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TOLERANCES ON q-LATTICES

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The concept of a q-lattice was introduced for the first time in [1] and some of its congruence properties were studied in [2] and [3]. Recall that an algebra $(A; \wedge, \vee)$ with two binary operations is a *q-lattice* if it satisfies the following axioms:

(associativity)	$x \vee (y \vee z) = (x \vee y) \vee z,$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z,$
(commutativity)	$x \vee y = y \vee x,$	$x \wedge y = y \wedge x,$
(weak absorption)	$x \vee (x \wedge y) = x \vee x,$	$x \wedge (x \vee y) = x \wedge x,$
(weak idempotence)	$x \vee (y \vee y) = x \vee y,$	$x \wedge (y \wedge y) = x \wedge y,$
(equalization)	$x \vee x = x \wedge x.$	

If, moreover, it satisfies also distributivity:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$

the q-lattice is called *distributive.*

In every q-lattice A we can distinguish two sorts of elements: *idempotents*, i.e. such $x \in A$ for which $x = x \vee x$ (and hence also $x = x \wedge x$), and *non-idempotents* (i.e. $x \neq x \vee x$). Denote by S_A the so called *skeleton of* A, i.e. S_A is the set of all idempotents of A. It is known (see e.g. [1] or [3]) that S_A is a sub-q-lattice of A which is a sublattice with respect to the *induced quasiorder* Q:

$$\langle a, b \rangle \Leftrightarrow a \vee b = b \vee b,$$

i.e. $Q \cap S_A^2$ is an order on S_A (for some details, see [1]).

The non-idempotents occur in A in the so called cells: a subset $C_x \subseteq A$ is called a *cell* (with the idempotent x) if $\operatorname{card} C_x > 1$ and for each a, $b \in C_x$, $a \vee a = b \vee b$ $(= x)$.

The aim of this paper is to characterize q-lattices with distributive lattices of tolerances.

By a *tolerance* on $(A;\wedge,\vee)$ we mean a reflexive and symmetric binary relation on A satisfying the substitution property with respect operations $\vee$ and $\wedge$. Denote by $\operatorname{Tol} A$ the lattice of all tolerance of $(A;\wedge,\vee)$ (for some details on $\operatorname{Tol} A$ and the basic properties of tolerances, see the monograph [4]). In particular, denote by ω (or ι) the least (greatest) element of $\operatorname{Tol} A$, i.e. ω is the identity relation on A and $\iota = A \times A$. If a, $b \in A$ denote by $T(a,b)$ the least tolerance on $(A;\wedge,\vee)$ containing the pair $\langle a,b\rangle$.

An algebra A is called *tolerance trivial* if every tolerance on A is a congruence, i.e. if $\operatorname{Tol} A = \operatorname{Con} A$ (e.g. every boolean or every relative complementary lattice is tolerance trivial, see [4]).

Proposition. *If a q-lattice $(A;\wedge,\vee)$ has at least one non-idempotent element and at least two idempotents, then it is not tolerance trivial.*

Proof. Suppose that $(A;\wedge,\vee)$ has at least one non-idempotent. Then $(A;\wedge,\vee)$ contains at least one cell C. Let S_A be the skeleton of A. Define a binary relation T on A as follows: $\langle x,y\rangle \in T$ if and only if either x, $y \in C$ or x, $y \in S_A$ or $x = y$. It is an easy exercise to show that $T \in \operatorname{Tol} A$. Let x be the unique idempotent of C, let $y \neq x$ be an idempotent of A and z a non-idempotent of C. Then x, $y \in S_A$, i.e. $\langle x,y\rangle \in T$, x, $z \in C$, i.e. $\langle x,z\rangle \in T$ but $\langle y,z\rangle \notin T$ which proves $T \notin \operatorname{Con} A$. □

Lemma. *Let $(A;\wedge,\vee)$ be a q-lattice and C its cell with the unique idempotent c.*

(i) *Let $p(x_1,\dots,x_n)$ be an n-ary term which is not a projection over $(A;\wedge,\vee)$, and let a, a_1, …, $a_n = A$ and $a_i \in C$ for some i. If $a = p(a_1,\dots,a_n)$ then*

$$a = p(a_1,\dots,a_{i-1},c,a_{i+1},\dots,a_n).$$

(ii) *If $T \in \operatorname{Tol} A$, $b \in C$, a is an idempotent and $\langle a,b\rangle \in T$, then $\langle a,c\rangle \in T$.*

