) \right) \right|.
\]

\(\Theta\) is \(\varphi_1\)-regular and \(\varphi_1\)-fine; by the first inequality in \((3.13)\) it is \(\varphi_j\)-regular and by \((3.17)\) it is \(\delta_j\)-fine. \((3.7)\) can be applied with \(k\) replaced by \(j\), so that

\[
(3.20) \quad \Gamma_5 \leq \xi_j.
\]

Further, we can write

\[
\Gamma_6 \leq \Gamma_7 + \Gamma_8 = \sum_{\Theta, M} \left| \sum_{\begin{array}{l}
\Delta_k \\
L \neq \emptyset, \forall (t, J) \in \Delta_k
\end{array}} \left( \frac{F(J)}{m(J)} m(M \cap J) - F(M \cap J) \right) \right|
\]

\[
+ \sum_{\Theta, M} \left| \sum_{\begin{array}{l}
\Delta_k \\
L \neq \emptyset, \forall (t, J) \in \Delta_k
\end{array}} \left( \frac{F(J)}{m(J)} m(M \cap J) - F(M \cap J) \right) \right|.
\]

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The first sum can be divided into three terms:

\[
\Gamma_7 \leq \Gamma_9 + \Gamma_{10} + \Gamma_{11} = \sum_{t \in \Delta_k} \left| \frac{F(J)}{m(J)} \right| \sum_{\Theta \in \mathcal{M}} m(M \cap J) \\
+ \sum_{\Theta \in \mathcal{M}} \sum_{d(J) \geq d(L)} |F(M \cap J)| + \sum_{\Theta \in \mathcal{M}} \sum_{d(L) > d(J)} |F(M \cap J)|.
\]

By (3.7) we obtain

\[
(3.21) \quad \Gamma_9 \leq \xi_k
\]

since the inner sum (for fixed \((t, J) \in \Delta_k\)) does not exceed \(m(J)\). Further,

\[
\Gamma_{10} \leq \sum_{\Theta} \sup \{|F(K)|; K \subset L\} \cdot \# \{(t, J) \in \Delta_k; J \cap L \neq \emptyset, d(J) \geq d(L)\}.
\]

By Lemma 2.5 the number of elements of \(\Delta_k\) on the righthand side of the inequality has the upper bound \(3^n A^{1-n}\) (since \(\zeta = 1\)), which together with (3.17) and (3.7) yields

\[
(3.22) \quad \Gamma_{10} \leq 3^n A^{1-n} \sum_{\Theta} \sup \{|F(K)|; K \subset L\} \leq 3^n A^{1-n} \xi_j.
\]

Similarly, with the role of \(\Delta_k\) and \(\Theta\) interchanged, we obtain

\[
(3.23) \quad \Gamma_{11} \leq \sum_{\Delta_k; t \in N} \sup\{|F(H); H \subset J\} \cdot \# \{(u, L) \in \Theta; L \cap J \neq \emptyset, d(L) > d(J)\} \leq 3^n A^{1-n} \xi_k.
\]

Returning to \(\Gamma_8\), we note that (2.6) and (3.5) yield for \((t, J) \in \Delta_k\) and \(t \in I \setminus N\)

\[
(3.24) \quad |F(J) - f(t)m(J)| \leq A^{1-n} \eta(d(J))m(J), \\
|F(M \cap J) - f(t)m(M \cap J)| \leq A^{1-n} \eta(d(J))m(J),
\]

hence

\[
(3.25) \quad \left| \frac{F(J)}{m(J)} m(M \cap J) - F(M \cap J) \right| \leq 2 A^{1-n} \eta(d(J))m(J).
\]
Consequently,
\[
\Gamma_8 \leq \Gamma_{12} + \Gamma_{13} = \sum_{\Delta_k \in \mathcal{I} \setminus N} \sum_{t \in \mathcal{I} \setminus T} \left| \frac{F(J)}{m(J)} m(M \cap J) - F(J \cap M) \right|
\]
\[
+ \sum_{\Theta, M} \sum_{\Delta_k \in \mathcal{I} \setminus N} \left| \frac{F(J)}{m(J)} m(M \cap J) - F(M \cap J) \right|.
\]