Proof. (i) If p is not a projection then p is a composition of operations $\vee$ and $\wedge$. Hence, $a = p(a_1,\dots,a_n)$ is an idempotent of $(A;\wedge,\vee)$. By induction over the rank of p, suppose first $p(x_1,\dots,x_n) = x_1 \vee x_2$, i.e. $a = a_1 \vee a_2$. If $a_1 \in C$, then clearly $a_1 \vee a_2 = c \vee a_2$; similarly for $i = 2$ and dually for the operation $\wedge$. By induction, we obtain the first assertion.

(ii) If $\langle a,b\rangle \in T$ and $b \in C$ and c is an idempotent of C, then $b \vee b = c$ and hence $\langle a,c\rangle = \langle a \vee a, b \vee b\rangle \in T$. □

Theorem 1. *Let $(A;\wedge,\vee)$ be a q-lattice with just one cell C, let S_A be its skeleton. If $\operatorname{Tol} S_A$ is distributive then also $\operatorname{Tol} A$ is distributive.*

Proof. Let R, S, $T \in \operatorname{Tol} A$ and x, $y \in A$. Suppose $\langle x,y\rangle \in R \wedge (S \vee T)$. Then $\langle x,y\rangle \in R$ and there exists an n-ary term $p(x_1,\dots,x_n)$ such that $x = p(a_1,\dots,a_n)$, $y = p(b_1,\dots,b_n)$, where $\langle a_i,b_i\rangle \in S$ or $\langle a_i,b_i\rangle \in T$, see e.g. [4].

(1) If at least one of the elements x, y is non-idempotent, then it cannot be the result of an operation, i.e. p is a projection, therefore $p(a_1, \ldots, a_n) = pr_i(a_1, \ldots, a_n) = a_i$, $p(b_1, \ldots, b_n) = pr_i(b_1, \ldots, b_n) = b_i$, thus $\langle x, y\rangle = \langle a_i, b_i\rangle$ and hence $\langle x, y\rangle \in S$ or $\langle x, y\rangle \in T$, i.e. $\langle x, y\rangle \in R \wedge S$ or $\langle x, y\rangle \in R \wedge T$, proving $\langle x, y\rangle \in (R \wedge S) \vee (R \wedge T)$.

(2) Suppose both x, y are idempotents. Then x, $y \in S_A$. By the Lemma, we can substitute all non-idempotents among a_1, …, a_n, b_1, …, b_n by a unique idempotent $c \in C$ because $(A; \wedge, \vee)$ has just one cell C.

If $\langle a_i, b_i\rangle \in S$ and b_i is a non-idempotent and a_i an idempotent, then $\langle a_i, c\rangle \in S$. Analogously for the converse case and also for T. If both a_i, b_i are non-idempotents, we have $\langle c, c\rangle \in S$ analogously for T. By the Lemma,

$$x = p(a_1^0, \ldots, a_n^0), \quad y = p(b_1^0, \ldots, b_n^0)$$

where

$$\begin{aligned} a_i^0 &= a_i \quad \text{if } a_i \text{ is an idempotent and} \\ a_i^0 &= c \quad \text{in the opposite case,} \\ b_i^0 &= b_i \quad \text{if } b_i \text{ is an idempotent and} \\ b_i^0 &= c \quad \text{in the opposite case.} \end{aligned}$$

By the Lemma, $\langle a_i^0, b_i^0\rangle \in S^0$ or T^0, where $S^0 = S \cap (S_A \times S_A)$, $T^0 = T \cap (S_A \times S_A)$ are the restrictions of S or T onto the skeleton. But x, $y \in S_A$ implies also $\langle x, y\rangle \in R^0 = R \cap (S_A \times S_A)$. Since $\operatorname{Tol} S_A$ is distributive, we have

$$\langle x, y\rangle \in (R^0 \wedge S^0) \vee (R^0 \wedge T^0) \subseteq (R \wedge S) \vee R \wedge T).$$