Estimating \( \Gamma_{12} \) with help of (3.25) and Lemma 2.5 we arrive at
\[
\Gamma_{12} \leq \sum_{\Delta_k \in \mathcal{I} \setminus N} 2A^{1-n} \eta(d(J))m(J)
\]
\[
\times \# \left\{ (u, L) \in \Theta; L \cap J \neq \emptyset, d(L) \geq \left[ \eta(d(J)) \right]^{\frac{3}{4n}} d(J) \right\}
\]
\[
\leq 2A^{1-n} \sum_{\Delta_k \in \mathcal{I} \setminus N} \eta(d(J))m(J)3^n A^{1-n} \left[ \eta(d(J)) \right]^{-\frac{3}{4}}
\]
and by (3.5) we obtain
\[
(3.26) \quad \Gamma_{12} \leq 2 \cdot 3^n A^{2-2n} \left[ \eta(\xi_k) \right]^\frac{1}{4}.
\]

In order to estimate \( \Gamma_{13} \) we use the first inequality (3.24):
\[
\Gamma_{13} \leq \Gamma_{14} + \Gamma_{15} + \Gamma_{16} = \sum_{\Delta_k \in \mathcal{I} \setminus N} |f(t)| \sum_{\Theta, M} \left[ \sum_{\Delta_k, t \in \mathcal{I} \setminus N} m(M \cap J)
\]
\[
+ A^{1-n} \sum_{\Delta_k \in \mathcal{I} \setminus N} \sum_{\Theta, M} \eta(d(J))m(M \cap J) + \sum_{\Theta, M} \sum_{\Delta_k \in \mathcal{I} \setminus N} \left| F(M \cap J) \right|.
\]

Now (3.4), (3.5) imply
\[
\Gamma_{14} \leq \sum_{\Delta_k} \left[ \eta(d(J)) \right]^{-\frac{1}{4n}} \sum_{\Theta, M} \left[ \sum_{\Delta_k \in \mathcal{I} \setminus N} m(L \cap J)
\]
\[
+ \sum_{\Theta, M} \sum_{\Delta_k \in \mathcal{I} \setminus N} \left[ \eta(d(J)) \right]^{-\frac{1}{4n}} d(L) \leq \left[ \eta(d(J)) \right]^{\frac{3}{4n}} d(J)
\]
and, assuming
\[
(3.27) \quad \left[ \eta(\xi_k) \right]^\frac{1}{4n} < \frac{1}{2} A
\]
we conclude by (3.4), (3.5) and Lemma 2.4

\[ (3.28) \quad \Gamma_{14} \leq \sum_{\Delta_k} [\eta(d(J))]^{-\frac{1}{n}} \kappa A^{1-n} m(J) \eta(d(J))^{\frac{1}{2n}} \]

\[ \leq \kappa A^{1-n} [\eta(\xi_k)]^{\frac{1}{2n}}. \]

Evidently,

\[ (3.29) \quad \Gamma_{15} \leq A^{1-n} \sum_{\Delta_k} \eta(d(J)) m(J) \leq A^{1-n} \eta(\xi_k) \]

and finally, by Lemma 2.5 and (3.7),

\[ (3.30) \quad \Gamma_{16} \leq \sum_{\varnothing} \sup\{|F(K)|; K \subset L\} \#\{(t, J) \in \Delta_k; J \cap L \neq \varnothing, d(J) > d(L)\} \]

\[ \leq 3^n A^{1-n} \xi_j. \]

Putting together the estimates (3.18)-(3.23), (3.26) and (3.28)-(3.30) we obtain

\[ \Sigma_1 \leq 3 \xi_j + \xi_k + 3^n A^{1-n} \xi_j + 3^n A^{1-n} \xi_k \]

\[ + 2 \cdot 3^n A^{2-2n} \eta(\xi_k) \frac{1}{2} + \kappa A^{1-n} [\eta(\xi_k)]^{\frac{1}{2n}} \]

\[ + A^{1-n} \eta(\xi_k) + 3^n A^{1-n} \xi_j. \]

This together with (3.13) implies that Proposition 3.2 holds for \( k \geq l_1 \) where \( l_1 \) is such that (3.27) and

\[ \xi_k (1 + 3^n A^{1-n}) + 2 \cdot 3^n A^{2-2n} \eta(\xi_k) \frac{1}{2} + \kappa A^{1-n} [\eta(\xi_k)]^{\frac{1}{2n}} + A^{1-n} \eta(\xi_k) < \frac{1}{2} \varepsilon \]

hold for every \( k \geq l_1 \). \( \square \)

**Proof of Proposition 3.3.** Given \( \varepsilon > 0 \) and \( i \in \mathbb{N} \), let us choose \( h \in \mathbb{N} \) such that

\[ (3.31) \quad \xi_h + A^{1-n} \eta(\xi_h) + 3^n A^{2-2n} \eta(2\xi_h) < \frac{1}{2} \varepsilon \]

and denote

\[ (3.32) \quad R(s) = \min\{k \in \mathbb{N}; 2\xi_k \leq \delta_h(s)\} \quad \text{for} \ s \in I \setminus N. \]

For \( k \in \mathbb{N} \) let a gauge \( \gamma_k : I \setminus N \to (0,1] \) be such that

\[ (3.33) \quad \sum_{\Xi, M} |G_k(M) - g_k(s)m(M)| \leq \xi_h \]

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is satisfied provided $\Xi = \{(s, K)\}$ is a $\gamma_k$-fine $\vartheta_t$-regular $(I \setminus N)$-tagged system and $\mathcal{M}$ an associated family of intervals (cf. Lemma 2.8). We choose a gauge $\vartheta_2 : I \setminus N \rightarrow (0, 1]$ satisfying the condition

\begin{align}
\vartheta_2(s) &\leq \gamma_k(s) \quad \text{for } k < R(s), \ s \in I \setminus N, \\
\vartheta_2(s) &\leq \delta_h(s) \quad \text{for } s \in I \setminus N.
\end{align}

According to the definition of the functions $g_k$ we have $g_k(s) = F(K)/m(K)$ where $(z, K) \in \Delta_k$, $s \in K^0$. If, moreover, $s \in I \setminus N$, $k > R(s)$, then $K \subset V(z, \delta_k(z))$, $d(K) \leq 2\delta_k(z) \leq \xi_k \leq \frac{1}{2}\delta_h(s) \leq \omega(s)$ (see (3.5) and (3.32)), hence $K \subset V(s, \delta_h(s)) \subset V(s, \omega(s))$, and putting $H = K$ in (2.6) we obtain

$$
|F(K) - f(s)m(K)| \leq A^{1-n}\eta(d(K))m(K)
$$

and consequently,

\begin{align}
|g_k(s) - f(s)| &\leq A^{1-n}\eta(\xi_k).
\end{align}

Now we start estimates leading to (3.11). Let $\Theta = \{(u, L)\}$ be a $\vartheta_2$-fine $\vartheta_t$-regular $(I \setminus N)$-tagged system and let $\mathcal{M}$ be an associated family of intervals. For $k \in \mathbb{N}$ we have (cf. (3.11))

$$
\Sigma_2 \leq \Gamma_{17} + \Gamma_{18} = \sum_{\Theta, \mathcal{M}} |G_k(M) - g_k(u)m(M)| + \sum_{\Theta, \mathcal{M}} |G_k(M) - g_k(u)m(M)|.
$$