Distributivity is proved in both the cases. □

Corollary. *Let $(A; \wedge, \vee)$ be a distributive q-lattice with at most one cell. Then* $\operatorname{Tol} A$ *is distributive.*

P r o o f. By [5], for every distributive lattice L, $\operatorname{Tol} L$ is also distributive. If $(A; \wedge, \vee)$ has no cell then $(A; \wedge, \vee)$ is a lattice and $\operatorname{Tol} A$ is therefore distributive. If $(A; \wedge, \vee)$ has just one cell then S_A is a distributive lattice and hence $\operatorname{Tol} S_A$ is distributive. By Theorem 1 we are done. □

Remark 1. If $(A; \wedge, \vee)$ is a q-lattice and C is its cell and S_A its skeleton, then for each $c \in C$ and each $x \in S_A$ there exists a tolerance $T \in \operatorname{Tol} A$ given by

$$T = \omega \cup \{\langle c, x\rangle, \langle x, c\rangle\} \cup (S_A \times S_A).$$

If $\operatorname{Tol} S_A = \{\omega_s, \iota_s\}$ only (i.e. S_A is tolerance simple, see [4]), then all tolerances on A are determined only by the pairs $\langle c, x\rangle$ as was shown before and by all tolerances on C. This is illustrated in the following

Example 1. Let A be a q-lattice with the diagram in Fig. 1.

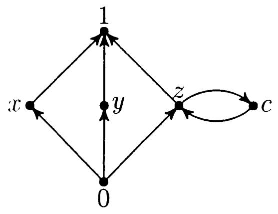

Fig. 1

It has just one cell $\{z, c\} = C$, z is an idempotent in C. It is evident that $\operatorname{Tol} S_A = \{\omega_s, \iota_s\}$, where $S_A = \{0, x, y, z, 1\}$. Henceforth, for every subset $B \subseteq S_A$ there exists a tolerance $T_B \in \operatorname{Tol} A$ given by

$$T_B = \omega \cup (S_A \times S_A) \cup \{\langle b, c\rangle, \langle c, b\rangle ; b \in B\}.$$

Since $\operatorname{card} S_A = 5$ we have 2^5 of such subsets; for $B = \emptyset$ we have $T_0 = \omega \cup (S_A \times S_A)$, i.e. it is the congruence collapsing S_A and having two blocks, namely S_A and $\{c\}$, i.e. $T_0 = \theta(0, 1)$. Moreover, $\operatorname{Tol} A$ also contains $\theta(z, c)$ collapsing the cell $C = \{z, c\}$ only and ω and ι, then $\operatorname{Tol} A$ has $2^5 + 2 = 34$ elements, see Fig. 2 (I denotes the two element lattice):

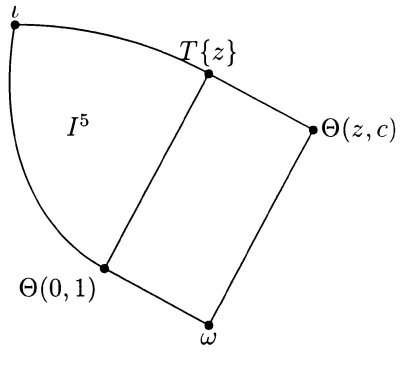

Fig. 2

Example 2. Although $(A; \wedge, \vee)$ can be "nice" and distributive, its $\operatorname{Tol} A$ is rather big in the case if $(A; \wedge, \vee)$ contains a cell. Such $\operatorname{Tol} A$ for a q-lattice visualized in Fig. 3 is the distributive lattice (by the foregoing Corollary) in Fig. 4. All tolerances of $\operatorname{Tol} A$ are listed in Fig. 5.