By (3.34) and (3.33) we have

\begin{align}
\Gamma_{17} &\leq \xi_h.
\end{align}

Further, we can write

$$
\Gamma_{18} \leq \Gamma_{19} + \Gamma_{20} = \sum_{\Theta, \mathcal{M}} |f(u) - g_k(u)|m(M) + \sum_{\Theta, \mathcal{M}} |G_k(M) - f(u)|m(M)
$$

and, by virtue of (3.35) we have

\begin{align}
\Gamma_{19} &\leq A^{1-n}\eta(\xi_k)
\end{align}

since $u \in I \setminus N$. Proceeding to $\Gamma_{20}$ we estimate it as

$$
\Gamma_{20} \leq \Gamma_{21} + \Gamma_{22} = \sum_{\Theta, \mathcal{M}} |F(M) - f(u)m(M)|
$$

$$
+ \sum_{\Theta, \mathcal{M}} \sum_{\Delta_k} \sum_{k \geq R(u)} \left| \frac{F(J)}{m(J)}m(M \cap J) - F(M \cap J) \right|.
$$
Applying (2.9) with \( M, u, L \) respectively instead of \( K, t, H \) and then (3.34) we conclude \((m(I) \leq 1 \text{ by Convention 3.4})\)

\[(3.38) \quad \Gamma_{21} \leq A^{1-n} \eta(\xi_k).\]

The term \( \Gamma_{22} \) is divided into three sums:

\[
\Gamma_{22} \leq \Gamma_{23} + \Gamma_{24} + \Gamma_{25} = \sum_{\Theta, M} \sum_{\Delta_k; \Delta_k \ni R(u)} \left| \sum_{J \ni \Theta \cap M \cap J} \frac{F(J)}{m(J)} m(M \cap J) - F(M \cap J) \right|
\]

\[
+ \sum_{\Theta, M} \sum_{\Delta_k; \Delta_k \ni R(u)} \left| \sum_{J \ni \Theta \cap M \cap J} \frac{F(J)}{m(J)} m(M \cap J) - F(M \cap J) \right|
\]

\[
+ \sum_{\Theta, M} \sum_{\Delta_k; \Delta_k \ni R(u)} \left| \sum_{J \ni \Theta \cap M \cap J} \frac{F(J)}{m(J)} m(M \cap J) - F(M \cap J) \right|
\]

where

\[
\Gamma_{23} \leq \Gamma_{26} + \Gamma_{27} = \sum_{\Theta, M} \sum_{\Delta_k; \Delta_k \ni R(u)} \left| \frac{F(J)}{m(J)} - f(u) \right| m(M \cap J)
\]

\[
+ \sum_{\Theta, M} \sum_{\Delta_k; \Delta_k \ni R(u)} \left| f(u) m(M \cap J) - F(M \cap J) \right|
\]

Let us estimate \( \Gamma_{26} \). The partition \( \Delta_k \) is \( \delta_k \)-fine so that \( d(J) \leq \xi_k \) by (3.5) and also \( d(L) \leq \xi_k \). Moreover, \( k \geq R(u) \) implies \( \xi_k \leq \frac{1}{2} \delta_h(u) \) (cf. (3.32)). If a summand in \( \Gamma_{26} \) is nonzero then necessarily \( L \cap J \neq \emptyset \), which implies \( J \subset V(u, d(L) + d(J)) \subset V(u, 2\xi_k) \subset V(u, \delta_h(u)) \subset V(u, \omega(u)) \) (see (3.5)). Replacing \( K, t, \nu \) in (2.6) by \( J, u, 2d(J) \), respectively, we obtain

\[
|F(J) - f(u)m(J)| \leq \eta(2d(J)) [2d(J)]^n
\]

and taking into account that \( m(J) \geq A^{n-1} (d(J))^n \), we conclude that

\[
|F(J) - f(u)m(J)| \leq 2^n A^{1-n} \eta(2d(J)) m(J)
\]

and, eventually,

\[
(3.39) \quad \Gamma_{26} \leq 2^n A^{1-n} \eta(2\xi_k).
\]
For the nonvanishing summands of $\Gamma_{27}$ we again use the fact that $J \cap L \neq \emptyset$, $u \in I \setminus N$, $M \cap J \subset L$, hence (2.9) yields with $t = u$, $K = M \cap J$, $H = L$