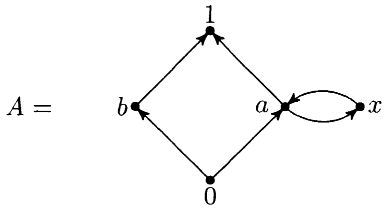

Fig. 3

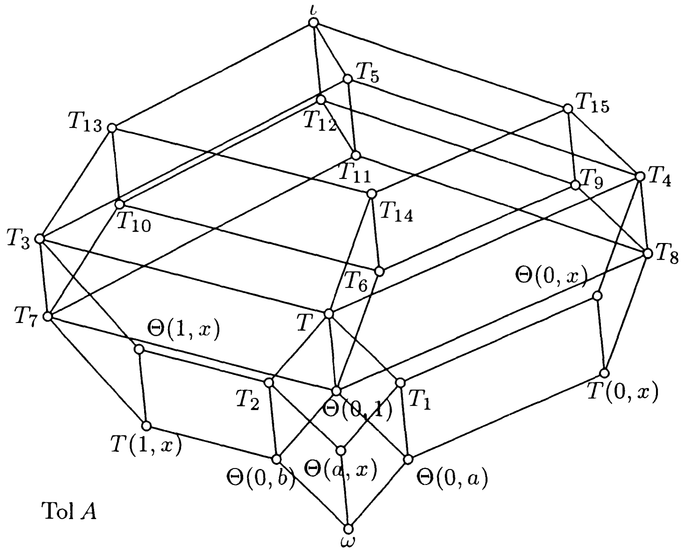

Fig. 4

Theorem 2. *If a q-lattice has at least two different cells then* Tol A *is not modular.*

P r o o f. Let A have cells $C_1 \neq C_2$, let c_i be the idempotent in C_i, $i = 1, 2$ and let $a \in C_1$, $b \in C_2$ be non-idempotents. Denote by $T(\langle u_1, v_1\rangle, \ldots, \langle u_n, v_n\rangle)$ the least tolerance of Tol A containing the pairs $\langle u_1, v_1\rangle$, …, $\langle u_n, v_n\rangle$. Now, put

$$\begin{aligned}
T_0 &= T(\langle a, b\rangle, \langle a, c_1\rangle),\\
T_x &= T(\langle a, b\rangle, \langle a, c_1\rangle, \langle b, c_1\rangle),\\
T_y &= T(\langle a, b\rangle, \langle a, c_1\rangle, \langle b, c_1\rangle, \langle b, c_2\rangle),\\
T_z &= T(\langle a, b\rangle, \langle a, c_1\rangle, \langle a, c_2\rangle),\\
T_1 &= T(\langle a, b\rangle, \langle a, c_1\rangle, \langle a, c_2\rangle, \langle b, c_1\rangle, \langle b, c_2\rangle).
\end{aligned}$$

Since $\langle a, b\rangle \in T_i$ for $i \in \{0, x, y, z, 1\}$ and a, b are non-idempotents, we have also $\langle c_1, c_2\rangle = \langle a \vee a, b \vee b\rangle \in T_i$.