$$|f(u)m(M \cap J) - F(M \cap J)| \leq A^{1-n} \eta(d(L))m(L)$$

since $L$ is $A$-regular (cf. Convention 3.4). By Lemma 2.5 we find

$$\Gamma_{27} \leq A^{1-n} \sum_{\Theta, M} \eta(d(L))m(L)$$

(3.40)

$$\# \{(t, J) \in \Delta_k; J \cap L \neq \emptyset, d(J) \geq d(L)\} \leq 3^n A^{2-2n} \eta(2\xi_k)$$

(see (3.34)).

In the sum $\Gamma_{24}$ we consider only terms with $t \in I \setminus N$, therefore we can use (2.9) replacing $H$ with $J$ and $K$ with $J$ or $J \cap M$. We arrive at the inequalities

$$|F(J) - f(t)m(J)| \leq A^{1-n} \eta(d(J))m(J),$$

$$|F(M \cap J) - f(t)m(M \cap J)| \leq A^{1-n} \eta(d(J))m(J),$$

which yield

$$\left| \frac{F(J)}{m(J)} m(M \cap J) - F(M \cap J) \right| \leq 2A^{1-n} \eta(d(J))m(J)$$

and, eventually, Lemma 2.5 implies

$$\Gamma_{24} \leq 2A^{1-n} \sum_{\Delta_k} \eta(d(J))m(J)$$

(3.41)

$$\times \# \{(u, L) \in \Theta; L \cap J \neq \emptyset, d(L) > d(J)\} \leq 2A^{1-n} \sum_{\Delta_k} \eta(d(J))m(J) \cdot 3^n A^{1-n}$$

$$\leq 2 \cdot 3^n A^{2-2n} \eta(\xi_k)$$

since $d(J) \leq 2\delta_k(t) \leq \xi_k$ by (3.5).

Finally, we write

$$\Gamma_{25} \leq \Gamma_{28} + \Gamma_{29} = \sum_{\Theta, M} \sum_{\Delta_k; t \in N} \frac{|F(J)|}{m(J)} m(M \cap J) + \sum_{\Theta, M} \sum_{\Delta_k; t \in N; d(L) > d(J)} |F(M \cap J)|.$$

By (3.7) we have

$$\Gamma_{28} \leq \sum_{\Delta_k, t \in N} |F(J)| \sum_{\Theta} \frac{m(L \cap J)}{m(J)} \leq \sum_{\Delta_k; t \in N} |F(J)| \leq \xi_k,$$

(3.42)
and again by (3.7) and Lemma 2.5 we conclude

\[
\Gamma_29 \leq \sum_{\Delta_k; \, \iota \in \mathbb{N}} \sup \{|F(K)|; \, K \subset J\} \\
\times \#\{(u, L) \in \Theta; \, L \cap J \neq \emptyset, d(L) > d(J)\} \\
\leq \xi_k \cdot 3^n A^{1-n}.
\]

Combining (3.36)–(3.43) we obtain

\[
\Sigma_2 \leq \xi_h + A^{1-n} \eta(\xi_k) + A^{1-n} \eta(\xi_h) \\
+ 2^n A^{1-n} \eta(2\xi_k) + 3^n A^{2-2n} \eta(2\xi_h) \\
+ 2 \cdot 3^n A^{2-2n} \eta(\xi_k) + \xi_k + 3^n A^{1-n} \xi_k.
\]

Since \( h \) satisfies (3.31), it is sufficient to choose \( l_2 \) such that

\[
(A^{1-n} + 2 \cdot 3^n A^{2-2n}) \eta(\xi_k) + 2^n A^{1-n} \eta(2\xi_k) + (1 + 3^n A^{1-n}) \xi_k < \frac{1}{2} \varepsilon
\]

is satisfied for all \( k \geq l_2 \). (3.11) holds and the proof of Proposition 3.3 is complete, since (3.12) holds by (3.25) and (3.1).

\[\square\]

References


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