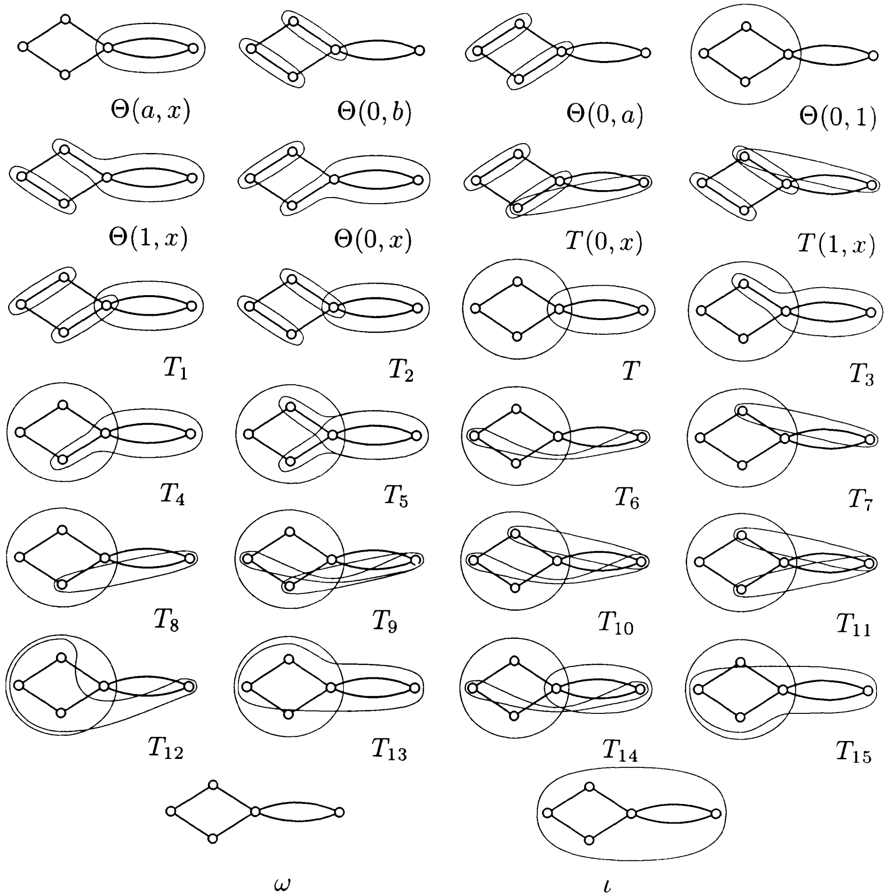

Fig. 5

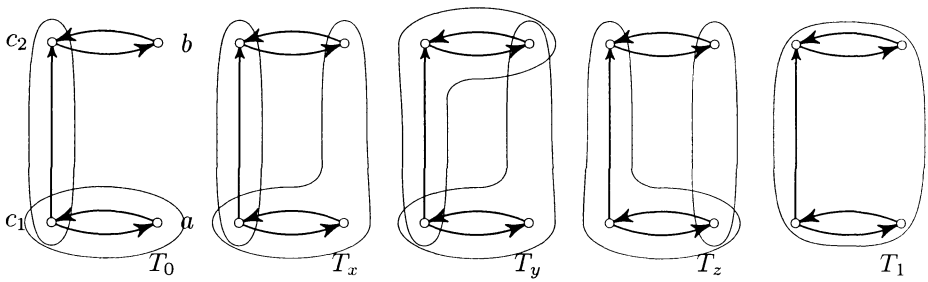

Fig. 6

(1) If $c_1 < c_2$, tolerances are visualized in Fig. 6:

(2) If c_1, c_2 are non-comparable elements (of the skeleton), the situation is visualized in Fig. 7.

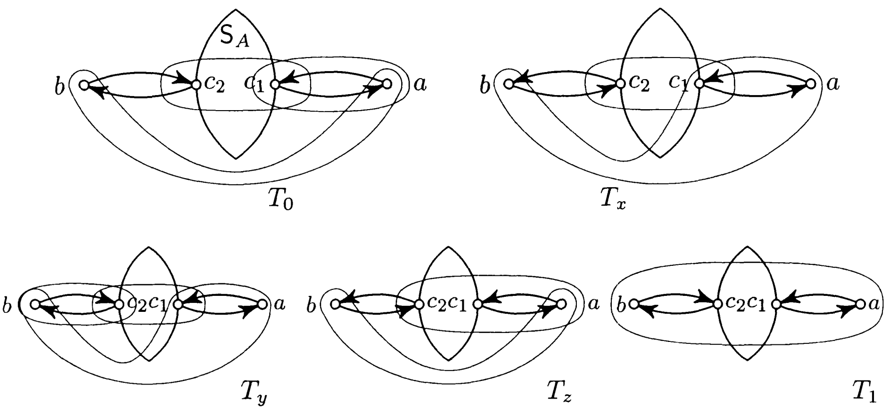

Fig. 7

It is routine to show that in both of the foregoing cases, tolerances T_0, T_x, T_y, T_z, T_1 form a sublattice N_5 of $\mathrm{Tol}\, A$, see Fig. 8. $\square$

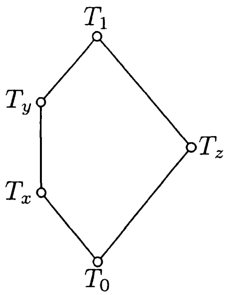

Fig. 8

Remark 2. If $(A; \wedge, \vee)$ is a q-lattice with a skeleton S_A and $\mathrm{Tol}\, S_A$ is not distributive then $\mathrm{Tol}\, A$ is not distributive either since $\mathrm{Tol}\, S_A$ is a sublattice of $\mathrm{Tol}\, A$.

Corollary. *For a distributive q-lattice $(A; \wedge, \vee)$, the following conditions are equivalent:*

(i) $\mathrm{Tol}\, A$ *is distributive;*

(ii) $(A; \wedge, \vee)$ *has at most one cell.*

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